Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

INTERVAL TYPE-2 FUZZY LINEAR PROGRAMMING PROBLEM WITH VAGUENESS IN THE RESOURCES VECTOR

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Article type: Research Article
(Received: 24 September 2023, Received in revised form 20 January 2024)
(Accepted: 09 February 2024, Published Online: 10 February 2024)

Abstract. One of the special cases of type-2 fuzzy sets is the interval type-2 fuzzy sets, which are less complicated and easier to understand than T2FSs. In this study, we explore the interval type-2 fuzzy linear programming problem with the resources vector that has imprecision of the vagueness type. These types of vagueness are expressed via membership functions. First, we review the three available methods, including the Figueroa and Sarani methods. Then, using the three ideas of Verdegay, Werners, and Guu and Wu for solving fuzzy linear programming problems with vagueness in the resources vector, we propose three new methods for solving interval type-2 fuzzy linear programming problems with vagueness in the resources vector. Finally, we demonstrate the effectiveness of our proposed methods by solving an example and comparing the results obtained with each other and with those of previous methods.

Keywords: Fuzzy linear programming, interval linear programming, interval type-2 fuzzy set, resources vector.
2020 MSC: Primary 03E72, 08A72.

1. Introduction

In real-world problems, it is assumed that the data have exact amounts, whereas the considered values of the data are often imprecise because of incomplete information. Real number arithmetic cannot be used to analyze real models with imprecise data. Therefore, interval, stochastic, or fuzzy approaches are employed to analyze them. First, Zadeh introduced the fuzzy sets (FSs) theory [33]. Because of the existence of membership functions (MFs) with precise membership degrees, fuzzy systems have limited capability to reduce the effect of uncertainty and are not suitable for solving complex problems with high uncertainties. In 1975, Zadeh proposed type-2 fuzzy sets (T2FSs) as a generalized version of FSs [34–36]. A specific case of T2FS is interval type-2 fuzzy sets (IT2FSs), which presents more information and uncertainties. The interval type-2 fuzzy linear programming (IT2FLP) problem has become more

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https://doi.org/10.22103/jmmr.2024.22238.1515

Publisher: Shahid Bahonar University of Kerman
How to cite: Sh. Sargolzaei and H. Mishmast Nehi, *Interval type-2 fuzzy linear programming problem with vagueness in the resources vector*, J. Mahani Math. Res. 2024; 13(2): 233 - 261.

important and has been used in many studies in recent years. As stated above, many researchers have studied and introduced several methods for solving different types of fuzzy linear programming (FLP) problems: Akram et al. proposed several methods for solving fuzzy and fully fuzzy linear programming problems [2–4]. Hao Li et al. [17] mentioned in their article that IT2FSs can be used to solve practical problems such as supplier selection [22, 30], portfolio allocation [31], COVID-19 forecasting [9], renewable energy evaluation [1, 20] and fuzzy control [8, 21, 25].

FLP has been expanded to deal with vagueness in the framework of optimization problems. According to the type of uncertainty in the problem, FLP problems are divided into flexible, possibilistic, and robust categories. Furthermore, in each of these divisions, concerning the position of uncertainty in the problem, various states can be created and different solution methods are provided for each case. Tanaka et al. [26] proposed the concept of the FLP problem on a general level in the fuzzy decision framework of Bellman and Zadeh [7]. Figueroa improved Zimmerman's method [37] for solving the IT2FLP problem with the right-hand side expressed as MFs [10–13]. In this case, MFs were used to represent the uncertainty in the degree of satisfaction of the objective function and the constraints. Golpaygani et al. presented a new method for solving two special cases of IT2FLP problems [14]. Furthermore, we explore the flexible single-objective IT2FLP problem with imprecision type of vagueness. This vagueness is accrued in the resources vector and is represented by the MF. Also, we investigate the IT2FLP problem with a maximization objective function and less than or equal constraints. It should be noted that in an interval linear programming problem, infinitely many linear programming problems must be solved. But based on the theorems related to interval programming, instead of solving infinitely many problems [5, 24], we try to obtain the upper bound on the biggest feasible region. In fact, according to the maximization of the objective function, we are looking for the best solution.

We review three methods used to solve IT2FLP problems. The first two methods are from Figueroa [11, 12] and the last one is from Sarani [23]. By solving a numerical example, we compare these three methods. We briefly state the advantages and main features of the two methods of Figueroa [12] and Sarani's method:

- The first one gets interval type-1 fuzzy set (T1FS) embedded on the footprint of uncertainty (FOU) of each IT2 right hand side, and then it is optimized. The second approach is a pseudo-optimal approach that reduces the complexity of the problem by using $\alpha-$cuts and interval optimization.
- These procedures can vary in their results because they are optimal in different ways. Moreover, the decision maker should select the best choice depending on the computational effort, amount of restrictions, and variables.

- The reader should keep in mind that the first method solves three linear programming (LP) problems before finding a solution, whereas the second method solves only one LP problem. This could be a disadvantage for large-scale problems.
- The second method is conditioned on the selection of the α value, given by an expert in the system; therefore, it does not provide the same solution as the first method.
- Sarani's method shows a higher degree of satisfaction of constraints than the two Figueroa methods. However, the optimization of this method is less. If the decision maker does not consider optimality and only the degree of satisfaction of the constraints is important, the Sarani's method is more suitable; otherwise, the methods of Figueroa are considered.

In the following, we propose three new methods for solving the IT2FLP problem with vagueness in the resources vector. The first new method is based on Verdegay's method for solving the FLP problem [27]. He proved that the optimal solution of an FLP can be found by the use of solving an equivalent LP problem assuming that the objective function is crisp. It assumes that the MFs of the fuzzy constraints are non-increasing and continuous, and the objective function is crisp. This method is non-symmetric. The second proposed method is based on Werners's idea to solve the FLP problem [29]. He suggested that the objective function should be fuzzy because of fuzzy inequality constraints and computed the lower and upper bounds of the optimal values by solving two crisp LP problems. The third proposed method is based on the idea of Guu and Wu [15]. They proposed a two-phase method for solving the FLP problem that not only pursues the highest membership degree in the objective but also enables better use of each constrained resource. Finally, to show the performance and efficiency of each of the new proposed methods, we solved the example used in the review methods and compared the obtained results with each other and with the review methods. In short, the main structure and advantages of this study are as follows:

- Reviewing and examining three solution methods for the IT2FLP problem: two methods of Figueroa and one method of Sarani;
- Investigating the performance of these three methods by solving a numerical example and comparing the obtained results;
- Proposing three new methods for solving the IT2FLP problem based on three ideas for solving the FLP problem, from Verdegay, Werners, and Guo and Wu;
- Examining the performance and efficiency of the three proposed methods by solving the same numerical example and comparing the results with those obtained by solving the three review methods.
- Since the Bellman-Zadeh operator is used to find a crisp solution to the IT2F constrained problem, our proposed methods are flexible and

interpretable. Hence, our proposal is appropriate for numerous similar problems. In addition, the third proposed method improves situations in which the max-min operator is not sufficient.

This study is arranged as follows: Firstly, in Section 2, the basic definitions and essential concepts are presented. Secondly, in Section 3, the MFs of the IT2F resources vector and the approaches introduced by Figueroa et al. [11, 12] and Sarani [23] for solving IT2FLP problems with vagueness in the resources vectors are reviewed. Then, in Section 4, three new approaches for solving IT2FLP problems with vagueness in the resource vectors are proposed. In Section 5, we present two numerical examples. To illustrate the efficiency and performance of our proposed method, we provide a concrete real-world example [13]. This application example is related to the classical transportation problem. The second example is that used by Figueroa and Sarani. Finally, the results of our methods are compared with those of review methods, also compared our methods with each other. Table 1 describes the symbols used in this study.

Table 1. Reference of the symbols used in this article, for example (a).

Symbols	Description
T2FS	$\tilde{\tilde{a}}$
Upper membership function (UMF)	$\overline{a}$
Lower membership function (LMF)	$\underline{a}$
Left MF	$a^{\vee}$
Right MF	$a^{\wedge}$

2. Basic definitions and preliminaries of the IT2FS

This section introduces the basic terminologies of IT2FSs and the interval linear programming (ILP) problem.

2.1. The IT2FSs.

A T2FS collects an infinite number of FSs and is characterized by two MFs. In this subsection, we present important and necessary definitions related to IT2Fs [18].

Definition 2.1. A T2FS $\tilde{\tilde{A}}$ is defined as follows:

$$\tilde{\tilde{A}} = \int_{x\in X}\int_{u\in J_x} f_x(u)/(x,u) = \int_{x\in X}\left[\int_{u\in J_x} f_x(u)/u\right]\Big/ x,$$

where $J_x \subseteq [0,1]$, $x \in X$, $u \in [0,1]$, $f_x(u) \in [0,1]$, $f_x(u) \in [0,1]$ is initial membership and $f_x(u)$ is the secondary grade. Fig. 1 shows the graphical image of the three-dimensional of an IT2FS.

Definition 2.2. An IT2FS, $\tilde{\tilde{A}}$, is defined as:

$$\tilde{\tilde{A}} = \int_{x\in X} \int_{u\in J_x} 1/(x,u) = \int_{x\in X} \left[\int_{u\in J_x} 1/u\right] \Big/ x,$$

where $J_x \subseteq [0,1]$, $x \in X$ and $u \in [0,1]$, (see Fig. 1).

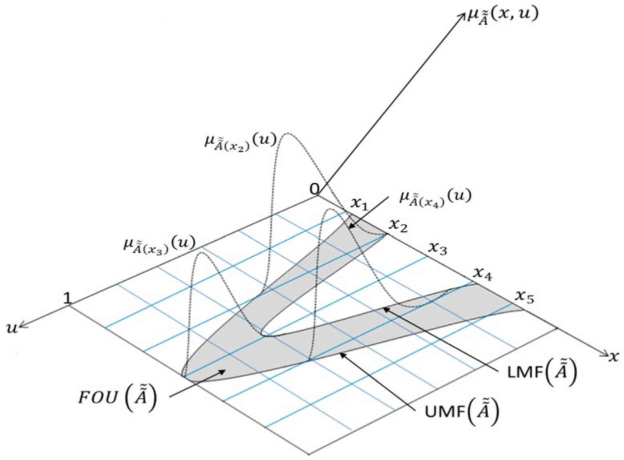

FIGURE 1. T2FS $\tilde{\tilde{A}}$.

Definition 2.3. ([19]) Uncertainty in the initial memberships of a T2FS $\tilde{\tilde{A}}$ consists of a bounded area that we name FOU. It is the union of all initial memberships, i.e.

$$FOU(\tilde{\tilde{A}}) = \bigcup_{x\in X} J_x.$$

A difference lies between the T1FS and T2FS in FOU of a T2FS which gets an infinite number of T1FSs.

Definition 2.4. The α-cut of an IT2FS is (see Fig. 2):

$$^{\alpha}\tilde{\tilde{a}}_{ij} = \left\{ \left(\tilde{\tilde{a}}_{ij}, u\right) \middle| \, J_{\tilde{\tilde{a}}_{ij}} \geq \alpha, u \in [0,1] \right\}.$$

It is shown by two parts as below:

$$^{\alpha}\overline{\mu}_{\tilde{\tilde{a}}_{ij}} = \left\{ \left(\tilde{\tilde{a}}_{ij}, u\right) \middle| \, \overline{\mu}_{\tilde{\tilde{a}}_{ij}} \geq \alpha \right\}, \tag{1}$$

$$^{\alpha}\underline{\mu}_{\tilde{\tilde{a}}_{ij}} = \left\{ \left(\tilde{\tilde{a}}_{ij}, u\right) \middle| \, \underline{\mu}_{\tilde{\tilde{a}}_{ij}} \geq \alpha \right\}, \tag{2}$$

where (1) is UMF $\tilde{\tilde{a}}_{ij}$ corresponding $\alpha \in [0,1]$ and (2) is LMF $\tilde{\tilde{a}}_{ij}$ corresponding $\alpha \in [0,1]$. So any T2FS can be decomposed into α-cuts through the use of $\overline{\mu}_{\tilde{\tilde{a}}_{ij}}$ and $\underline{\mu}_{\tilde{\tilde{a}}_{ij}}$.

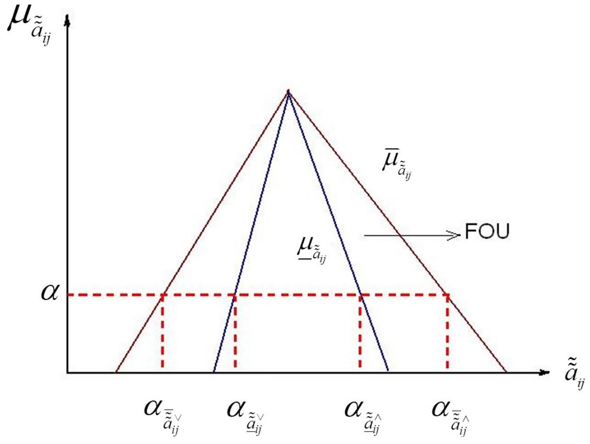

FIGURE 2. The α-cut of an IT2FS.

2.2. **The uncertainty in resources vector.** In this subsection, referring to uncertain resources vector when using IT2FS [12].

Definition 2.5. Consider a set of parameters to the right of the constraints $\tilde{\tilde{\mathbf{b}}}$ from an FLP problem that is defined as IT2FSs. The MF which represents $\tilde{\tilde{\mathbf{b}}}_i$ is:

$$\tilde{\tilde{\mathbf{b}}}_i = \int_{\mathbf{b}_i \in \mathbb{R}} \left[\int_{u \in J_{\mathbf{b}_i}} 1/u \right] \Big/ \mathbf{b}_i, \;\; i \in N_m, \;\; J_{\mathbf{b}_i} \subseteq [0,1] \,.$$

Notice that $\tilde{\tilde{\mathbf{b}}}$ is bounded by two primary LMF and UMF called $\underline{\mu}_{\tilde{\tilde{\mathbf{b}}}}(x)$ with components $\underline{\mathbf{b}}^{\vee}$ and $\underline{\mathbf{b}}^{\wedge}$ and $\overline{\mu}_{\tilde{\tilde{\mathbf{b}}}}(x)$ with components $\overline{\mathbf{b}}^{\vee}$ and $\overline{\mathbf{b}}^{\wedge}$, respectively.

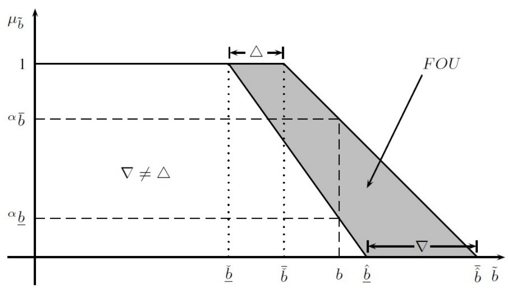

FIGURE 3. IT2FS with joint uncertain $\triangle$ and $\bigtriangledown$.

In Fig. 3, $\tilde{\tilde{\mathbf{b}}}$ is an IT2FS with linear MFs $\underline{\mu}_{\tilde{\tilde{b}}}(x)$ and $\overline{\mu}_{\tilde{\tilde{b}}}(x)$. A particular value b projects an interval of infinite membership degrees $u \in J_b$, as follows:

$$J_b \in [\alpha_{\overline{b}}, \alpha_{\underline{b}}]\forall b \in \mathbb{R},$$

where J_b is the set of all possible membership degrees associated with $b \in \Re$.

2.3. **The ILP.** This subsection presents the necessary definition and theorem for the ILP problem [5, 16] and introduces the method to solve it.

Definition 2.6. ([5]) An interval number appears as $[\underline{x}, \overline{x}]$ in which the $\underline{x} \leq \overline{x}$ condition is satisfied. If it is $\underline{x} = \overline{x}$, x will be destroyed.

The basic form of the ILP problem is defined as follows [16, 32]:

$$\begin{aligned} \max \quad & z = \sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j]x_j \\ s.t. \quad & \sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}]x_j \leq [\underline{b}_i, \overline{b}_i], \quad i = 1, 2, ..., m, \\ & x_j \geq 0, \quad j = 1, 2, ..., n, \end{aligned} \tag{3}$$

where $\underline{c}_j$,$\overline{c}_j$, $\underline{a}_{ij}$, $\overline{a}_{ij}$, $\underline{b}_i$ and $\overline{b}_i$ are real numbers. The characteristic model of problem is as follows:

$$\begin{aligned} \max \quad & z = \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad i = 1, 2, ..., m, \\ & x_j \geq 0, \quad j = 1, 2, ..., n, \end{aligned}$$

where $c_j \in [\underline{c}_j, \overline{c}_j]$, $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$, and $b_i \in [\underline{b}_i, \overline{b}_i]$. Several methods have been proposed to solve the IP problem [6, 28, 38]. One of the basic methods to solve this type of problem is the best worst cases (BWC) method [24]. This method divides model (3) into two best and worst sub-models.

Theorem 2.7. *([16]) Consider* $\sum_{j=1}^{n} \left[\underline{a}_{ij}, \overline{a}_{ij}\right] x_j \leq \left[\underline{b}_i, \overline{b}_i\right]$, *for* $i = 1, 2, ..., m$. *The biggest feasible region is* $\sum_{j=1}^{n} \underline{a}_{ij}x_j \leq \overline{b}_i$ *and the smallest feasible region is* $\sum_{j=1}^{n} \overline{a}_{ij}x_j \leq \underline{b}_i$.

The following theorem obtains the worst and best values of the objective function.

Theorem 2.8. *([16]) The best value of the objective function of the ILP problem* (3) *is acquired by:*

$$\begin{array}{lll} \max & = \sum_{j=1}^{n} \underline{c}_j x_j & \\ s.t. & \sum_{j=1}^{n} \overline{a}_{ij} x_j \leq \underline{b}_i, & i = 1, 2, ..., m, \\ & x_j \geq 0, & j = 1, 2, ..., n, \end{array}$$

and the worst values of the objective function of the ILP problem (3) *results in:*

$$\begin{array}{lll} \max & \overline{z} = \sum_{j=1}^{n} \overline{c}_j x_j & \\ s.t. & \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \overline{b}_i, & i = 1, 2, ..., m, \\ & x_j \geq 0, & j = 1, 2, ..., n. \end{array}$$

3. IT2FLP problem with vagueness in the resources vector

This section includes two parts. In the first subsection, we express the MFs of the IT2F resources vector. An IT2FLP model with the maximization objective function and IT2F resources vector is presented as follows:

$$(4) \qquad \begin{array}{lll} max & \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j & \\ s.t. & \sum_{j=1}^{n} a_{ij} \mathbf{x}_j \leq \tilde{\tilde{\mathbf{b}}}_{\mathbf{i}}, & i = 1, ..., m, \\ & \mathbf{x}_j \geq 0, & j = 1, 2, ..., n, \end{array}$$

where $\mathbf{c}^t, \mathbf{x} \in \mathbb{R}^n$, A is $n \times m$ matrix of real elements denoted by a_{ij}, and $\tilde{\tilde{\mathbf{b}}}_i$ is $m \times 1$ vector of IT2FSs.

3.1. **MFs of IT2F resources vector.** As we know, $\tilde{\tilde{\mathbf{b}}}$ is an IT2FS defined by its two primary MFs $\underline{\mu}_{\tilde{\tilde{\mathbf{b}}}}(x)$ and $\overline{\mu}_{\tilde{\tilde{\mathbf{b}}}}(x)$. There are two MFs the LMF and the UMF for $\leq$. Also, there are two MFs the LMF and the UMF for $\geq$. This means that there are four possible relative orders $\leq$ and $\geq$ with IT2 fuzzy resources vector (see Fig. 3).
The LMF for $\leq$ is:

$$\underline{\mu}_{\tilde{\tilde{b}}_i}((A\mathbf{x})_i; \underline{b}_i^{\vee}; \underline{b}_i^{\wedge}) = \begin{cases} 1, & (A\mathbf{x})_i \leq \underline{b}_i^{\vee}, \\ \frac{\underline{b}_i^{\wedge} - (A\mathbf{x})_i}{\underline{b}_i^{\wedge} - \underline{b}_i^{\vee}}, & \underline{b}_i^{\vee} \leq (A\mathbf{x})_i \leq \underline{b}_i^{\wedge}, \\ 0, & (A\mathbf{x})_i \geq \underline{b}_i^{\wedge}. \end{cases}$$

The UMF for $\leq$ is:

$$\overline{\mu}_{\tilde{\tilde{b}}_i}\left((A\mathbf{x})_i; \overline{b}_i^{\vee}; \overline{b}_i^{\wedge}\right) = \begin{cases} 1, & (A\mathbf{x})_i \leq \overline{b}_i^{\vee}, \\ \frac{\overline{b}_i^{\wedge} - (A\mathbf{x})_i}{\overline{b}_i^{\wedge} - \overline{b}_i^{\vee}}, & \overline{b}_i^{\vee} \leq (A\mathbf{x})_i \leq \overline{b}_i^{\wedge}, \\ 0, & (A\mathbf{x})_i \geq \overline{b}_i^{\wedge}. \end{cases}$$

The LMF for $\geq$ is:

$$\underline{\mu}_{\tilde{b}_i}\left((A\mathbf{x})_i;\underline{b}_i^{\vee};\underline{b}_i^{\wedge}\right)=\begin{cases}1, & (A\mathbf{x})_i\geq\underline{b}_i^{\vee},\\ \frac{(A\mathbf{x})_i-\underline{b}_i^{\wedge}}{\underline{b}_i^{\vee}-\underline{b}_i^{\wedge}}, & \underline{b}_i^{\wedge}\leq(A\mathbf{x})_i\leq\underline{b}_i^{\vee},\\ 0, & (A\mathbf{x})_i\leq\underline{b}_i^{\wedge}.\end{cases}$$

The UMF for $\geq$ is:

$$\overline{\mu}_{\tilde{b}_i}\left((A\mathbf{x})_i;\overline{b}_i^{\vee};\overline{b}_i^{\wedge}\right)=\begin{cases}1, & (A\mathbf{x})_i\geq\overline{b}_i^{\vee},\\ \frac{(A\mathbf{x})_i-\overline{b}_i^{\wedge}}{\overline{b}_i^{\vee}-\overline{b}_i^{\wedge}}, & \overline{b}_i^{\wedge}\leq(A\mathbf{x})_i\leq\overline{b}_i^{\vee},\\ 0, & (A\mathbf{x})_i\leq\overline{b}_i^{\wedge}.\end{cases}$$

3.2. **Review of the three methods used for solving the IT2FLP problem with vagueness in the resources vector.** In this subsection, we review three methods to solve LP problems in which the resources vector is the IT2FSs. The first two approaches were proposed by Figueroa [11, 12], and the third method was proposed by Sarani [23].

3.2.1. *The first method of Figueroa.* In this sub-subsection, we discuss the first method of Figueroa.

1. In the beginning, the lower optimal FS (shown as $\underline{\tilde{z}}$) is computed as follows:

(a). The lower bound which refers to minimum $\underline{z}$ (shown as $\underline{z}^{\vee}$) is computed by using $\underline{\mathbf{b}}^{\vee}$ as a bound;
(b). The upper bound named maximum $\underline{z}$ (shown as $\underline{z}^{\wedge}$) is computed by using $\underline{\mathbf{b}}^{\wedge}$ as a bound;
(c). FS $\underline{\tilde{z}}(x^*)$ is defined by bonds as $\underline{z}^{\vee}$, $\underline{z}^{\wedge}$, and trapezoidal MF;
(d). If the purpose is to maximize the function, then its MF is as:

$$\underline{\mu}_{\tilde{z}}(\mathbf{cx};\underline{z}^{\vee},\underline{z}^{\wedge})=\begin{cases}0, & \mathbf{cx}\leq\underline{z}^{\vee},\\ \frac{\mathbf{cx}-\underline{z}^{\vee}}{\underline{z}^{\wedge}-\underline{z}^{\vee}}, & \underline{z}^{\vee}\leq\mathbf{cx}\leq\underline{z}^{\wedge},\\ 1, & \mathbf{cx}\geq\underline{z}^{\wedge}.\end{cases}$$

(e). If the purpose is to minimize the function, then its MF is as:

$$\underline{\mu}_{\tilde{z}}(\mathbf{cx};\underline{z}^{\vee},\underline{z}^{\wedge})=\begin{cases}0, & \mathbf{cx}\leq\underline{z}^{\vee},\\ \frac{\underline{z}^{\vee}-\mathbf{cx}}{\underline{z}^{\wedge}-\underline{z}^{\vee}}, & \underline{z}^{\vee}\leq\mathbf{cx}\leq\underline{z}^{\wedge},\\ 1, & \mathbf{cx}\geq\underline{z}^{\wedge}.\end{cases}$$

2. The upper optimal FS (shown as $\overline{\tilde{z}}$) is computed as follows:

(a). The lower bound called minimum $\underline{z}$ (shown as $\overline{z}^{\vee}$) is computed by using $\overline{\mathbf{b}}^{\vee}$ as a bound;
(b). The upper bound named maximum $\underline{z}$ (shown as $\overline{z}^{\wedge}$) is computed by using $\overline{\mathbf{b}}^{\wedge}$ as a bound;
(c). FS $\overline{\tilde{z}}(x^*)$ is defined by bounds as $\overline{z}^{\vee}$, $\overline{z}^{\wedge}$, and trapezoidal MF;

(d). If the purpose is to maximize the function, then its MF is as:

$$\overline{\mu}_{\tilde{\tilde{z}}}(\mathbf{cx};\overline{z}^{\vee},\overline{z}^{\wedge})=\begin{cases}0, & \mathbf{cx}\leq\overline{z}^{\vee},\\ \frac{\mathbf{cx}-\overline{z}^{\vee}}{\overline{z}^{\wedge}-\overline{z}^{\vee}}, & \overline{z}^{\vee}\leq\mathbf{cx}\leq\overline{z}^{\wedge},\\ 1, & \mathbf{cx}\geq\overline{z}^{\wedge}.\end{cases}$$

(e). If the purpose is to minimize the function, then its MF is as:

$$\overline{\mu}_{\tilde{\tilde{z}}}(\mathbf{cx};\overline{z}^{\vee},\overline{z}^{\wedge})=\begin{cases}1, & \mathbf{cx}\leq\overline{z}^{\vee},\\ \frac{\overline{z}^{\vee}-\mathbf{cx}}{\overline{z}^{\wedge}-\overline{z}^{\vee}}, & \overline{z}^{\vee}\leq\mathbf{cx}\leq\overline{z}^{\wedge},\\ 0, & \mathbf{cx}\geq\overline{z}^{\wedge}.\end{cases}$$

3. The optimal α-cut for the lower and upper MFs that are called $\underline{\alpha}$ and $\overline{\alpha}$, respectively, are Find. If the purpose is to maximize the objective function, then its optimal $\underline{\alpha}$ would be:

$$\begin{array}{rl} max & \underline{\alpha} \\ s.t. & \mathbf{cx}-\underline{\alpha}(\underline{z}^{\wedge}-\underline{z}^{\vee})=\underline{z}^{\vee}, \\ & A\mathbf{x}+\underline{\alpha}(\underline{b}^{\wedge}-\underline{b}^{\vee})\leq\underline{b}^{\wedge}, \\ & x,\underline{\alpha}\geq 0. \end{array} \tag{5}$$

By minimizing the objective function, its optimal $\underline{\alpha}$ would be:

$$\begin{array}{rl} max & \underline{\alpha} \\ s.t. & \mathbf{cx}+\underline{\alpha}(\underline{z}^{\wedge}-\underline{z}^{\vee})=\underline{z}^{\wedge}, \\ & A\mathbf{x}-\underline{\alpha}(\underline{b}^{\vee}-\underline{b}^{\wedge})\geq\underline{b}^{\vee}, \\ & \mathbf{x},\underline{\alpha}\geq 0. \end{array} \tag{6}$$

By maximizing the objective function, its optimal $\overline{\alpha}$ would be:

$$\begin{array}{rl} max & \overline{\alpha} \\ s.t. & \mathbf{cx}-\overline{\alpha}(\overline{z}^{\wedge}-\overline{z}^{\vee})=\overline{z}^{\vee}, \\ & A\mathbf{x}+\overline{\alpha}(\overline{b}^{\wedge}-\overline{b}^{\vee})\leq\overline{b}^{\wedge}, \\ & \mathbf{x},\overline{\alpha}\geq 0. \end{array} \tag{7}$$

By minimizing the objective function, then its optimal $\overline{\alpha}$ would be:

$$\begin{array}{rl} max & \overline{\alpha} \\ s.t. & \mathbf{cx}+\overline{\alpha}(\overline{z}^{\wedge}-\overline{z}^{\vee})=\overline{z}^{\wedge}, \\ & A\mathbf{x}-\overline{\alpha}(\overline{b}^{\vee}-\overline{b}^{\wedge})\geq\overline{b}^{\vee}, \\ & \mathbf{x},\overline{\alpha}\geq 0. \end{array} \tag{8}$$

In the problems (5), (6), (7), and (8), we have $x\in\mathbb{R}^n$ and $\underline{\alpha},\overline{\alpha}\in[0,1]$.

4. we select the best solution by using a comparison between $\underline{\alpha}^*$ and $\overline{\alpha}^*$ as optimal α-cuts.

3.2.2. *The second method of Figueroa.* In the sub-subsection, we review the second method of Figueroa.

1. The lower bound called minimum Z (shown as $z^{\vee}$) is computed by using $\underline{\mathbf{b}}^{\vee} + \Delta$ as bound in the model where Δ is an auxiliary set of variables that show the lower uncertainty intervals in this way:

$$\Delta \leq \overline{\mathbf{b}}^{\vee} - \underline{\mathbf{b}}^{\vee}.$$

2. The upper bound called maximum Z (shown as $z^{\wedge}$) is computed by using $\overline{\mathbf{b}}^{\wedge} + \nabla$ as bound in the model where ∇ is an auxiliary set of variables that shows the lower uncertainty intervals in this way:

$$\nabla \leq \overline{\mathbf{b}}^{\wedge} - \underline{\mathbf{b}}^{\wedge}.$$

3. Obtain the optimal α-cut, consider $b^{\vee} = \underline{\mathbf{b}}^{\vee} + \Delta$ and $b^{\wedge} = \overline{\mathbf{b}}^{\wedge} + \nabla$. by maximizing the objective function, then its optimal α would be:

$$\begin{array}{ll} \max & \alpha \\ s.t. & \mathbf{cx} - \alpha(z^{\wedge} - z^{\vee}) = z^{\vee}, \\ & A\mathbf{x} + \alpha(b^{\wedge} - b^{\vee}) \leq b^{\wedge}, \\ & \mathbf{x} \geq 0,\ \alpha \in [0,1]. \end{array}$$

By minimizing the objective function, then its optimal α would be:

$$\begin{array}{ll} \max & \alpha \\ s.t. & \mathbf{cx} + \alpha(z^{\wedge} - z^{\vee}) = z^{\wedge}, \\ & A\mathbf{x} - \alpha(b^{\vee} - b^{\wedge}) \geq b^{\vee}, \\ & \mathbf{x} \geq 0,\ \alpha \in [0,1]. \end{array}$$

This method may result in multiple solutions due to given conditions in (5) and (6). Therefore, it is recommended to prevent this situation indicate a proper weight for each variable based on its beneficiary increase of specific resources.

3.2.3. *The Sarani method.* In this sub-subsection, we review the Sarani method [23]:

1. Start computing the lower optimal FS named $\underline{\tilde{z}}$, as follows:

(a.) The lower bound called minimum $\underline{z}$ (shown as $\underline{z}^{\vee}$) is computed by using $\underline{\mathbf{b}}^{\vee}$ as a bound;

(b.) The upper bound called maximum $\underline{z}$ (shown as $\underline{z}^{\wedge}$) is computed by using $\underline{\mathbf{b}}^{\wedge}$ as a bound;

(c.) FS $\underline{\tilde{z}}(x^*)$ is defined by trapezoidal MF, bound $\underline{z}^{\wedge}$ and bound $\underline{z}^{\vee}$;

(d.) Assume the objective function is to maximize, its MF would be:

$$\underline{\mu}_{\underline{\tilde{z}}}(\mathbf{cx}; \underline{z}^{\vee}, \underline{z}^{\wedge}) = \begin{cases} 0, & \mathbf{cx} \leq \underline{z}^{\vee}, \\ \frac{\mathbf{cx} - \underline{z}^{\vee}}{\underline{z}^{\wedge} - \underline{z}^{\vee}}, & \underline{z}^{\vee} \leq \mathbf{cx} \leq \underline{z}^{\wedge}, \\ 1. & \mathbf{cx} \geq \underline{z}^{\wedge}. \end{cases}$$

(e.) Assume the objective function is to minimize, its MF would be:

$$\underline{\mu}_{\underline{\tilde{z}}}(\mathbf{cx}; \underline{z}^{\vee}, \underline{z}^{\wedge}) = \begin{cases} 0, & \mathbf{cx} \leq \underline{z}^{\vee}, \\ \frac{\underline{z}^{\vee} - \mathbf{cx}}{\underline{z}^{\wedge} - \underline{z}^{\vee}}, & \underline{z}^{\vee} \leq \mathbf{cx} \leq \underline{z}^{\wedge}, \\ 1. & \mathbf{cx} \geq \underline{z}^{\wedge}. \end{cases}$$

2. Start computing the upper optimal FS (shown as $\overline{\tilde{z}}$) as follows:

(a.) The lower bound called minimum $\underline{z}$ (shown as $\overline{z}^{\vee}$) is computed by using $\overline{\mathbf{b}}^{\vee}$ as a bound;

(b.) The upper bound called maximum $\underline{z}$ (shown as $\overline{z}^{\wedge}$) is computed by using $\overline{\mathbf{b}}^{\wedge}$ as a bound;

(c.) FS $\overline{\tilde{z}}(\mathbf{x}^*)$ is defined by trapezoidal MF, bounds $\overline{z}^{\vee}$ and $\overline{z}^{\wedge}$;

(d.) If the purpose is to maximize the function, then its MF would be:

$$\overline{\mu}_{\tilde{z}}(\mathbf{cx}; \overline{z}^{\vee}, \overline{z}^{\wedge}) = \begin{cases} 0, & \mathbf{cx} \leq \overline{z}^{\vee}, \\ \frac{\mathbf{cx} - \overline{z}^{\vee}}{\overline{z}^{\wedge} - \overline{z}^{\vee}} & \overline{z}^{\vee} \leq \mathbf{cx} \leq \overline{z}^{\wedge}, \\ 1, & \mathbf{cx} \geq \overline{z}^{\wedge}. \end{cases}$$

(e.) If the purpose is to minimize the function, then its MF would be:

$$\overline{\mu}_{\tilde{z}}(\mathbf{cx}; \overline{z}^{\vee}, \overline{z}^{\wedge}) = \begin{cases} 1, & \mathbf{cx} \leq \overline{z}^{\vee}, \\ \frac{\overline{z}^{\vee} - \mathbf{cx}}{\overline{z}^{\wedge} - \overline{z}^{\vee}}, & \overline{z}^{\vee} \leq \mathbf{cx} \leq \overline{z}^{\wedge}, \\ 0, & \mathbf{cx} \geq \overline{z}^{\wedge}. \end{cases}$$

3. Using the above two steps, the objective function of the problem (4) is converted to a fuzzy objective function. Since the objective function is definitive, the optimal solution obtained by solving the problem should be definitive as well. So, the Bellman-Zadeh operator [7] is applied. According to the Bellman-Zadeh operator, we have

$$\mu_D(\mathbf{x}^*) = MaxMin\{\mu((A\mathbf{x})_i, u), \mu(\mathbf{cx}, u)\}.$$

For linearization, it is necessary to consider $\alpha = Min\{\mu((A\mathbf{x})_i, u), \mu(\mathbf{cx}, u)\}$. So, we can write the LP problem as follows:

$$\begin{aligned} & max \;\; \alpha \\ & s.t. \;\; \alpha \leq \mu(\mathbf{cx}, u), \\ & \qquad \alpha \leq \mu((A\mathbf{x})_i, u), \\ & \qquad \mathbf{x} \geq 0, \; \alpha \in [0, 1]. \end{aligned} \tag{9}$$

Considering the objective function maximization and the less-than-or-equal constraints. The MFs of which are presented as follows, respectively [23]:

$$\mu_{(B\mathbf{x})_0} = \begin{cases} 1, & (B\mathbf{x})_0 \geq z_0, \\ \frac{(B\mathbf{x})_0 - (z_0 - \Delta_0)}{\Delta_0}, & (z_0 - \Delta_0) \leq (B\mathbf{x})_0 \leq z_0, \\ 0, & (B\mathbf{x})_0 \leq z_0 - \Delta_0, \end{cases} \tag{10}$$

and for $i = 1, ..., m$

$$\mu_{(A\mathbf{x})_i} = \begin{cases} 1, & (A\mathbf{x})_i \leq b_i \\ \frac{b_i + \Delta_i - (A\mathbf{x})_i}{\Delta_i}, & b_i \leq (A\mathbf{x})_i \leq b_i + \Delta_i, \\ 0. & (A\mathbf{x})_i \leq b_i + \Delta_i, \end{cases} \tag{11}$$

where $\mathbf{b}_i \in [\underline{b}_i^{\vee}, \bar{b}_i^{\vee}]$, $z_0 \in [\underline{z}^{\wedge}, \overline{z}^{\wedge}]$, $\Delta_i \in [\underline{\Delta}_i, \overline{\Delta}_i]$, $\Delta_0 \in [\underline{\Delta}_0, \overline{\Delta}_0]$. By substitution the MFs of (11) and (10) in problem (9) and according to the Bellman-Zadeh operator, we have

$$(12) \qquad \begin{array}{ll} max & \alpha \\ s.t. & \alpha \leq \frac{\mathbf{b}_i + \Delta_i - (A\mathbf{x})_i}{\Delta_i},\ i = 1, ..., m, \\ & \alpha \leq \frac{(B\mathbf{x})_0 - (z_0 - \Delta_0)}{\Delta_0}, \\ & \mathbf{x} \geq 0,\ \alpha \in [0, 1]. \end{array}$$

The model (12) can be written as:

$$(13) \qquad \begin{array}{ll} max & \alpha \\ s.t. & (A\mathbf{x})_i + \Delta_i \alpha \leq \mathbf{b}_i + \Delta_i, \quad i = 1, ..., m, \\ & (B\mathbf{x})_0 - \Delta_0 \alpha \geq z_0 - \Delta_0, \\ & \mathbf{x} \geq 0,\ \alpha \in [0, 1], \end{array}$$

according to the ILP problem, the problem (13) is denoted as follows:

$$(14) \qquad \begin{array}{ll} max & \alpha \\ s.t. & (A\mathbf{x})_i + [\underline{\Delta}_i \alpha, \bar{\Delta}_i \alpha] \leq [\underline{\mathbf{b}}_i, \bar{\mathbf{b}}_i], \quad i = 1, ..., m, \\ & (B\mathbf{x})_0 + [-\underline{\Delta}_0 \alpha, -\bar{\Delta}_0 \alpha] \geq [\underline{z}_0, \bar{z}_0], \\ & \mathbf{x} \geq 0,\ \alpha \in [0, 1], \end{array}$$

the best solution of (14), according to the ILP problem, is obtained by solving the following problem:

$$\begin{array}{ll} max & \alpha \\ s.t. & (A\mathbf{x})_i + \underline{\Delta}_i \alpha \leq \overline{\mathbf{b}}_i^{\wedge}, \quad i = 1, ..., m, \\ & (B\mathbf{x})_0 + (-\underline{\Delta}_0 \alpha) \geq \underline{z}^{\vee}, \\ & \mathbf{x} \geq 0,\ \alpha \in [0, 1]. \end{array}$$

Refer to [23] to see other cases in this method.

4. Three new methods for solving IT2FLP problem with vagueness in the resources vector

In this section, we propose three new methods for solving the FLP problem with the IT2F resources vector (4).

4.1. **First method.** The idea of this method is taken from Verdegay's method in solving the type-1 fuzzy linear programming (T1FLP) with vagueness in the resources vector [27]. It assumes that the MFs of the fuzzy constraints are non-increasing and continuous, and the objective function is crisp. Hence, this method is nonsymmetric. Then, we can formulate the MFs of the fuzzy inequality constraints of problem (4) as follows. Since the resources vector is composed of fuzzy parameters, due to the Bellman-Zadeh operator, the MF of all constraints of the problem (4) is given by:

$$\mu_D(\mathbf{x}^*) = MaxMin\{\mu((A\mathbf{x})_i, u)\},$$

assuming $\alpha = Min\{\mu((A\mathbf{x})_i, u)\}$, The IT2FLP problem is:

$$
\begin{aligned}
&max \quad \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \\
&s.t. \quad \alpha \le \mu((A\mathbf{x})_i, u), \\
&\qquad \mathbf{x} \ge 0,\ \alpha \in [0,1]\,.
\end{aligned} \tag{15}
$$

In this case, the constraints are in the $(\le)$ form, and the objective function is maximized. The MFs of constraints are equivalent to:

$$
\mu(A\mathbf{x})_i = \begin{cases} 1, & (A\mathbf{x})_i \le \mathbf{b}_i, \\ \frac{\mathbf{b}_i + \Delta_i - (A\mathbf{x})_i}{\Delta_i}, & \mathbf{b}_i \le (A\mathbf{x})_i \le \mathbf{b}_i + \Delta_i, \\ 0, & (A\mathbf{x})_i \ge \mathbf{b}_i + \Delta_i, \end{cases} \tag{16}
$$

where $\mathbf{b}_i \in \left[\underline{b}_i^{\vee}, \bar{b}_i^{\vee}\right]$ and $\Delta_i \in \left[\underline{\Delta}_i, \bar{\Delta}_i\right]$. By substitution the MF of (16) in the problem (15), we have

$$
\begin{aligned}
&max \quad \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \\
&s.t. \quad \alpha \le \tfrac{\mathbf{b}_i + \Delta_i - (A\mathbf{x})_i}{\Delta_i}, \quad i = 1, ..., m, \\
&\qquad \mathbf{x} \ge 0,\ \alpha \in [0,1]\,.
\end{aligned} \tag{17}
$$

Note that, for $i = 1, ..., m$, we have $(A\mathbf{x})_i = \sum_{j=1}^{n} a_{ij} x_j$. Therefore, the LP problem (17) is achieved as follows:

$$
\begin{aligned}
&max \quad \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \\
&s.t. \quad \sum_{j=1}^{n} a_{ij} x_j \le \mathbf{b}_i + (1-\alpha)\Delta_i, \quad i = 1, ..., m, \\
&\qquad \mathbf{x_j} \ge 0,\ j = 1, ..., n,\ \alpha \in [0,1]\,.
\end{aligned} \tag{18}
$$

Now, we substitute $\left[\underline{b}_i^{\vee}, \bar{b}_i^{\vee}\right]$ and $\left[\underline{\Delta}_i, \bar{\Delta}_i\right]$ instead of $\mathbf{b}_i$ and Δ_i into the problem (18), respectively. We obtain an ILP problem as follows:

$$
\begin{aligned}
&max \quad \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \\
&s.t. \quad \sum_{j=1}^{n} a_{ij} x_j \le \left[\underline{b}_i^{\vee}, \bar{b}_i^{\vee}\right] + (1-\alpha)\left[\underline{\Delta}_i, \bar{\Delta}_i\right], \quad i = 1, ..., m, \\
&\qquad \mathbf{x_j} \ge 0,\ j = 1, ..., n,\ \alpha \in [0,1]\,,
\end{aligned} \tag{19}
$$

the best solution of (19) is:

$$
\begin{aligned}
&max \quad \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \\
&s.t. \quad \sum_{j=1}^{n} a_{ij} x_j + \underline{\Delta}_i \alpha \le \bar{b}_i^{\wedge}, \quad i = 1, ..., m, \\
&\qquad \mathbf{x_j} \ge 0,\ j = 1, ..., n,\ \alpha \in [0,1]\,.
\end{aligned} \tag{20}
$$

Note that, for each $\alpha \in [0,1]$, the problem (20) obtained an optimal solution with a degree of MF.

4.2. **Second method.** The idea behind this method is taken from Werner's method for solving the T1FLP with vagueness in the resources vector [29]. The previous method was asymmetric, but this method is symmetric, which means that the objective function of problem (4) should be fuzzy because of fuzzy inequality constraints. We computed the lower and upper bounds of the optimal values by solving two LP problems. Consider the general form of an IT2FLP problem with vagueness in the resources vector in the model (4). Because the fuzzy resources vector MF has two upper and lower MFs, we have $(\underline{\mathbf{b}}_i^{\vee}, \bar{\mathbf{b}}_i^{\vee}, \underline{\mathbf{b}}_i^{\wedge}, \bar{\mathbf{b}}_i^{\wedge})$. Therefore, we have four optimal values of the objective function, resulting from four IT2FLP models $(\underline{z}^{\vee}, \underline{z}^{\wedge}, \bar{z}^{\vee}, \bar{z}^{\wedge})$. Then, according to the ILP rules, the maximum and minimum values of the objective functions are $\bar{z}^{\wedge}$ and $\underline{z}^{\vee}$, are presented as follows (see Fig. 4):

$$
\begin{aligned}
&\bar{z}^{\wedge} = max \quad \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \\
&s.t. \quad \sum_{j=1}^{n} a_{ij}\mathbf{x}_j \leq \bar{\mathbf{b}}_i^{\wedge}, \quad i = 1, ..., m, \\
&\qquad \mathbf{x_j} \geq 0, \qquad\qquad j = 1, 2, ..., n,
\end{aligned} \tag{21}
$$

and

$$
\begin{aligned}
&\underline{z}^{\vee} = max \quad \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \\
&s.t. \quad \sum_{j=1}^{n} a_{ij}\mathbf{x}_j \leq \underline{\mathbf{b}}_i^{\vee}, \quad i = 1, ..., m, \\
&\qquad \mathbf{x_j} \geq 0, \qquad\qquad j = 1, 2, ..., n.
\end{aligned} \tag{22}
$$

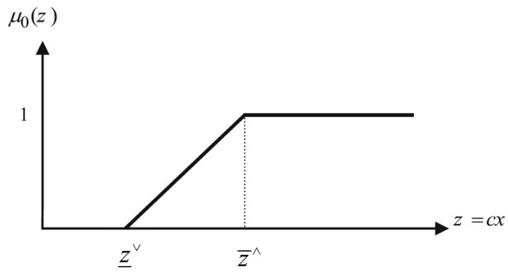

FIGURE 4. The MF of the objective function $\mu_0(z)$.

Defining the MF of the objective function $\mu_0(z)$ as (23):

$$\mu_0(z) = \begin{cases} 1, & \mathbf{cx} > \bar{z}^{\wedge}, \\ 1 - \frac{\bar{z}^{\wedge} - \mathbf{cx}}{\bar{z}^{\wedge} - \underline{z}^{\vee}}, & \underline{z}^{\vee} \leq \mathbf{cx} \leq \bar{z}^{\wedge}, \\ 0, & \mathbf{cx} < \underline{z}^{\vee}. \end{cases} \tag{23}$$

Now, problem (4) can be solved by solving problem (24) using the Bellman-Zadeh operator, i.e., the purpose is to obtain a solution that satisfies the objective and the constraints with the highest degree:

$$\begin{aligned} & max \quad \alpha \\ & s.t. \quad \alpha \leq \mu_0(z), \\ & \qquad \alpha \leq \mu((A\mathbf{x})_i, u), \\ & \qquad \mathbf{x} \geq 0,\ \alpha \in [0,1]. \end{aligned} \tag{24}$$

The MF of the objective function given in (23) is putting into the problem (24), then the problem (25) is obtaining:

$$\begin{aligned} & max \quad \alpha \\ & s.t. \quad \alpha \leq 1 - \frac{\bar{z}^{\wedge} - \mathbf{cx}}{\bar{z}^{\wedge} - \underline{z}^{\vee}}, \\ & \qquad \alpha \leq \frac{\mathbf{b}_i + \Delta_i - (A\mathbf{x})_i}{\Delta_i}, \quad i = 1, ..., m, \\ & \qquad \mathbf{x} \geq 0,\ \alpha \in [0,1]. \end{aligned} \tag{25}$$

As mentioned above, for $i = 1, ..., m$, we consider the $(A\mathbf{x})_i = \sum_{j=1}^{n} a_{ij}x_j$ and $\mathbf{cx} = \sum_{j=1}^{n} c_j x_j$, therefore the model (25) became as follows:

$$\begin{aligned} & max \quad \alpha \\ & s.t. \quad \sum_{j=1}^{n} c_j x_j \geq \bar{z}^{\wedge} - (1-\alpha)(\bar{z}^{\wedge} - \underline{z}^{\vee}), \\ & \qquad \sum_{j=1}^{n} a_{ij} x_j \leq \mathbf{b}_i + (1-\alpha)\Delta_i, \quad i = 1, ..., m, \\ & \qquad \mathbf{x_j} \geq 0,\ j = 1, ..., n,\ \alpha \in [0,1]. \end{aligned} \tag{26}$$

Therefore, by substituting $\left[\underline{\mathbf{b}}^{\vee}{}_i, \bar{\mathbf{b}}_i^{\vee}\right]$ and $\left[\underline{\Delta}_i, \bar{\Delta}_i\right]$ instead of $\mathbf{b}_i$ and Δ_i, into the problem (26), respectively; we obtain an ILP problem as follows:

$$\begin{aligned} & max \quad \alpha \\ & s.t. \quad \sum_{j=1}^{n} c_j x_j \geq \bar{z}^{\wedge} - (1-\alpha)(\bar{z}^{\wedge} - \underline{z}^{\vee}), \\ & \qquad \sum_{j=1}^{n} a_{ij} x_j \leq \left[\underline{\mathbf{b}}_i^{\vee}, \bar{\mathbf{b}}_i^{\vee}\right] + (1-\alpha)\left[\underline{\Delta}_i, \bar{\Delta}_i\right], \quad i = 1, ..., m, \\ & \qquad \mathbf{x_j} \geq 0,\ j = 1, ..., n,\ \alpha \in [0,1], \end{aligned} \tag{27}$$

the best solution of (27) is obtained by solving the LP problem (28):

$$
\begin{aligned}
&max \quad \alpha \\
&s.t. \quad \sum_{j=1}^{n} c_j x_j \geq \bar{z}^{\wedge} - (1-\alpha)(\bar{z}^{\wedge} - \underline{z}^{\vee}), \\
&\qquad \sum_{j=1}^{n} a_{ij} x_j + \alpha \underline{\Delta}_i \leq \bar{\mathbf{b}}_i^{\wedge}, \quad i = 1, ..., m, \\
&\qquad \mathbf{x_j} \geq 0, \, j = 1, ..., n, \, \alpha \in [0,1]\,.
\end{aligned} \tag{28}
$$

4.3. **Third method.** In this approach, the idea of Guu and Wu in solving T1FLP with vagueness in the resources vector is applied [15]. This method is a two-phase approach that fixes situations in which the max-min operator is not efficient. The first phase solves the problem using the max-min operator. In the second phase, a solution is obtained that is at least better than the solution obtained by the max-min operator. In other words, this method follows the highest degree of membership in the objective and better usage of each constrained resource. According to this method, to solve the IT2FLP problem, we first solve it using the max-min operator (problem (28)). If the optimal solution of problem (28) is unique, it is a fuzzy efficient solution for problem (4). This means that if the problem (28) has multiple optimal solutions, the solution obtained by the max-min operator of Bellman-Zadeh may not be efficient. In the second phase, assuming problem (28) has an optimal solution $(\mathbf{x}^*, \alpha^*)$, we have

$$
\begin{aligned}
&max \quad \sum_{i=0}^{m} \alpha_i \\
&s.t. \quad \alpha_0 \geq \mu_0(z = \mathbf{c}\mathbf{x}^*), \\
&\qquad \alpha_i \geq \mu((A\mathbf{x}^*)_i, u), \quad i = 1, ..., m, \\
&\qquad \alpha_0 \leq \mu_0(z = \mathbf{c}\mathbf{x}), \\
&\qquad \alpha_i \leq \mu((A\mathbf{x})_i, u), \quad i = 1, ..., m, \\
&\qquad \mathbf{x} \geq 0, \alpha_i \in [0,1]\,, \quad i = 0, 1, ..., m,
\end{aligned} \tag{29}
$$

therefore, by substituting the MFs of (16) and (23) in problem (29), we have

$$
\begin{aligned}
&max \quad \sum_{i=0}^{m} \alpha_i \\
&s.t. \quad \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j^* \leq \bar{z}^{\wedge} - (1-\alpha_0)(\bar{z}^{\wedge} - \underline{z}^{\vee}), \\
&\qquad \sum_{j=1}^{n} a_{ij} x_j^* \geq \mathbf{b}_i + (1-\alpha_i)\Delta_i, \quad i = 1, ..., m, \\
&\qquad \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \geq \bar{z}^{\wedge} - (1-\alpha_0)(\bar{z}^{\wedge} - \underline{z}^{\vee}), \\
&\qquad \sum_{j=1}^{n} a_{ij} x_j \leq \mathbf{b}_i + (1-\alpha_i)\Delta_i, \quad i = 1, ..., m, \\
&\qquad \mathbf{x_j} \geq 0, \, j = 1, ..., n, \, \alpha_i \in [0,1]\,, \;\; i = 0, 1, ...m,
\end{aligned} \tag{30}
$$

by putting $\left[\underline{\mathbf{b}}_i^{\vee}, \bar{\mathbf{b}}_i^{\wedge}\right]$ and $\left[\underline{\Delta}_i, \bar{\Delta}_i\right]$ instead of $\mathbf{b}_i$ and Δ_i , respectively, then the problem (30) becomes as following IT2FLP problem:

$$
\begin{array}{llr}
max & \sum_{i=0}^{m} \alpha_i & \\
s.t. & \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j^* \leq \bar{z}^{\wedge} - (1-\alpha_0)(\bar{z}^{\wedge} - \underline{z}^{\vee}), & \\
& \sum_{j=1}^{n} a_{ij} x_j^* \geq \left[\underline{\mathbf{b}}_i^{\vee}, \bar{\mathbf{b}}_i^{\vee}\right] + (1-\alpha_i)\left[\underline{\Delta}_i, \bar{\Delta}_i\right], & i = 1, ..., m, \\
& \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \geq \bar{z}^{\wedge} - (1-\alpha_0)(\bar{z}^{\wedge} - \underline{z}^{\vee}), & \\
& \sum_{j=1}^{n} a_{ij} x_j \leq \left[\underline{\mathbf{b}}_i^{\vee}, \bar{\mathbf{b}}_i^{\vee}\right] + (1-\alpha_i)\left[\underline{\Delta}_i, \bar{\Delta}_i\right], & i = 1, ..., m, \\
& \mathbf{x_j} \geq 0,\ j = 1, ..., n,\ \alpha_i \in [0,1], & i = 0, 1, ..., m,
\end{array} \tag{31}
$$

the best solution of (31) is obtained as follows:

$$
\begin{array}{llr}
max & \sum_{i=0}^{m} \alpha_i & \\
s.t. & \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j^* \leq \bar{z}^{\wedge} - (1-\alpha_0)(\bar{z}^{\wedge} - \underline{z}^{\vee}) & \\
& \sum_{j=1}^{n} a_{ij} x_j^* + \underline{\Delta}_i \alpha_i \geq \bar{\mathbf{b}}_i^{\wedge}, & i = 1, ..., m, \\
& \sum_{j=1}^{n} \mathbf{c}_j \mathbf{x}_j \geq \bar{z}^{\wedge} - (1-\alpha_0)(\bar{z}^{\wedge} - \underline{z}^{\vee}) & \\
& \sum_{j=1}^{n} a_{ij} x_j + \alpha_i \bar{\Delta}_i \leq \underline{\mathbf{b}}_i^{\vee}, & i = 1, ..., m, \\
& \mathbf{x_j} \geq 0,\ j = 1, ..., n,\ \alpha_i \in [0,1], & i = 0, 1, ..., m,
\end{array} \tag{32}
$$

the optimal solution of (32) is $(\mathbf{x}^{**}, \alpha_0^{**}, \alpha_1^{**}, ..., \alpha_m^{**})$.

Theorem 4.1. *[15] The optimal solution $\mathbf{x}^{**}$ of the problem* (32) *is a fuzzy efficient solution of the problem* (4)*.*

5. Numerical Examples

In this section, we consider two examples. We implement and evaluate our proposed methods using these two examples. First, we provide an application example related to the transportation problem from [13] and solve it using the three proposed methods. The second example was previously presented by Figueroa and Sarani. Now, we consider this example to better evaluate our proposed methods [11, 12, 23].

Example 5.1. ***(Application Example)*** *To demonstrate the functionality of the three proposed methods, we consider a classical transportation problem given in the article [13]. In which its demands and supplies are defined by the perspectives of system experts on two fronts: experts in customer behavior and experts in supplier capabilities. Sometimes, the experts provide opinions using words instead of numbers through sentences such as "I think that the* *demand of the product X should be between b_1 and b_2", where b_1 and b_2 become $b_1^{\vee}$ and $b_2^{\wedge}$ [13]. Consequently, when different experts provide ideas based on their prior knowledge, the challenge lies in managing the information they have provided. In cases where different experts hold varying viewpoints on the same concept, linguistic uncertainty emerges, and T2FSs emerge as an alternative for handling this type of uncertainty. This is how we define the demands and supplies of the system by the experts, with the primary goal being to minimize the system's shipping costs. The general IT2FLP transportation model is as follows:*

$$
\begin{aligned}
&\min z = c_{ij}x_{ij} \\
&s.t. \quad -\sum_{i=1}^{m} x_{ij} \geq -\tilde{\tilde{t}}_j, \quad \forall \;\; j \in \mathbb{N}_n, \quad \mathbb{N}_n = 1, ..., n, \\
&\qquad\quad \sum_{j=1}^{n} x_{ij} \geq \tilde{\tilde{s}}_i, \qquad \forall \;\; i \in \mathbb{N}_m, \quad \mathbb{N}_m = 1, ..., m,
\end{aligned}
\tag{33}
$$

where c_{ij}, $x_{ij} \in \mathbb{R}^{n,m}$, $\tilde{\tilde{t}}_j$ and $\tilde{\tilde{s}}_i$ are IT2FS. $\mathbb{N}_m$ is the set of all i resources. $\mathbb{N}_n$ is the set of all j products.
x_{ij}: The amount of product that will be sent from supplier i to customer j.
t_j =: The amount of product that is accessible from supplier j.
s_i =: The amount of product needed by customer i.
Note that the vector $\tilde{\tilde{b}}$ is made up of two vectors: $\tilde{\tilde{s}}$ with parameters $\underline{s}^{\vee}$, $\underline{s}^{\wedge}$, $\bar{s}^{\vee}$, and $\bar{s}^{\wedge}$, representing the customers' demands, and $\tilde{\tilde{b}}$ with parameters $\underline{t}^{\vee}$, $\underline{t}^{\wedge}$, $\bar{t}^{\vee}$, and $\bar{t}^{\wedge}$, representing the availabilities provided by suppliers.

$$
\bar{s}^{\vee} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}, \; \underline{s}^{\vee} = \begin{bmatrix} 13 \\ 14 \\ 18 \end{bmatrix}, \; \bar{s}^{\wedge} = \begin{bmatrix} 12 \\ 13 \\ 15 \end{bmatrix}, \; \underline{s}^{\wedge} = \begin{bmatrix} 16 \\ 17 \\ 21 \end{bmatrix},
$$

$$
\bar{t}^{\vee} = \begin{bmatrix} 24 \\ 37 \\ 29 \end{bmatrix}, \; \underline{t}^{\vee} = \begin{bmatrix} 16 \\ 30 \\ 23 \end{bmatrix}, \; \bar{t}^{\wedge} = \begin{bmatrix} 20 \\ 32 \\ 24 \end{bmatrix}, \; \underline{t}^{\wedge} = \begin{bmatrix} 14 \\ 25 \\ 18 \end{bmatrix},
$$

$$
c_{ij}^{T} = \begin{bmatrix} 2 & 3 & 2 & 4 & 1 & 3 & 2 & 4 & 2 \end{bmatrix}, \; \Delta = \begin{bmatrix} 3 \\ 3 \\ 6 \\ 8 \\ 7 \\ 6 \end{bmatrix}.
$$

This example consists of three suppliers and three customers, and their parameters are determined by experts using IT2FS.

Solving Example 5.1 using the first proposed method: *Using the problem* (20), *We have*

(34)

$$\begin{array}{ll} \min & z = 2x_{11} + 3x_{12} + 2x_{13} + 4x_{21} + x_{22} + 3x_{23} + 2x_{31} + 4x_{32} + 2x_{33} \\ s.t. & x_{11} + x_{21} + x_{31} + 8\alpha \leq 24, \\ & x_{12} + x_{22} + 4x_{32} + 7\alpha \leq 37, \\ & x_{13} + x_{23} + x_{33} + 6\alpha \leq 29, \\ & x_{11} + x_{12} + x_{13} - 3\alpha \geq 10, \\ & x_{21} + x_{22} + x_{23} - 3\alpha \geq 11, \\ & x_{31} + 4x_{32} + 2x_{33} - 6\alpha \geq 12, \\ & \alpha \in [0, 1], \end{array}$$

by solving the problem (34) *for each* $\alpha \in [0, 1]$, *the optimal objective value and optimal solutions are shown in Table 2.*

Solving Example 5.1 using the second proposed method: *In this process, we need to compute* $\underline{z}^{\vee}$ *and* $\bar{z}^{\wedge}$. *They obtained as* $\underline{z}^{\vee} = 55$ *and* $\bar{z}^{\wedge} = 99$. *Then, using the problem* (28), *we get*

$$\begin{array}{ll} \max & \alpha \\ s.t. & 2x_{11} + 3x_{12} + 2x_{13} + 4x_{21} + x_{22} \\ & \quad +3x_{23} + 2x_{31} + 4x_{32} + 2x_{33} \leq 96 - 41\alpha, \\ & g_1(x) = x_{11} + x_{21} + x_{31} \leq 24 - 8\alpha, \\ & g_2(x) = x_{12} + x_{22} + 4x_{32} \leq 37 - 7\alpha, \\ & g_3(x) = x_{13} + x_{23} + x_{33} \leq 29 - 6\alpha, \\ & g_4(x) = x_{11} + x_{12} + x_{13} \geq 10 + 3\alpha, \\ & g_5(x) = x_{21} + x_{22} + x_{23} \geq 11 + 3\alpha, \\ & g_6(x) = x_{31} + 4x_{32} + 2x_{33} \geq 12 + 6\alpha, \\ & \quad \alpha \in [0, 1], \end{array} \tag{35}$$

by solving the problem (35), *the results are shown in Table 2.*

TABLE 2. The result of the first and second suggested methods.

Methods	α^*	x^*_{11}	x^*_{12}	x^*_{13}	x^*_{21}	x^*_{22}	x^*_{23}	x^*_{31}	x^*_{32}	x^*_{33}	z^*
Our first method	*0*	*0*	*0*	*10*	*0*	*11*	*0*	*0*	*0*	*12*	*55*
	0.1	*0*	*0*	*9.7*	*0*	*10.7*	*0*	*0*	*0*	*11.4*	*52.9*
	0.2	*0*	*0*	*9.4*	*0*	*10.4*	*0*	*0*	*0*	*10.8*	*50.8*
	0.3	*0*	*0*	*9.1*	*0*	*10.1*	*0*	*0*	*0*	*10.2*	*48.7*
	0.4	*0*	*0*	*8.8*	*0*	*9.8*	*0*	*0*	*0*	*9.6*	*46.6*
	0.5	*0*	*0*	*8.5*	*0*	*9.5*	*0*	*0*	*0*	*9*	*44.5*
	0.6	*0*	*0*	*8.2*	*0*	*9.2*	*0*	*0*	*0*	*8.4*	*42.4*
	0.7	*0*	*0*	*7.9*	*0*	*8.9*	*0*	*0*	*0*	*7.8*	*40.3*
	0.8	*0*	*0*	*7.6*	*0*	*8.6*	*0*	*0*	*0*	*7.2*	*38.2*
	0.9	*0*	*0*	*7.3*	*0*	*8.3*	*0*	*0*	*0*	*6.6*	*36.1*
	1	*0*	*0*	*7*	*0*	*8*	*0*	*0*	*0*	*6*	*34*
Our second method	*0.629*	*3.435*	*0*	*8.452*	*0*	*12.887*	*0*	*0*	*0*	*16.774*	*70.21*

In the first method, the best optimal value of the objective function ($z^* = 34$) occurs at $\alpha = 1$, Since the goal of the main problem is to minimize transportation costs between the buyer and seller, the optimal value of the objective function indicates a significant reduction in costs. In the second method, the best level to satisfy the objective function and the constraints of the problem is equal to $\alpha = 0.629$, providing a crisp solution to the problem. In this method, more products are transported between suppliers and customers; therefore, the cost of transportation has increased slightly.

Solving Example 5.1 using the third proposed method: *The optimal solutions obtained from solving problem (35) are placed on the left side of its constraints and the values $g_1(x^*) = 5.6$, $g_2(x^*) = 12.99$, $g_3(x^*) = 25.23$, $g_4(x^*) = 11.89$, $g_5(x^*) = 12.89$, and $g_6(x^*) = 16.77$ are obtained. Also, the limited degrees of constraints satisfaction are $0.629 \leq \alpha_0, \alpha_3, \alpha_4, \alpha_5 \leq 1$, $\alpha_1 = \alpha_2 = 1$, and $0.79 \leq \alpha_6 \leq 1$. Now, as second phase, using the problem (32), we have*

$$\begin{array}{ll}
\max & \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \\
s.t. & 2x_{11} + 3x_{12} + 2x_{13} + 4x_{21} + x_{22} \\
& \quad +3x_{23} + 2x_{31} + 4x_{32} + 2x_{33} \leq 96 - 41\alpha_0, \\
& g_1(x) = x_{11} + x_{21} + x_{31} \leq 24 - 8\alpha_1, \\
& g_2(x) = x_{12} + x_{22} + 4x_{32} \leq 37 - 7\alpha_2, \\
& g_3(x) = x_{13} + x_{23} + x_{33} \leq 29 - 6\alpha_3, \\
& g_4(x) = x_{11} + x_{12} + x_{13} \geq 10 + 3\alpha_4, \\
& g_5(x) = x_{21} + x_{22} + x_{23} \geq 11 + 3\alpha_5, \\
& g_6(x) = x_{31} + 4x_{32} + 2x_{33} \geq 12 + 6\alpha_6, \\
& 0.629 \leq \alpha_0, \alpha_3, \alpha_4, \alpha_5 \leq 1, \ \alpha_1 = \alpha_2 = 1, \\
& 0.79 \leq \alpha_6 \leq 1,
\end{array} \tag{36}$$

by solving the problem (36), the quantities of shipments, denoted as x_{ij}^{**}, that need to be delivered from suppliers to customers are presented as below:

$$\begin{array}{lll}
x_{11}^{**} = 0, & x_{21}^{**} = 0, & x_{31}^{**} = 5.627, \\
x_{12}^{**} = 0, & x_{22}^{**} = 12.957, & x_{32}^{**} = 0, \\
x_{13}^{**} = 11.887, & x_{23}^{**} = 0, & x_{33}^{**} = 11.113,
\end{array}$$

substituting these values into the objective function of the problem (34), we obtain $z^* = 70.21$ and into the constraints of it, we have $g_1(x^{**}) = 5.6$, $g_2(x^{**}) = 12.96$, $g_3(x^{**}) = 23$, $g_4(x^{**}) = 11.89$, $g_5(x^{**}) = 12.96$, and $g_6(x^{**}) = 16.77$, also $\sum_{i=0}^{6} \alpha_i = 5.7$. As we have seen, not only $\boldsymbol{x}^{**}$ achieve the optimal objective value, but it also obtains a maximum degree in the third constraint. Furthermore, it utilizes 23 units of the third resource, while the solution x^* of the max-min operator requires 25.23, units of the third resource. In the remaining resources, there has also been a slight saving. It means that the use of resources has been saved.

By solving this example, we have effectively demonstrated the efficiency of our proposed methods in application problems. Our proposed methods are flexible and interpretable, such that the Bellman-Zadeh operator is used to find a crisp solution to the IT2F constrained problem. This makes our proposal suitable for solving many similar problems. Furthermore, in the third proposed method, we fix situations in which the max-min operator is not efficient.

Example 5.2. *Consider an FLP problem aimed at maximizing the objective function with less-than-or-equal constraints with an IT2F resources vector [11, 12, 23].*

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 10 & 4 & 9 \\ 4 & 6 & 3 \\ 2 & 7 & 7 \\ 5 & 6 & 11 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 12 \\ 17 \\ 9 \end{bmatrix}, \ \underline{\mathbf{b}}^{\vee} = \begin{bmatrix} 50 \\ 70 \\ 40 \\ 60 \\ 40 \end{bmatrix}, \ \underline{\mathbf{b}}^{\wedge} = \begin{bmatrix} 72 \\ 104 \\ 65 \\ 95 \\ 80 \end{bmatrix}, \ \overline{\mathbf{b}}^{\vee} = \begin{bmatrix} 60 \\ 80 \\ 55 \\ 75 \\ 57 \end{bmatrix}, \text{ and } \overline{\mathbf{b}}^{\wedge} = \begin{bmatrix} 95 \\ 110 \\ 77 \\ 102 \\ 98 \end{bmatrix}.$$

Solving Example 5.2 using the three review method (Subsection 3.2): *By using the first method of Figueroa,* $\tilde{\tilde{z}}(\boldsymbol{x}^*) = 175.76$ *and* $\alpha_{\tilde{\tilde{z}}} = 0.51$ *are obtained. Also by using the second method of Figueroa,* $\tilde{\tilde{z}}(\boldsymbol{x}^*) = 175.76$ *and* $\alpha_{\tilde{\tilde{z}}} = 0.5106$ *are computed. Furthermore, the result of using the Sarani's method is presented by* $(\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{x}_3^*) = (6.2495, 5.5812, 0)$*,* $\tilde{\tilde{z}}^* = 169.8743$ *and* $\alpha^*_{\tilde{\tilde{z}}} = 0.74059$*. Sarani's method has a higher degree of satisfaction in terms of the constraints than the two Figueroa's methods. See Table 3.*

Solving Example 5.2 using the first proposed method: *Assumes the same data as Example 5.2. Using problem* (20)*, and* $\Delta = \bar{b}^{\vee} - \underline{b}^{\vee} = \begin{bmatrix} 10 \\ 10 \\ 15 \\ 15 \\ 17 \end{bmatrix}$.

We have

$$\begin{aligned} \max \quad & 12\mathbf{x}_1 + 17\mathbf{x}_2 + 9\mathbf{x}_3 \\ s.t. \quad & 5\mathbf{x}_1 + 3\mathbf{x}_2 + 7\mathbf{x}_3 + 10\alpha \leq 95, \\ & 10\mathbf{x}_1 + 4\mathbf{x}_2 + 9\mathbf{x}_3 + 10\alpha \leq 110, \\ & 4\mathbf{x}_1 + 6\mathbf{x}_2 + 3\mathbf{x}_3 + 15\alpha \leq 77, \\ & 2\mathbf{x}_1 + 7\mathbf{x}_2 + 7\mathbf{x}_3 + 15\alpha \leq 102, \\ & 5\mathbf{x}_1 + 6\mathbf{x}_2 + 11\mathbf{x}_3 + 17\alpha \leq 98, \\ & \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \geq 0, \ \ \alpha \in [0, 1], \end{aligned} \tag{37}$$

by solving the problem (37) *for each* $\alpha \in [0,1]$, *the optimal objective value and optimal solutions are shown in Table 3.*

TABLE 3. Comparison of the results of our first method and the results of the review methods

Methods	α^*	$\boldsymbol{x}_1^*$	$\boldsymbol{x}_2^*$	$\boldsymbol{x}_3^*$	z^*
Our first method	*0*	*8.0000*	*7.5000*	*0*	*223.5*
	0.1	*8.0000*	*7.2500*	*0*	*219.2500*
	0.2	*8.0000*	*7.0000*	*0*	*215.0000*
	0.3	*8.0000*	*6.7500*	*0*	*210.7500*
	0.4	*8.0000*	*6.5000*	*0*	*206.5000*
	0.5	*8.0000*	*6.2500*	*0*	*202.2500*
	0.6	*8.0000*	*6.0000*	*0*	*198.0000*
	0.7	*8.0000*	*5.7500*	*0*	*193.7500*
	0.8	*8.0000*	*5.5000*	*0*	*189.5000*
	0.9	*8.0000*	*5.2500*	*0*	*185.2500*
	1	*8.0000*	*5.0000*	*0*	*181.0000*
First method of Figueroa	*0.51*	-	-	-	*190.99*
Second method of Figueroa	*0.5106*	-	-	-	175.76
Sarani method	*0.741*	*6.2495*	*5.5812*	*0*	*169.8743*

By comparing the results of our first method with those of the two Figueroa methods and Sarani's method: it is observed that for $\alpha^* = 0.5$, *the optimal objective function value in our first method is better than those of the two methods of Figueroa. In addition, at* $\alpha^* = 0.7$, *the optimal objective function value obtained by the first method is better than that obtained by Sarani's method. In addition, in the two methods of Figueroa, only the* α^* *value and the optimal objective function are obtained. However, in our proposed method, the decision-maker can choose a value for* $\alpha \in [0,1]$ *according to the conditions of the problem, and by solving it, he can also calculate the optimal values of the decision variables and the objective function.*

Solving Example 5.2 using the second proposed method: *Consider the FLP problem* (37) *of Example 5.2. We solve two problems* (21) *and* (22), *therefore we obtain* $\bar{z}^{\wedge} = 223.5000$ *and* $\underline{z}^{\vee} = 113.3333$. *Then, by replacing* *these values in* (28)*, we have*

$$
\begin{aligned}
& \max \quad \alpha \\
& s.t. \quad -12x_1 - 17x_2 - 9x_3 \le 113.3333 - 110.1667\alpha, \\
& \qquad 5x_1 + 3x_2 + 7x_3 \le 95 - 10\alpha, \\
& \qquad 10x_1 + 4x_2 + 9x_3 \le 110 - 10\alpha, \\
& \qquad 4x_1 + 6x_2 + 3x_3 \le 77 - 15\alpha, \\
& \qquad 2x_1 + 7x_2 + 7x_3 \le 102 - 15\alpha, \\
& \qquad 5x_1 + 6x_2 + 11x_3 \le 98 - 17\alpha, \\
& \qquad x_1, x_2, x_3 \ge 0, \quad \alpha \in [0,1],
\end{aligned} \tag{38}
$$

by solving the problem (38)*, the values of* $(x_1^*, x_2^*, x_3^*) = (8, 5.6960, 0)$ *and* $\alpha^* = 0.7216$ *are obtained. Then, we substitute the values of* $\mathbf{x}^*$ *into the objective function of the problem* (37)*. The optimal value of the objective function becomes* $z^* = 192.8320$*. Table 4 shows a comparison between the results of our first and second methods, the first two methods of Figueroa, and Sarani's method.*

Table 4. Comparison of the results of our second method and the results of the review methods.

Methods	α^*	x_1^*	x_2^*	x_3^*	z^*
Our first method	$\alpha = 0.5$	*8.0000*	*6.2500*	*0*	*202.2500*
	$\alpha = 0.7$	*8.0000*	*5.7500*	*0*	*193.7500*
Our second method	*0.7216*	*8*	*5.696*	*0*	*192.8320*
First method of Figueroa	*0.51*	-	-	-	*190.99*
Second method of Figueroa	*0.5106*	-	-	-	175.76
Sarani method	*0.741*	*6.2495*	*5.5812*	*0*	*169.8743*

For a better comparison, we have added the values of rows $\alpha = 0.5$ *and* $\alpha = 0.7$ *from Table 3 in Table 4. As can be seen from it, the values obtained from our two proposed methods are better than those of the first two Figueroa and Sarani methods. Furthermore, the optimal value obtained for* $\alpha^* = 0.7216$ *from our second method is better than those obtained from the first two Figueroa methods and is almost equal to the value obtained from Sarani's method. The optimal value obtained for* $z^* = 192.8320$ *is much better than those obtained from the first two methods of Figueroa and Sarani's method and is slightly lower than our first method. In this method, in addition to the optimal values of* α^* *and the objective function, the optimal values of the decision variables are also obtained.*

Solving Example 5.2 using the third proposed method: *First, we rewrite problem* (38) *as follows:*

$$
\begin{aligned}
&\max \quad \alpha\\
&s.t. \quad -12x_1-17x_2-9x_3 \le 113.3333-110.1667\alpha,\\
&\quad g_1(x)=5x_1+3x_2+7x_3 \le 95-10\alpha,\\
&\quad g_2(x)=10x_1+4x_2+9x_3 \le 110-10\alpha,\\
&\quad g_3(x)=4x_1+6x_2+3x_3 \le 77-15\alpha,\\
&\quad g_4(x)=2x_1+7x_2+7x_3 \le 102-15\alpha,\\
&\quad g_5(x)=5x_1+6x_2+11x_3 \le 98-17\alpha,\\
&\qquad x_1,x_2,x_3 \ge 0, \ \ \alpha\in[0,1],
\end{aligned} \tag{39}
$$

as we know that the optimal solutions to the problem (39) *are* $\boldsymbol{x}^*=(x_1^*,x_2^*,x_3^*)=(8,5.696,0)$ *and* $\alpha^*=0.7216$. *By substituting these values into the objective function of the problem* (37), *we obtain* $z(\boldsymbol{x}^*)=z^*=192.8320$, *and into the constraints of it , we have* $g_1(x^*)=57.088$, $g_2(x^*)=102.784$, $g_3(x^*)=66.176$, $g_4(x^*)=55.872$, *and* $g_5(x^*)=74.176$. *In addition, the limited degrees of constraints satisfaction are* $0.6335\le\alpha_0\le 1$, $\alpha_1=\alpha_4=\alpha_5=1$, $0.7216\le\alpha_2\le 1$, *and* $0.7216\le\alpha_3\le 1$. *Now, according to the problem* (32), *as second phase, we apply these limits to the problem* (32), *as a result we have*

$$
\begin{aligned}
&\max \quad \alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,\\
&s.t. \quad -12x_1-17x_2-9x_3+110.1667\alpha_0 \le 113.3333,\\
&\quad g_1(x)=5x_1+3x_2+7x_3+10\alpha_1 \le 95,\\
&\quad g_2(x)=10x_1+4x_2+9x_3+10\alpha_2 \le 110,\\
&\quad g_3(x)=4x_1+6x_2+3x_3+15\alpha_3 \le 77,\\
&\quad g_4(x)=2x_1+7x_2+7x_3+15\alpha_4 \le 102,\\
&\quad g_5(x)=5x_1+6x_2+11x_3+17\alpha_5 \le 98,\\
&\qquad 0.6325\le\alpha_0\le 1, \ \alpha_1=1,\\
&\qquad 0.7216\le\alpha_2\le 1, \ 0.7216\le\alpha_3\le 1,\\
&\qquad \alpha_4=1, \ \alpha_5=1,\\
&\qquad x_1,x_2,x_3\ge 0,
\end{aligned} \tag{40}
$$

by solving the problem (40), *we obtain* $\boldsymbol{x}^{**}=(x_1^{**},x_2^{**},x_3^{**})=(7.9304,5.1741,0)$.

Substituting these values into the objective function of the problem (37), *we obtain* $z^*=182.726$ *and into the constraints of it, we have* $g_1(x^{**})=55.142$, $g_2(x^{**})=100$, $g_3(x^{**})=62.627$, $g_4(x^{**})=46.741$, *and* $g_5(x^{**})=70.750$. *In addition, we obtain* $\alpha_0^*=0.6335$, $\alpha_1^*=\alpha_2^*=\alpha_4^*=\alpha_5^*=1$, $\alpha_3^*=0.958$, *and* $\sum_{i=0}^{m}\alpha_i^*=5.588$.

Not only $\boldsymbol{x}^{**}$ *achieve the optimal objective value, but it also obtains a maximum degree in all constraints. Furthermore, it utilizes* 55.142 *units of the first resource,* 100 *units of the second resource,* 62.627 *units of the third resource,* 46.741 *units of the fourth resource, and* 70.57 *of the fifth resource, while the solution* x^* *of the max-min operator requires* 57.088, 102.782, 66.176, 55.872, *and* 74.176 *units of the first, second, third, fourth, and fifth resources, respectively. It means that the use of resources has been saved. Also, the optimal objective function value obtained from this method is better than the second method of Figueroa and Sarani's methods (see Table 4).*

6. **Conclusions**

This article aims to investigate and solve the IT2FLP problem with vagueness in the resource vectors. To obtain this aim, firstly, the three available methods for solving these problems, such as Figueroa's and Sarani's methods, are discussed. In the following, we proposed three new methods for solving the IT2FLP problem with vagueness in the resources vector. The first new method is based on Verdegay's method for solving the FLP problem. Then, we calculated that the optimal solution of an IT2FLP problem which can be found by solving an equivalent LP problem, assuming that the objective function is crisp. The second proposed method is based on Werners's idea to solve the FLP problem. In this method, the asymmetric problem is converted to a symmetric problem. Next, using the max-min operator of Bellman and Zadeh, the maximum degree satisfying the objective and constraints of the problem is calculated. The third proposed method is based on the ideas of Guu and Wu. We proposed the two-phase method for solving the IT2FLP problem that performs the highest membership degree in the objective and enables better use of each constrained resource. Additionally, to illustrate the performance of our proposed methods, we present two numerical examples. The first example is an application case related to the classic transportation problem. The second example is that used by Figueroa and Sarani. We use this example to compare the proposed methods with the three reviewed methods. Finally, the results of our methods are compared with those of previous methods. The advantages of this study and the proposed method can be summarized as follows:

- Reviewing and comparing three solution methods for the IT2FLP problem, including the first two methods of Figueroa and one method of Sarani;
- Proposing three new methods for solving the IT2FLP problem based on three ideas for solving the FLP problem, from Verdagai, Werners, and Guo and Wu;
- Examining the efficiency of the three proposed methods by solving the same numerical example and comparing the results with those obtained by solving the three review methods;
- Discussion the performance and efficiency of the three proposed methods by solving an application example;
- Proposed methods are more efficient and have better optimal solutions;
- One advantage of the third proposed method is that its optimal solution not only obtains the optimal value of the objective function, but also helps to save resources by obtaining the highest degree in the MF of constraints.
- In our first proposed method, the decision-maker can choose a value for $\alpha \in [0, 1]$ according to the conditions of the problem, and by solving it, he can also obtained the optimal values of the decision variables and the objective function;

- Computational simplicity and desirable results.
- Our proposed methods are flexible and interpretable, such that the Bellman-Zadeh operator is used to find a crisp solution to the IT2F constrained problem. This makes our proposal suitable for solving many similar problems.
- Furthermore, in the third proposed method, we fix situations in which the max-min operator is not efficient;

Finally, as mentioned, imprecision in a linear programming problem can take two forms: vagueness and ambiguity. The imprecision of the vagueness type is displayed as a membership function. Furthermore, depending on the conditions and the position of the vagueness, different states occur. The vagueness can be in the objective function vector, technological coefficients, resources vector, or any possible combination of them. These problems can be investigated as attractive topics for future research. Alternatively, if the uncertainty in the problem is of the ambiguity type, the parameters can be considered as fuzzy numbers and the problem can be modeled using the possibilistic linear programming problem. In these cases, considering the position of ambiguity in the problem, we are faced with different situations. These findings can be considered attractive topics for future research.

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