Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

ALGEBRA FUZZY NORMS GENERATED BY HOMOMORPHISMS

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Article type: Research Article
(Received: 11 December 2023, Received in revised form 17 February 2024)
(Accepted: 01 March 2024, Published Online: 02 March 2024)

Abstract. As a new approach, for a nonzero normed algebra A, we will define some different classes of algebra fuzzy norms on A generated by homomorphisms and continuous homomorphisms. Also as a source of examples and counterexamples in the field of fuzzy normed algebras, separate continuity of the elements within each class are investigated.
Keywords: Linear functional, Fuzzy normed algebra, Homomorphism, Separate continuity.
2020 MSC: Primary 46S40, 26A15

1. Introduction

The idea of the fuzzy norm on a linear space was introduced by Katsaras [11] in 1984. Also a type of fuzzy metric introduced by O. Kaleva and S. Seikkala [10] in 1984. In 1992, Felbin [7] introduced an idea of the fuzzy norm on a linear space, such that its corresponding fuzzy metric is of type of introduced by O. Kaleva and S. Seikkala. Another idea of a fuzzy norm on a linear space was introduced by Cheng and Mordeson [6] in 1994. Following Cheng and Mordeson, a definition of a fuzzy norm whose associated fuzzy metric is similar to Kramosil and Michalek type [12], was introduced by T. Bag and S. K. Samanta [1] in 2003. A large number of papers concerning fuzzy norms have been published by different authors such as [2–5, 8, 9]. Some fuzzy notions on the sequence spaces are investigated in [14, 15].
The concept of fuzzy normed algebra is different from the notion of fuzzy normed linear space in one step that had to be done.
In this paper, we will consider A as a normed algebra over the field $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. Also let $Hom(A, \mathbb{F})$ be the set of all algebra homomorphisms from A into $\mathbb{F}$.
Given a normed algebra A, we will introduce some different classes of fuzzy normed algebras. Also, separate continuity of the elements within each class are investigated. The motivation of this paper is to produce algebra fuzzy norms by a new method and approach which while being different from the previous findings, are significantly simpler. In this method, by choosing an arbitrary algebra homomorphism, a different algebra fuzzy norm can be produced. The

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https://doi.org/10.22103/jmmr.2024.22657.1550

Publisher: Shahid Bahonar University of Kerman
How to cite: A.R. Khoddami, F. Kouhsari, *Algebra fuzzy norms generated by homomorphisms*, J. Mahani Math. Res. 2025; 14(1): 23 - 44.

results of this paper can be applied as a source of examples and counterexamples in the field of fuzzy normed algebras.

2. CONSTRUCTING ALGEBRAIC FUZZY NORMS

Definition 2.1. [1] Let X be a linear space and let $N : X \times \mathbb{R} \longrightarrow [0,1]$ be a function such that for all $x, y \in X$ and for all $s, t \in \mathbb{R}$,

(1) $N(x,t) = 0$ for all $t \leq 0$,
(2) $N(x,t) = 1$ for all $t > 0$ if and only if $x = 0$,
(3) $N(\alpha x, t) = N(x, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$,
(4) $N(x+y, s+t) \geq \min(N(x,s), N(y,t))$,
(5) for each fixed $x \in X, N(x,\cdot) : \mathbb{R} \longrightarrow [0,1]$ is an increasing function, and $\lim_{t \longrightarrow \infty} N(x,t) = 1$.
Then (X, N) is called a fuzzy normed linear space.

Definition 2.2. [13] Let A be an algebra and let $N : A \times \mathbb{R} \longrightarrow [0,1]$ be a function such that for all $a, b \in A$ and for all $s, t \in \mathbb{R}$,

(1) $N(a,t) = 0$ for all $t \leq 0$,
(2) $N(a,t) = 1$ for all $t > 0$ if and only if $a = 0$,
(3) $N(\alpha a, t) = N(a, \frac{t}{|\alpha|})$ for all $\alpha \neq 0$,
(4) $N(a+b, s+t) \geq \min(N(a,s), N(b,t))$,
(5) for each fixed $a \in A, N(a,\cdot) : \mathbb{R} \longrightarrow [0,1]$ is an increasing function, and $\lim_{t \longrightarrow \infty} N(a,t) = 1$,
(6) $N(ab, st) \geq N(a,s)N(b,t)$.
Then (A, N) is called a fuzzy normed algebra.

Remark 2.3. The increasing property of Definition 2.1 and Definition 2.2 is a result of other conditions. Indeed, for $x \in X$ and $s < t$ we have,

$$\begin{aligned} N(x,t) &= N(x+0, s+(t-s)) \\ &\geq \min(N(x,s), N(0,t-s)) \\ &= \min(N(x,s), 1) \\ &= N(x,s). \end{aligned}$$

Example 2.4. *If A is a normed algebra, then $N : A \times \mathbb{R} \longrightarrow [0,1]$ defined by*

$$N(a,t) = \begin{cases} 0, & t \leq \|a\| \\ 1, & t > \|a\| \end{cases}$$

is an algebra fuzzy norm on A.

Proposition 2.5. *Let A be a normed algebra over the field $\mathbb{F}$ and $\varphi \in Hom(A, \mathbb{F})$. Define $N_\varphi : A \times \mathbb{R} \longrightarrow [0,1]$ by*

$$N_\varphi(a,t)=\begin{cases}0, & t\leq \|a\|+|\varphi(a)|\\ \frac{t-\|a\|-|\varphi(a)|}{t+\|a\|+|\varphi(a)|}, & t>\|a\|+|\varphi(a)|.\end{cases}$$

Then N_φ is an algebra fuzzy norm on A.

Proof. We only prove parts (2), (4), (5), and (6) of Definition 2.2.
(2): If $a=0$, then clearly $N_\varphi(a,t)=1$ for all $t>0$. For the converse let $N_\varphi(a,t)=1$ for all $t>0$ and suppose by contradiction that $a\neq 0$.
If $t=\|a\|+|\varphi(a)|$, then $N_\varphi(a,t)=0$ that is a contradiction.
(4): Let $a,b\in A$ and $s,t\in\mathbb{R}$. If $s+t\leq 0$, then $s\leq 0$ or $t\leq 0$. So $N_\varphi(a,s)=0$ or $N_\varphi(b,t)=0$.
Hence, $0=N_\varphi(a+b,s+t)\geq \min(N_\varphi(a,s),N_\varphi(b,t))=0$.
If $s+t>0$ and $s+t\leq \|a+b\|+|\varphi(a+b)|$, then $s\leq\|a\|+|\varphi(a)|$ or $t\leq \|b\|+|\varphi(b)|$. Therefore,

$$0=N_\varphi(a+b,s+t)\geq \min(N_\varphi(a,s),N_\varphi(b,t))=0.$$

If $s+t>\|a+b\|+|\varphi(a+b)|$, then $N_\varphi(a+b,s+t)=\frac{s+t-\|a+b\|-|\varphi(a)+\varphi(b)|}{s+t+\|a+b\|+|\varphi(a)+\varphi(b)|}$.
In this case if $s\leq\|a\|+|\varphi(a)|$ or $t\leq\|b\|+|\varphi(b)|$, then clearly

$$\frac{s+t-\|a+b\|-|\varphi(a)+\varphi(b)|}{s+t+\|a+b\|+|\varphi(a)+\varphi(b)|}\geq \min(N_\varphi(a,s),N_\varphi(b,t))=0.$$

If $s>\|a\|+|\varphi(a)|$ and $t>\|b\|+|\varphi(b)|$, then $N_\varphi(a,s)=\frac{s-\|a\|-|\varphi(a)|}{s+\|a\|+|\varphi(a)|}$ and $N_\varphi(b,t)=\frac{t-\|b\|-|\varphi(b)|}{t+\|b\|+|\varphi(b)|}$.
One can easily verify that if $N_\varphi(a,s)\leq N_\varphi(b,t)$, then

$$s(\|b\|+|\varphi(b)|)\leq t(\|a\|+|\varphi(a)|),$$

and if $N_\varphi(b,t)\leq N_\varphi(a,s)$, then $t(\|a\|+|\varphi(a)|)\leq s(\|b\|+|\varphi(b)|)$.
A straightforward calculation reveals that
if $s(\|b\|+|\varphi(b)|)\leq t(\|a\|+|\varphi(a)|)$, then

$$\begin{aligned}N_\varphi(a+b,s+t)&=\frac{s+t-\|a+b\|-|\varphi(a)+\varphi(b)|}{s+t+\|a+b\|+|\varphi(a)+\varphi(b)|}\\&\geq\frac{s-\|a\|-|\varphi(a)|}{s+\|a\|+|\varphi(a)|}\\&=\min(N_\varphi(a,s),N_\varphi(b,t)),\end{aligned}$$

and if $t(\|a\|+|\varphi(a)|)\leq s(\|b\|+|\varphi(b)|)$, then

$$\begin{aligned}N_\varphi(a+b,s+t)&=\frac{s+t-\|a+b\|-|\varphi(a)+\varphi(b)|}{s+t+\|a+b\|+|\varphi(a)+\varphi(b)|}\\&\geq\frac{t-\|b\|-|\varphi(b)|}{t+\|b\|+|\varphi(b)|}\\&=\min(N_\varphi(a,s),N_\varphi(b,t)).\end{aligned}$$

Therefore, $N_\varphi(a+b, s+t) \geq \min(N_\varphi(a,s), N_\varphi(b,t))$ for all $a, b \in A$ and $s, t \in \mathbb{R}$.
(5): Let $a \in A$ and $s \leq t$. If $N_\varphi(a,s) = 0$, then clearly $N_\varphi(a,s) \leq N_\varphi(a,t)$. If $N_\varphi(a,s) \neq 0$, then $N_\varphi(a,s) = \frac{s-\|a\|-|\varphi(a)|}{s+\|a\|+|\varphi(a)|}$ and $s >\|a\| + |\varphi(a)|$. Hence, $t > \|a\| + |\varphi(a)|$ and $N_\varphi(a,t) = \frac{t-\|a\|-|\varphi(a)|}{t+\|a\|+|\varphi(a)|}$. Since $s \leq t$,

$$s(\|a\| + |\varphi(a)|) \leq t(\|a\| + |\varphi(a)|).$$

Therefore, a straightforward calculation reveals that

$$N_\varphi(a,s) = \frac{s - \|a\| - |\varphi(a)|}{s + \|a\| + |\varphi(a)|} \leq \frac{t - \|a\| - |\varphi(a)|}{t + \|a\| + |\varphi(a)|} = N_\varphi(a,t).$$

This shows that $N_\varphi(a, \cdot)$ is an increasing function for all $a \in A$.
Also $\lim\limits_{t\to\infty} N_\varphi(a,t) = \lim\limits_{t\to\infty} \frac{t-\|a\|-|\varphi(a)|}{t+\|a\|+|\varphi(a)|} = 1$.
(6): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $N_\varphi(a,s) = 0$ or $N_\varphi(b,t) = 0$, then $N_\varphi(ab, st) \geq N_\varphi(a,s)N_\varphi(b,t) = 0$.
If $N_\varphi(a,s) > 0$ and $N_\varphi(b,t) > 0$, then $s > \|a\| + |\varphi(a)|$ and $t > \|b\| + |\varphi(b)|$. So $st > \|ab\| + |\varphi(ab)|$. It is easy but tedious to show that the inequality

$$N_\varphi(ab, st) \geq N_\varphi(a,s)N_\varphi(b,t) \tag{1}$$

is equivalent to

$$\begin{aligned} st\Big(s(\|b\| + |\varphi(b)|) + t(\|a\| + |\varphi(a)|\Big) &\geq st|\varphi(a)||\varphi(b)| + st\|ab\| \\ &+ \|ab\|\|a\|\|b\| + \|ab\|\|a\||\varphi(b)| \\ &+ \|ab\|\|b\||\varphi(a)| + \|ab\||\varphi(a)||\varphi(b)| \\ &+ \|a\|\|b\||\varphi(a)||\varphi(b)| + \|a\||\varphi(a)||\varphi(b)|^2 \\ &+ \|b\||\varphi(b)||\varphi(a)|^2 + |\varphi(a)|^2|\varphi(b)|^2. \end{aligned}$$

So by hypothesis, we have

$$
\begin{aligned}
& st\Big(s(\|b\|+|\varphi(b)|)+t(\|a\|+|\varphi(a)|)\Big)\\
& \geq st\Big((\|a\|+|\varphi(a)|)(\|b\|+|\varphi(b)|)+(\|b\|+|\varphi(b)|)(\|a\|+|\varphi(a)|)\Big)\\
& = 2st\Big((\|a\|+|\varphi(a)|)(\|b\|+|\varphi(b)|)\Big)\\
& = (st+st)\Big(\|a\|\|b\|+\|a\||\varphi(b)|+\|b\||\varphi(a)|+|\varphi(a)||\varphi(b)|\Big)\\
& = st\Big(\|a\|\|b\|+\|a\||\varphi(b)|+\|b\||\varphi(a)|+|\varphi(a)||\varphi(b)|\Big)\\
& + st\Big(\|a\|\|b\|+\|a\||\varphi(b)|+\|b\||\varphi(a)|+|\varphi(a)||\varphi(b)|\Big)\\
& \geq st\Big(\|a\|\|b\|+|\varphi(a)||\varphi(b)|\Big)\\
& + \Big(\|ab\|+|\varphi(a)\varphi(b)|\Big)\Big(\|a\|\|b\|+\|a\||\varphi(b)|+\|b\||\varphi(a)|+|\varphi(a)||\varphi(b)|\Big)\\
& \geq st\Big(\|ab\|+|\varphi(a)||\varphi(b)|\Big)\\
& + \Big(\|ab\|+|\varphi(a)\varphi(b)|\Big)\Big(\|a\|\|b\|+\|a\||\varphi(b)|+\|b\||\varphi(a)|+|\varphi(a)||\varphi(b)|\Big)\\
& = st|\varphi(a)||\varphi(b)|+st\|ab\|+\|ab\|\|a\|\|b\|+\|ab\|\|a\||\varphi(b)|\\
& + \|ab\|\|b\||\varphi(a)|+\|ab\||\varphi(a)||\varphi(b)|+\|a\|\|b\||\varphi(a)||\varphi(b)|\\
& + \|a\||\varphi(a)||\varphi(b)|^2+\|b\||\varphi(b)||\varphi(a)|^2+|\varphi(a)|^2|\varphi(b)|^2.
\end{aligned}
$$

This shows that inequality 1 holds for all $a, b \in A$ and $s, t \in \mathbb{R}$. Hence, N_φ is an algebra fuzzy norm on A. □

Proposition 2.6. *Let A be a normed algebra, $\varphi : A \longrightarrow \mathbb{F}$ be a continuous homomorphism, and $\varepsilon > 0$ be an element such that $(\varepsilon + \|\varphi\|) \geq 1$. Define $N_{\varphi,\varepsilon} : A \times \mathbb{R} \longrightarrow [0, 1]$ by*

$$
N_{\varphi,\varepsilon}(a,t) = \begin{cases} 0, & t \leq \|a\|(\varepsilon+\|\varphi\|) \\ \frac{t-|\varphi(a)|}{t+|\varphi(a)|}, & t > \|a\|(\varepsilon+\|\varphi\|). \end{cases}
$$

Then $N_{\varphi,\varepsilon}$ is an algebra fuzzy norm on A.

Proof. We only prove parts (2), (4), (5), and (6) of Definition 2.2.
(2) : If $a = 0$, then clearly $N_{\varphi,\varepsilon}(a, t) = 1$ for all $t > 0$. For the converse let $N_{\varphi,\varepsilon}(a, t) = 1$ for all $t > 0$ and suppose by contradiction that $a \neq 0$.
If $t = \|a\|(\varepsilon + \|\varphi\|)$, then $N_{\varphi,\varepsilon}(a, t) = 0$ that is a contradiction.

(4) : Let $a,b \in A$ and $s,t \in \mathbb{R}$. If $s+t \leq 0$, then $s \leq 0$ or $t \leq 0$. So $N_{\varphi,\varepsilon}(a,s)=0$ or $N_{\varphi,\varepsilon}(b,t)=0$.
Hence, $0 = N_{\varphi,\varepsilon}(a+b,s+t) \geq \min(N_{\varphi,\varepsilon}(a,s), N_{\varphi,\varepsilon}(b,t)) = 0$.
If $s+t>0$ and $s+t \leq \|a+b\|(\varepsilon+\|\varphi\|)$, then $s \leq \|a\|(\varepsilon+\|\varphi\|)$ or $t \leq \|b\|(\varepsilon+\|\varphi\|)$. Therefore,

$$0 = N_{\varphi,\varepsilon}(a+b,s+t) \geq \min(N_{\varphi,\varepsilon}(a,s), N_{\varphi,\varepsilon}(b,t)) = 0.$$

If $s+t > \|a+b\|(\varepsilon+\|\varphi\|)$, then $N_{\varphi,\varepsilon}(a+b,s+t) = \frac{s+t-|\varphi(a)+\varphi(b)|}{s+t+|\varphi(a)+\varphi(b)|}$.
In this case if $s \leq \|a\|(\varepsilon+\|\varphi\|)$ or $t \leq \|b\|(\varepsilon+\|\varphi\|)$, then clearly

$$\frac{s+t-|\varphi(a)+\varphi(b)|}{s+t+|\varphi(a)+\varphi(b)|} \geq \min(N_{\varphi,\varepsilon}(a,s), N_{\varphi,\varepsilon}(b,t)) = 0.$$

If $s > \|a\|(\varepsilon+\|\varphi\|)$ and $t > \|b\|(\varepsilon+\|\varphi\|)$, then $N_{\varphi,\varepsilon}(a,s) = \frac{s-|\varphi(a)|}{s+|\varphi(a)|}$ and $N_{\varphi,\varepsilon}(b,t) = \frac{t-|\varphi(b)|}{t+|\varphi(b)|}$.
One can easily verify that if $N_{\varphi,\varepsilon}(a,s) \leq N_{\varphi,\varepsilon}(b,t)$, then $s|\varphi(b)| \leq t|\varphi(a)|$, and if $N_{\varphi,\varepsilon}(b,t) \leq N_{\varphi,\varepsilon}(a,s)$, then $t|\varphi(a)| \leq s|\varphi(b)|$.
A straightforward calculation reveals that if $s|\varphi(b)| \leq t|\varphi(a)|$, then

$$\begin{aligned} N_{\varphi,\varepsilon}(a+b,s+t) &= \frac{s+t-|\varphi(a)+\varphi(b)|}{s+t+|\varphi(a)+\varphi(b)|} \\ &\geq \frac{s-|\varphi(a)|}{s+|\varphi(a)|} \\ &= \min(N_{\varphi,\varepsilon}(a,s), N_{\varphi,\varepsilon}(b,t)), \end{aligned}$$

and if $t|\varphi(a)| \leq s|\varphi(b)|$, then

$$\begin{aligned} N_{\varphi,\varepsilon}(a+b,s+t) &= \frac{s+t-|\varphi(a)+\varphi(b)|}{s+t+|\varphi(a)+\varphi(b)|} \\ &\geq \frac{t-|\varphi(b)|}{t+|\varphi(b)|} \\ &= \min(N_{\varphi,\varepsilon}(a,s), N_{\varphi,\varepsilon}(b,t)). \end{aligned}$$

Therefore, $N_{\varphi,\varepsilon}(a+b,s+t) \geq \min(N_{\varphi,\varepsilon}(a,s), N_{\varphi,\varepsilon}(b,t))$ for all $a,b \in A$ and $s,t \in \mathbb{R}$.
(5) : Let $a \in A$ and $s \leq t$. If $N_{\varphi,\varepsilon}(a,s) = 0$, then clearly $N_{\varphi,\varepsilon}(a,s) \leq N_{\varphi,\varepsilon}(a,t)$. If $N_{\varphi,\varepsilon}(a,s) \neq 0$, then $N_{\varphi,\varepsilon}(a,s) = \frac{s-|\varphi(a)|}{s+|\varphi(a)|}$ and $s > \|a\|(\varepsilon+\|\varphi\|)$. Hence, $t > \|a\|(\varepsilon+\|\varphi\|)$ and $N_{\varphi,\varepsilon}(a,t) = \frac{t-|\varphi(a)|}{t+|\varphi(a)|}$.
Since $s \leq t$, $2s|\varphi(a)| \leq 2t|\varphi(a)|$. Therefore, a straightforward calculation reveals that $N_{\varphi,\varepsilon}(a,s) = \frac{s-|\varphi(a)|}{s+|\varphi(a)|} \leq \frac{t-|\varphi(a)|}{t+|\varphi(a)|} = N_{\varphi,\varepsilon}(a,t)$. This shows that $N_{\varphi,\varepsilon}(a,\cdot)$ is an increasing function for all $a \in A$. Also

$$\lim_{t\to\infty} N_{\varphi,\varepsilon}(a,t) = \lim_{t\to\infty} \frac{t-|\varphi(a)|}{t+|\varphi(a)|} = 1.$$

(6): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $N_{\varphi,\varepsilon}(a, s) = 0$ or $N_{\varphi,\varepsilon}(b, t) = 0$, then

$$N_{\varphi,\varepsilon}(ab, st) \geq N_{\varphi,\varepsilon}(a, s)N_{\varphi,\varepsilon}(b, t) = 0.$$

If $N_{\varphi,\varepsilon}(a, s) > 0$ and $N_{\varphi,\varepsilon}(b, t) > 0$, then $s > \|a\|(\varepsilon+\|\varphi\|)$ and $t > \|b\|(\varepsilon+\|\varphi\|)$. So the condition $\varepsilon + \|\varphi\| \geq 1$ implies $st > \|ab\|(\varepsilon + \|\varphi\|)$. It is easy to see that the inequality

$$N_{\varphi,\varepsilon}(ab, st) \geq N_{\varphi,\varepsilon}(a, s)N_{\varphi,\varepsilon}(b, t) \tag{2}$$

is equivalent to $st|\varphi(a)||\varphi(b)| + |\varphi(a)|^2|\varphi(b)|^2 \leq s^2t|\varphi(b)| + st^2|\varphi(a)|$.
Since $s > \|a\|(\varepsilon + \|\varphi\|) \geq \|a\|\|\varphi\| \geq |\varphi(a)|$ and $t > \|b\|(\varepsilon + \|\varphi\|) \geq \|b\|\|\varphi\| \geq |\varphi(b)|$, we have

$$\begin{aligned} st|\varphi(a)||\varphi(b)| + |\varphi(a)|^2|\varphi(b)|^2 &= st|\varphi(a)||\varphi(b)| + |\varphi(b)|^2|\varphi(a)||\varphi(a)| \\ &\leq st(s)|\varphi(b)| + t^2s|\varphi(a)| \\ &= s^2t|\varphi(b)| + st^2|\varphi(a)|. \end{aligned}$$

This shows that inequality 2 holds for all $a, b \in A$ and $s, t \in \mathbb{R}$. □

Remark 2.7. Note that the condition $\varepsilon+\|\varphi\| \geq 1$ in Proposition 2.6 is necessary. Indeed, for $A = \mathbb{R}$, $\varphi = 0$, and $0 < \varepsilon < 1$ we have $\varepsilon + \|\varphi\| = \varepsilon < 1$. Also $N_{0,\varepsilon}(1\cdot 1, \sqrt{\varepsilon}\cdot\sqrt{\varepsilon}) = N_{0,\varepsilon}(1, \varepsilon) = 0$ and $N_{0,\varepsilon}(1, \sqrt{\varepsilon}) = 1$. Since, $\varepsilon \leq |1|(\varepsilon+0)$ and $\sqrt{\varepsilon} > |1|(\varepsilon + 0)$. So the inequality $N_{0,\varepsilon}(1 \cdot 1, \sqrt{\varepsilon} \cdot \sqrt{\varepsilon}) \geq N_{0,\varepsilon}(1, \sqrt{\varepsilon})N_{0,\varepsilon}(1, \sqrt{\varepsilon})$ does not hold. Hence, $N_{0,\varepsilon}$ is not an algebra fuzzy norm.

Proposition 2.8. *Let A be a normed algebra and $\psi \in Hom(A, \mathbb{F})$. Then the maps*

$$N_\psi^{(1)} : A \times \mathbb{R} \longrightarrow [0, 1]$$
$$N_\psi^{(1)}(a, t) = \begin{cases} 0, & t \leq \|a\| + |\psi(a)| \\ \frac{t}{t+\|a\|+|\psi(a)|}, & t > \|a\| + |\psi(a)|, \end{cases}$$

and

$$N_\psi^{(2)} : A \times \mathbb{R} \longrightarrow [0, 1]$$
$$N_\psi^{(2)}(a, t) = \begin{cases} 0, & t \leq |\psi(a)| \\ \frac{t}{t+\|a\|+|\psi(a)|}, & t > |\psi(a)|, \end{cases}$$

are algebra fuzzy norms on A.
Also if $\ker\psi = \{0\}$, then the map

$$N_\psi^{(3)} : A \times \mathbb{R} \longrightarrow [0, 1]$$
$$N_\psi^{(3)}(a, t) = \begin{cases} 0, & t \leq |\psi(a)| \\ \frac{t}{t+|\psi(a)|}, & t > |\psi(a)|, \end{cases}$$

is an algebra fuzzy norm on A.

Proof. First, we prove that $N_\psi^{(2)}$ is an algebra fuzzy norm. In the end, we will only prove part (6) of Definition 2.2 for $N_\psi^{(1)}$ and $N_\psi^{(3)}$. The proof of other parts is similar. Note that for investigation of the fuzzy norm properties in the case $N_\psi^{(3)}$, the condition $\ker\psi = \{0\}$ will be used in part (2) of Definition 2.2, since, $|\psi(a)| = 0$ implies $a = 0$.

(1): If $t \leq 0$, then clearly $N_\psi^{(2)}(a,t) = 0$ for all $a \in A$.

(2): If $a = 0$, then $\psi(a) = 0$ and consequently for all $t > 0 = |\psi(a)|$, $N_\psi^{(2)}(a,t) = \frac{t}{t+0+0} = 1$. For the converse let $N_\psi^{(2)}(a,t) = 1$ for all $t > 0$. First, we show that $\psi(a) = 0$. Suppose by contradiction that $\psi(a) \neq 0$.

If $t = |\psi(a)|$, then $N_\psi^{(2)}(a,t) = 0$ that is a contradiction. This shows that $\psi(a) = 0$. So by hypothesis, for all $t > 0 = |\psi(a)|$, $\frac{t}{t+\|a\|+0} = N_\psi^{(2)}(a,t) = 1$. It follows that $\|a\| = 0$ that implies $a = 0$.

(3): If $\alpha \neq 0$, then

$$\begin{aligned} N_\psi^{(2)}(\alpha a,t) &= \begin{cases} 0, & t \leq |\psi(\alpha a)| \\ \frac{t}{t+\|\alpha a\|+|\psi(\alpha a)|}, & t > |\psi(\alpha a)| \end{cases} \\ &= \begin{cases} 0, & \frac{t}{|\alpha|} \leq |\psi(a)| \\ \frac{\frac{t}{|\alpha|}}{\frac{t}{|\alpha|}+\|a\|+|\psi(a)|}, & \frac{t}{|\alpha|} > |\psi(a)| \end{cases} \\ &= N_\psi^{(2)}\big(a, \frac{t}{|\alpha|}\big), a \in A, t \in \mathbb{R}. \end{aligned}$$

(4): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $s + t \leq |\psi(a+b)| = |\psi(a) + \psi(b)|$, then $s \leq |\psi(a)|$ or $t \leq |\psi(b)|$. So

$$0 = N_\psi^{(2)}(a+b, s+t) \geq \min(N_\psi^{(2)}(a,s), N_\psi^{(2)}(b,t)) = 0.$$

Let $s + t > |\psi(a) + \psi(b)|$ and $s \leq |\psi(a)|$ or $t \leq |\psi(b)|$. Then clearly,

$$\begin{aligned} N_\psi^{(2)}(a+b, s+t) &= \frac{s+t}{s+t+\|a+b\| + |\psi(a)+\psi(b)|} \\ &> 0 \\ &= \min(N_\psi^{(2)}(a,s), N_\psi^{(2)}(b,t)). \end{aligned}$$

Let $s+t > |\psi(a)+\psi(b)|$ and $s > |\psi(a)|$ and $t > |\psi(b)|$. Then $N_\psi^{(2)}(a+b, s+t) = \frac{s+t}{s+t+\|a+b\|+|\psi(a)+\psi(b)|}$ and

$N_\psi^{(2)}(a,s) = \frac{s}{s+\|a\|+|\psi(a)|}$ and $N_\psi^{(2)}(b,t) = \frac{t}{t+\|b\|+|\psi(b)|}$.

If $\frac{s}{s+\|a\|+|\psi(a)|} \leq \frac{t}{t+\|b\|+|\psi(b)|}$, then

$$s(\|b\| + |\psi(b)|) \leq t(\|a\| + |\psi(a)|). \tag{3}$$

If $\frac{t}{t+\|b\|+|\psi(b)|} \leq \frac{s}{s+\|a\|+|\psi(a)|}$, then

$$t(\|a\| + |\psi(a)|) \leq s(\|b\| + |\psi(b)|). \tag{4}$$

For the case $N_\psi^{(2)}(a,s) \leq N_\psi^{(2)}(b,t)$, inequality 3 implies

$$\begin{aligned}\frac{s+t}{s+t+\|a+b\|+|\psi(a)+\psi(b)|} &= N_\psi^{(2)}(a+b, s+t)\\ &\geq \frac{s}{s+\|a\|+|\psi(a)|}\\ &= \min(N_\psi^{(2)}(a,s), N_\psi^{(2)}(b,t)).\end{aligned}$$

Also for the case $N_\psi^{(2)}(b,t) \leq N_\psi^{(2)}(a,s)$, inequality 4 implies
$N_\psi^{(2)}(a+b, s+t) \geq \min(N_\psi^{(2)}(a,s), N_\psi^{(2)}(b,t))$.
(5): Let $s \leq t$. If $N_\psi^{(2)}(a,s) = 0$, then clearly $N_\psi^{(2)}(a,s) \leq N_\psi^{(2)}(a,t)$. Let $N_\psi^{(2)}(a,s) \neq 0$. Then $s > |\psi(a)|$. It follows that $t > |\psi(a)|$.
Since $s(\|a\| + |\psi(a)|) \leq t(\|a\| + |\psi(a)|)$, $N_\psi^{(2)}(a,s) \leq N_\psi^{(2)}(a,t)$. So $N_\psi^{(2)}(a,\cdot)$ is an increasing function and also

$$\lim_{t\to\infty} N_\psi^{(2)}(a,t) = \lim_{t\to\infty} \frac{t}{t+\|a\|+|\psi(a)|} = 1.$$

(6): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $N_\psi^{(2)}(a,s) = 0$ or $N_\psi^{(2)}(b,t) = 0$, then

$$N_\psi^{(2)}(ab, st) \geq N_\psi^{(2)}(a,s)N_\psi^{(2)}(b,t) = 0.$$

If $N_\psi^{(2)}(a,s) > 0$ and $N_\psi^{(2)}(b,t) > 0$, then $s > |\psi(a)|$ and $t > |\psi(b)|$. So $st > |\psi(ab)|$. Clearly

$$st + \|ab\|+|\psi(a)||\psi(b)| \leq (s + \|a\|+|\psi(a)|)(t + \|b\|+|\psi(b)|).$$

So $\frac{st}{st+\|ab\|+|\psi(a)||\psi(b)|} \geq \frac{st}{(s+\|a\|+|\psi(a)|)(t+\|b\|+|\psi(b)|)}$. Hence,

$$N_\psi^{(2)}(ab, st) \geq N_\psi^{(2)}(a,s)N_\psi^{(2)}(b,t). \tag{5}$$

Therefore, inequality 5 holds for all $a, b \in A$ and $s, t \in \mathbb{R}$.
Now we will investigate part (6) of Definition 2.2 for $N_\psi^{(1)}$ and $N_\psi^{(3)}$.

$$N_\psi^{(1)} : A \times \mathbb{R} \longrightarrow [0,1]$$
$$N_\psi^{(1)}(a,t) = \begin{cases} 0, & t \leq \|a\| + |\psi(a)| \\ \frac{t}{t+\|a\|+|\psi(a)|}, & t > \|a\| + |\psi(a)|, \end{cases}$$

(6): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $N_\psi^{(1)}(a,s) = 0$ or $N_\psi^{(1)}(b,t) = 0$, then

$$N_\psi^{(1)}(ab, st) \geq N_\psi^{(1)}(a,s)N_\psi^{(1)}(b,t) = 0.$$

If $N_\psi^{(1)}(a,s)>0$ and $N_\psi^{(1)}(b,t)>0$, then $s>\|a\|+|\psi(a)|$ and $t>\|b\|+|\psi(b)|$. So $st>\|ab\|+|\psi(ab)|$. Clearly $st+\|ab\|+|\psi(a)||\psi(b)|\le(s+\|a\|+|\psi(a)|)(t+\|b\|+|\psi(b)|)$. So $\frac{st}{st+\|ab\|+|\psi(a)||\psi(b)|}\ge\frac{st}{(s+\|a\|+|\psi(a)|)(t+\|b\|+|\psi(b)|)}$. Hence,

$$N_\psi^{(1)}(ab,st)\ge N_\psi^{(1)}(a,s)N_\psi^{(1)}(b,t). \tag{6}$$

Therefore, inequality 6 holds for all $a,b\in A$ and $s,t\in\mathbb{R}$.

$$N_\psi^{(3)}:A\times\mathbb{R}\longrightarrow[0,1]$$

$$N_\psi^{(3)}(a,t)=\begin{cases}0, & t\le|\psi(a)|\\ \frac{t}{t+|\psi(a)|}, & t>|\psi(a)|,\end{cases}$$

(6): Let $a,b\in A$ and $s,t\in\mathbb{R}$. If $N_\psi^{(3)}(a,s)=0$ or $N_\psi^{(3)}(b,t)=0$, then

$$N_\psi^{(3)}(ab,st)\ge N_\psi^{(3)}(a,s)N_\psi^{(3)}(b,t)=0.$$

If $N_\psi^{(3)}(a,s)>0$ and $N_\psi^{(3)}(b,t)>0$, then $s>|\psi(a)|$ and $t>|\psi(b)|$. So $st>|\psi(ab)|$. Clearly $st+|\psi(a)||\psi(b)|\le(s+|\psi(a)|)(t+|\psi(b)|)$. So $\frac{st}{st+|\psi(a)||\psi(b)|}\ge\frac{st}{(s+|\psi(a)|)(t+|\psi(b)|)}$ and consequently

$$N_\psi^{(3)}(ab,st)\ge N_\psi^{(3)}(a,s)N_\psi^{(3)}(b,t). \tag{7}$$

This shows that for all $a,b\in A$ and $s,t\in\mathbb{R}$ inequality 7 holds. □

Proposition 2.9. *Let A be a normed algebra and $\eta\in Hom(A,\mathbb{F})$. Then the map*

$$N_\eta^{(1)}:A\times\mathbb{R}\longrightarrow[0,1]$$

$$N_\eta^{(1)}(a,t)=\begin{cases}0, & t\le\|a\|+|\eta(a)|\\ \frac{t-\|a\|-|\eta(a)|}{t}, & t>\|a\|+|\eta(a)|,\end{cases}$$

is an algebra fuzzy norm on A.
If $\ker\eta=\{0\}$, then the map

$$N_\eta^{(2)}:A\times\mathbb{R}\longrightarrow[0,1]$$

$$N_\eta^{(2)}(a,t)=\begin{cases}0, & t\le|\eta(a)|\\ \frac{t-|\eta(a)|}{t}, & t>|\eta(a)|,\end{cases}$$

is an algebra fuzzy norm on A.

Proof. We only prove part (6) of Definition 2.2 for $N_\eta^{(1)}$ and $N_\eta^{(2)}$.

$$N_\eta^{(1)}:A\times\mathbb{R}\longrightarrow[0,1]$$

$$N_\eta^{(1)}(a,t)=\begin{cases}0, & t\le\|a\|+|\eta(a)|\\ \frac{t-\|a\|-|\eta(a)|}{t}, & t>\|a\|+|\eta(a)|,\end{cases}$$

(6): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $N_\eta^{(1)}(a, s) = 0$ or $N_\eta^{(1)}(b, t) = 0$, then

$$N_\eta^{(1)}(ab, st) \geq N_\eta^{(1)}(a, s)N_\eta^{(1)}(b, t) = 0.$$

If $N_\eta^{(1)}(a, s) > 0$ and $N_\eta^{(1)}(b, t) > 0$, then $s > \|a\| + |\eta(a)|$ and $t > \|b\| + |\eta(b)|$. So $st > \|ab\| + |\eta(ab)|$. We show that the inequality

$$N_\eta^{(1)}(ab, st) \geq N_\eta^{(1)}(a, s)N_\eta^{(1)}(b, t) \tag{8}$$

holds. It is easy to see that inequality 8 is equivalent to

$$s(\|b\|+|\eta(b)|)+t(\|a\|+|\eta(a)|) \geq \|a\|\|b\|+\|ab\|+\|a\||\eta(b)|+\|b\||\eta(a)|+2|\eta(a)||\eta(b)|.$$

In this case

$$s(\|b\|+|\eta(b)|) \geq (\|a\|+|\eta(a)|)(\|b\|+|\eta(b)|)$$

and

$$t(\|a\|+|\eta(a)|) \geq (\|b\|+|\eta(b)|)(\|a\|+|\eta(a)|).$$

Hence,

$$s(\|b\|+|\eta(b)|) + t(\|a\|+|\eta(a)|) \geq 2(\|a\|\|b\|+\|a\||\eta(b)| + \|b\||\eta(a)| + |\eta(a)||\eta(b)|).$$

Therefore,

$$\begin{aligned} s(\|b\|+|\eta(b)|) + t(\|a\|+|\eta(a)|) &\geq (\|a\|\|b\|+\|a\||\eta(b)| + \|b\||\eta(a)| + 2|\eta(a)||\eta(b)|) \\ &\quad + (\|a\|\|b\|) + (\|a\||\eta(b)| + \|b\||\eta(a)|) \\ &\geq (\|a\|\|b\|+\|ab\|+\|a\||\eta(b)| + \|b\||\eta(a)| + 2|\eta(a)||\eta(b)|) \\ &\quad + (\|a\||\eta(b)| + \|b\||\eta(a)|) \\ &\geq \|a\|\|b\|+\|ab\|+\|a\||\eta(b)| + \|b\||\eta(a)| + 2|\eta(a)||\eta(b)|. \end{aligned}$$

So inequality 8 holds for all $a, b \in A$ and $s, t \in \mathbb{R}$.

$$N_\eta^{(2)} : A \times \mathbb{R} \longrightarrow [0, 1]$$

$$N_\eta^{(2)}(a, t) = \begin{cases} 0, & t \leq |\eta(a)| \\ \frac{t-|\eta(a)|}{t}, & t > |\eta(a)|, \end{cases}$$

(6): Let $a, b \in A$ and $s, t \in \mathbb{R}$. If $N_\eta^{(2)}(a, s) = 0$ or $N_\eta^{(2)}(b, t) = 0$, then

$$N_\eta^{(2)}(ab, st) \geq N_\eta^{(2)}(a, s)N_\eta^{(2)}(b, t) = 0.$$

If $N_\eta^{(2)}(a, s) > 0$ and $N_\eta^{(2)}(b, t) > 0$, then $s > |\eta(a)|$ and $t > |\eta(b)|$. So $st > |\eta(ab)|$. It is easy to see that the inequality

$$N_\eta^{(2)}(ab, st) \geq N_\eta^{(2)}(a, s)N_\eta^{(2)}(b, t) \tag{9}$$

is equivalent to $s|\eta(b)| + t|\eta(a)| \geq 2|\eta(a)||\eta(b)|$. So in this case we have,

$$s|\eta(b)| \geq |\eta(a)||\eta(b)|$$

and

$$t|\eta(a)| \geq |\eta(b)||\eta(a)|.$$

Hence,

$$s|\eta(b)| + t|\eta(a)| \geq 2|\eta(a)||\eta(b)|.$$

This shows that for all $a, b \in A$ and $s, t \in \mathbb{R}$ inequality 9 holds. $\square$

Example 2.10. *Let*

$$C([0,1]) = \{f : [0,1] \longrightarrow \mathbb{C}, \quad f \text{ is continuous}\}$$

be the complex-valued, bounded and continuous functions on $[0,1]$ *with the uniform norm* $\|f\|_\infty = \sup\{|f(x)|, \quad x \in [0,1]\}$. *Clearly,* $(C([0,1]), \|\cdot\|_\infty)$ *is a Banach algebra. Let* $s \in [0,1]$ *be a fixed element. Define* $\varphi_s : C([0,1]) \longrightarrow \mathbb{C}$ *by* $\varphi_t(f) = f(t)$ *for all* $f \in C([0,1])$. *Obviously,* φ_s *is a continuous algebra homomorphism and* $\|\varphi_s\| = 1$. *Let* $\varepsilon > 0$. *By Proposition 2.5, Proposition 2.6, and Proposition 2.8,*

$$N_{\varphi_s} : C([0,1]) \times \mathbb{R} \longrightarrow [0,1]$$

$$N_{\varphi_s}(f,t) = \begin{cases} 0, & t \leq \|f\|_\infty + |f(s)| \\ \frac{t - \|f\|_\infty - |f(s)|}{t + \|f\|_\infty + |f(s)|}, & t > \|f\|_\infty + |f(s)| \end{cases},$$

$$N_{\varphi_s,\varepsilon} : C([0,1]) \times \mathbb{R} \longrightarrow [0,1]$$

$$N_{\varphi_s,\varepsilon}(f,t) = \begin{cases} 0, & t \leq \|f\|_\infty(\varepsilon + 1) \\ \frac{t - |f(s)|}{t + |f(s)|}, & t > \|f\|_\infty(\varepsilon + 1) \end{cases},$$

$$N^{(1)}_{\varphi_s} : C([0,1]) \times \mathbb{R} \longrightarrow [0,1]$$

$$N^{(1)}_{\varphi_s}(f,t) = \begin{cases} 0, & t \leq \|f\|_\infty + |f(s)| \\ \frac{t}{t + \|f\|_\infty + |f(s)|}, & t > \|f\|_\infty + |f(s)| \end{cases},$$

and

$$N^{(2)}_{\varphi_s} : C([0,1]) \times \mathbb{R} \longrightarrow [0,1]$$

$$N^{(2)}_{\varphi_s}(f,t) = \begin{cases} 0, & t \leq |f(s)| \\ \frac{t}{t + \|f\|_\infty + |f(s)|}, & t > |f(s)| \end{cases}$$

are algebra fuzzy norms on $C([0,1])$.

3. The separate continuity of N_φ, $N_{\varphi,\varepsilon}$, $N^{(i)}_\psi$, $N^{(j)}_\eta$, $1 \leq i \leq 3$, $1 \leq j \leq 2$

In this section, we characterize separate continuity of the algebra fuzzy norms N_φ, $N_{\varphi,\varepsilon}$, $N^{(i)}_\psi$, $N^{(j)}_\eta$, $1 \leq i \leq 3$, $1 \leq j \leq 2$, where $\varepsilon > 0$ and φ, ψ, η are continuous homomorphisms from A into $\mathbb{F}$.

Theorem 3.1. *Let A be a normed algebra and $\varphi : A \longrightarrow \mathbb{F}$ be a continuous homomorphism. If*
$N_\varphi : A \times \mathbb{R} \longrightarrow [0, 1]$ *is defined by*

$$N_\varphi(a,t) = \begin{cases} 0, & t \leq \|a\| + |\varphi(a)| \\ \frac{t - \|a\| - |\varphi(a)|}{t + \|a\| + |\varphi(a)|}, & t > \|a\| + |\varphi(a)|, \end{cases}$$

then the map

$$\begin{aligned} N_\varphi(\cdot, t) : A &\longrightarrow [0,1] \\ a &\longrightarrow N_\varphi(a,t), \end{aligned}$$

is continuous for all $t \in \mathbb{R}$.
Also the map

$$\begin{aligned} N_\varphi(a, \cdot) : \mathbb{R} &\longrightarrow [0,1] \\ t &\longrightarrow N_\varphi(a,t), \end{aligned}$$

is continuous for all $a \in A$ except $a = 0$.

Proof. If $t \leq 0$, then $N_\varphi(a,t) = 0$ for all $a \in A$. So $N_\varphi(\cdot, t) : A \longrightarrow [0,1]$ is a constant function and consequently is continuous on A for all $t \leq 0$.
For a fixed $t > 0$, let $a \in A$ and $\{a_n\}_{n=1}^\infty$ be a sequence such that $a_n \longrightarrow a$, as $n \longrightarrow \infty$. If $t = \|a\| + |\varphi(a)|$, then for all subsequences $\{a_{n_k}\}_{k=1}^\infty \subseteq \{a_n\}_{n=1}^\infty$ satisfying $t \leq \|a_{n_k}\| + |\varphi(a_{n_k})|$, $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty} N_\varphi(a_{n_k}, t) = \lim_{k\to\infty} 0 = 0 = N_\varphi(a,t).$$

Also for all subsequences $\{a_{n_k}\}_{k=1}^\infty \subseteq \{a_n\}_{n=1}^\infty$ satisfying $t > \|a_{n_k}\| + |\varphi(a_{n_k})|, k \in \mathbb{N}$, we have

$$\begin{aligned} \lim_{k\to\infty} N_\varphi(a_{n_k}, t) &= \lim_{k\to\infty} \frac{t - \|a_{n_k}\| - |\varphi(a_{n_k})|}{t + \|a_{n_k}\| + |\varphi(a_{n_k})|} \\ &= \frac{t-t}{t+t} \\ &= 0 \\ &= N_\varphi(a,t). \end{aligned}$$

If $t < \|a\| + |\varphi(a)|$, then there exists an $n_0 \in \mathbb{N}$ such that $t < \|a_n\| + |\varphi(a_n)|$ for all $n \geq n_0$. So $\lim\limits_{n\to\infty} N_\varphi(a_n, t) = \lim\limits_{n\to\infty} 0 = 0 = N_\varphi(a,t)$.
If $t > \|a\| + |\varphi(a)|$, then there exists an $n_0 \in \mathbb{N}$ such that $t > \|a_n\| + |\varphi(a_n)|$ for all $n \geq n_0$. So

$$\begin{aligned} \lim_{n\to\infty} N_\varphi(a_n, t) &= \lim_{n\to\infty} \frac{t - \|a_n\| - |\varphi(a_n)|}{t + \|a_n\| + |\varphi(a_n)|} \\ &= \frac{t - \|a\| - |\varphi(a)|}{t + \|a\| + |\varphi(a)|} \\ &= N_\varphi(a,t). \end{aligned}$$

Hence, for each $t > 0$, $N_\varphi(\cdot, t)$ is continuous at every $a \in A$. So $N_\varphi(\cdot, t)$ is continuous on A for all $t \in \mathbb{R}$.
For any fixed $a \neq 0$, let $t \in \mathbb{R}$ and $t_n \longrightarrow t$, as $n \longrightarrow \infty$. If $t = \|a\|+|\varphi(a)|$, then for all subsequences $\{t_{n_k}\}_{k=1}^{\infty} \subseteq \{t_n\}_{n=1}^{\infty}$ satisfying $t_{n_k} \leq\|a\| + |\varphi(a)|$ we have

$$\lim_{k\to\infty} N_\varphi(a, t_{n_k}) = \lim_{k\to\infty} 0 = 0 = N_\varphi(a, t).$$

Also for all subsequences $\{t_{n_k}\}_{k=1}^{\infty} \subseteq \{t_n\}_{n=1}^{\infty}$ satisfying $t_{n_k} >\|a\| + |\varphi(a)|$ we have

$$\begin{aligned} \lim_{k\to\infty} N_\varphi(a, t_{n_k}) &= \lim_{k\to\infty} \frac{t_{n_k} - \|a\| - |\varphi(a)|}{t_{n_k} + \|a\| + |\varphi(a)|} \\ &= \frac{t - \|a\| - |\varphi(a)|}{t + \|a\| + |\varphi(a)|} \\ &= \frac{t-t}{t+t} \\ &= 0 \\ &= N_\varphi(a, t). \end{aligned}$$

If $t <\|a\| + |\varphi(a)|$, then there exists an $n_0 \in \mathbb{N}$ such that $t_n <\|a\| + |\varphi(a)|$ for all $n \geq n_0$. So $\lim\limits_{n\to\infty} N_\varphi(a, t_n) = \lim\limits_{n\to\infty} 0 = 0 = N_\varphi(a, t)$.
If $t >\|a\| + |\varphi(a)|$, then there exists an $n_0 \in \mathbb{N}$ such that $t_n >\|a\| + |\varphi(a)|$ for all $n \geq n_0$. So

$$\begin{aligned} \lim_{n\to\infty} N_\varphi(a, t_n) &= \lim_{n\to\infty} \frac{t_n - \|a\| - |\varphi(a)|}{t_n + \|a\| + |\varphi(a)|} \\ &= \frac{t - \|a\| - |\varphi(a)|}{t + \|a\| + |\varphi(a)|} \\ &= N_\varphi(a, t). \end{aligned}$$

It follows that for any fixed $a \neq 0$, $t_n \longrightarrow t$, as $n \longrightarrow \infty$, implies $\lim\limits_{n\longrightarrow\infty} N_\varphi(a, t_n) = N_\varphi(a, t)$. Hence, $N_\varphi(a, \cdot) : \mathbb{R} \longrightarrow [0, 1]$ is continuous for all $a \neq 0$.
We shall show that $N_\varphi(0, \cdot) : \mathbb{R} \longrightarrow [0, 1]$ is not continuous at $t = 0$.
Since

$$N_\varphi(0, t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0, \end{cases}$$

$\lim\limits_{t\longrightarrow 0^+} N_\varphi(0, t) = 1 \neq N_\varphi(0, 0) = 0$. This shows that $N_\varphi(0, \cdot) : \mathbb{R} \longrightarrow [0, 1]$ is not continuous on $\mathbb{R}$. □

Theorem 3.2. *Let A be a normed algebra, $\varphi : A \longrightarrow \mathbb{F}$ be a continuous homomorphism, and $\varepsilon > 0$ be an element such that $(\varepsilon + \|\varphi\|) \geq 1$. If*

$N_{\varphi,\varepsilon} : A \times \mathbb{R} \longrightarrow [0,1]$ *is defined by*

$$N_{\varphi,\varepsilon}(a,t) = \begin{cases} 0, & t \leq \|a\|(\varepsilon + \|\varphi\|) \\ \frac{t-|\varphi(a)|}{t+|\varphi(a)|}, & t > \|a\|(\varepsilon + \|\varphi\|), \end{cases}$$

then

(1) *the map*

$$\begin{aligned} N_{\varphi,\varepsilon}(\cdot,t) : A &\longrightarrow [0,1] \\ a &\longrightarrow N_{\varphi,\varepsilon}(a,t), \end{aligned}$$

is continuous for all $t \leq 0$.

(2) *if* $t > 0$*, then* $N_{\varphi,\varepsilon}(\cdot,t)$ *is continuous at every* $a \in A \setminus S$*, where* $S = \{a \in A \mid \|a\| = \frac{t}{\varepsilon+\|\varphi\|}\}$.

(3) *the map* $N_{\varphi,\varepsilon}(a,\cdot)$ *is continuous at every* $t \in \mathbb{R} \setminus T$*, where* $T = \{t \in \mathbb{R} \mid t = \|a\|(\varepsilon+\|\varphi\|)\}$.

Proof. (1) If $t \leq 0$, then $N_{\varphi,\varepsilon}(a,t) = 0$ for all $a \in A$. So $N_{\varphi,\varepsilon}(\cdot,t)$ is a constant function on A that is continuous.

(2) If $t > 0$ and $a \in S$, then $\|a\| = \frac{t}{\varepsilon+\|\varphi\|}$ and $N_{\varphi,\varepsilon}(a,t) = 0$. Set $a_n = \frac{t-\frac{t}{2n}}{\|a\|(\varepsilon+\|\varphi\|)}a$ for all $n \in \mathbb{N}$. So $a_n \longrightarrow a$, as $n \longrightarrow \infty$, and
$\|a_n\| = \frac{t-\frac{t}{2n}}{\|a\|(\varepsilon+\|\varphi\|)}\|a\| < \frac{t}{(\varepsilon+\|\varphi\|)}$ for all $n \in \mathbb{N}$. This shows that
$t > \|a_n\|(\varepsilon+\|\varphi\|)$ for all $n \in \mathbb{N}$.
Hence,

$$\begin{aligned} \lim_{n\to\infty} N_{\varphi,\varepsilon}(a_n,t) &= \lim_{n\to\infty} \frac{t-|\varphi(a_n)|}{t+|\varphi(a_n)|} \\ &= \frac{t-|\varphi(a)|}{t+|\varphi(a)|} \\ &\neq N_{\varphi,\varepsilon}(a,t) = 0, \end{aligned}$$

since, $t = \|a\|(\varepsilon+\|\varphi\|) > \|a\|\|\varphi\| \geq |\varphi(a)|$. Therefore, $N_{\varphi,\varepsilon}(\cdot,t)$ is discontinuous at every $a \in S$.
Let $a \notin S$ and let $a_n \longrightarrow a$, as $n \longrightarrow \infty$. So $t > \|a\|(\varepsilon+\|\varphi\|)$ or $t < \|a\|(\varepsilon+\|\varphi\|)$. If $t > \|a\|(\varepsilon+\|\varphi\|)$, then there exists an $n_0 \in \mathbb{N}$ such that $t > \|a_n\|(\varepsilon+\|\varphi\|)$ for all $n \geq n_0$. Hence,

$$\begin{aligned}\lim_{n\to\infty} N_{\varphi,\varepsilon}(a_n,t) &= \lim_{n\to\infty}\frac{t-|\varphi(a_n)|}{t+|\varphi(a_n)|}\\ &= \frac{t-|\varphi(a)|}{t+|\varphi(a)|}\\ &= N_{\varphi,\varepsilon}(a,t).\end{aligned}$$

If $t <\|a\|(\varepsilon+\|\varphi\|)$, then there exists an $n_0 \in \mathbb{N}$ such that $t <\|a_n\|(\varepsilon+\|\varphi\|)$ for all $n \geq n_0$. So

$$\lim_{n\to\infty} N_{\varphi,\varepsilon}(a_n,t) = \lim_{n\to\infty} 0 = 0 = N_{\varphi,\varepsilon}(a,t).$$

Consequently for the case $t > 0$, $N_{\varphi,\varepsilon}(\cdot,t)$ is continuous at every point $a \in A \setminus S$.

(3) Let $t \in T$. So $t =\|a\|(\varepsilon+\|\varphi\|)$ and $N_{\varphi,\varepsilon}(a,t) = 0$. Set $t_n = (\|a\| + \frac{1}{n})(\varepsilon+\|\varphi\|)$ for all $n \in \mathbb{N}$. Clearly $t_n \longrightarrow t$, as $n \longrightarrow \infty$, and $t_n >\|a\|(\varepsilon+\|\varphi\|)$ for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}\lim_{n\to\infty} N_{\varphi,\varepsilon}(a,t_n) &= \lim_{n\to\infty}\frac{t_n-|\varphi(a)|}{t_n+|\varphi(a)|}\\ &= \begin{cases}1, & a=0\\ \frac{t-|\varphi(a)|}{t+|\varphi(a)|}, & a\neq 0.\end{cases}\end{aligned}$$

It follows that $\lim_{n\to\infty} N_{\varphi,\varepsilon}(a,t_n) \neq N_{\varphi,\varepsilon}(a,t) = 0$. Note that if $t = \|a\|(\varepsilon+\|\varphi\|)$ and $a \neq 0$, then $t - |\varphi(a)| \neq 0$. Since,

$$t =\|a\|(\varepsilon+\|\varphi\|) >\|a\|\|\varphi\| \geq |\varphi(a)|.$$

We shall show that $N_{\varphi,\varepsilon}(a,\cdot)$ is continuous at every $t \in \mathbb{R} \setminus T$.
Let $t \in \mathbb{R} \setminus T$ and let $t_n \longrightarrow t$, as $n \longrightarrow \infty$. Then $t <\|a\|(\varepsilon+\|\varphi\|)$ or $t >\|a\|(\varepsilon+\|\varphi\|)$. If $t <\|a\|(\varepsilon+\|\varphi\|)$, then there exists an $n_0 \in \mathbb{N}$ such that $t_n <\|a\|(\varepsilon+\|\varphi\|)$ for all $n \geq n_0$. Hence,

$$\lim_{n\to\infty} N_{\varphi,\varepsilon}(a,t_n) = \lim_{n\to\infty} 0 = 0 = N_{\varphi,\varepsilon}(a,t).$$

If $t >\|a\|(\varepsilon+\|\varphi\|)$, then there exists an $n_0 \in \mathbb{N}$ such that $t_n >\|a\|(\varepsilon+\|\varphi\|)$ for all $n \geq n_0$. So

$$\begin{aligned}\lim_{n\to\infty} N_{\varphi,\varepsilon}(a,t_n) &= \lim_{n\to\infty}\frac{t_n-|\varphi(a)|}{t_n+|\varphi(a)|}\\ &= \frac{t-|\varphi(a)|}{t+|\varphi(a)|}\\ &= N_{\varphi,\varepsilon}(a,t).\end{aligned}$$

This shows that $N_{\varphi,\varepsilon}(a,\cdot)$ is continuous at every $t \in \mathbb{R} \setminus T$.

□

Theorem 3.3. *Let A be a normed algebra and $\psi : A \longrightarrow \mathbb{F}$ be a continuous homomorphism. If*
$N_\psi^{(1)} : A \times \mathbb{R} \longrightarrow [0,1]$*and* $N_\psi^{(2)} : A \times \mathbb{R} \longrightarrow [0,1]$ *are defined by*

$$N_\psi^{(1)}(a,t) = \begin{cases} 0, & t \leq \|a\| + |\psi(a)| \\ \frac{t}{t+\|a\|+|\psi(a)|}, & t > \|a\| + |\psi(a)|, \end{cases}$$

$$N_\psi^{(2)}(a,t) = \begin{cases} 0, & t \leq |\psi(a)| \\ \frac{t}{t+\|a\|+|\psi(a)|}, & t > |\psi(a)|, \end{cases}$$

and for the case $\ker\psi = \{0\}$, $N_\psi^{(3)} : A \times \mathbb{R} \longrightarrow [0,1]$ *is defined by*

$$N_\psi^{(3)}(a,t) = \begin{cases} 0, & t \leq |\psi(a)| \\ \frac{t}{t+|\psi(a)|}, & t > |\psi(a)|, \end{cases}$$

then

(1) *the maps $N_\psi^{(1)}(\cdot,t)$, $N_\psi^{(2)}(\cdot,t)$, and $N_\psi^{(3)}(\cdot,t)$ are continuous on A for all $t \leq 0$,*
(2) *if $t > 0$, then the map $N_\psi^{(1)}(\cdot,t)$ is continuous at every $a \in A \setminus S_1$, where $S_1 = \{a \in A \mid t = \|a\| + |\psi(a)|\}$,*
(3) *if $t > 0$, then the maps $N_\psi^{(2)}(\cdot,t)$ and $N_\psi^{(3)}(\cdot,t)$ are continuous at every $a \in A \setminus S_2$, where $S_2 = \{a \in A \mid t = |\psi(a)|\}$,*
(4) *for $a \in A$, the map $N_\psi^{(1)}(a,\cdot)$ is continuous at every $t \in \mathbb{R} \setminus T_1$, where $T_1 = \{t \in \mathbb{R} \mid t = \|a\| + |\psi(a)|\}$,*
(5) *for $a \neq 0$, the map $N_\psi^{(2)}(a,\cdot)$ is continuous at every $t \in \mathbb{R} \setminus T_2$, where $T_2 = \{t > 0 \mid t = |\psi(a)|\}$, also the map $N_\psi^{(2)}(0,\cdot)$ is continuous at every $t \in \mathbb{R}$ except $t = 0$,*
(6) *for $a \in A$, the map $N_\psi^{(3)}(a,\cdot)$ is continuous at every $t \in \mathbb{R} \setminus T_3$, where $T_3 = \{t \in \mathbb{R} \mid t = |\psi(a)|\}$.*

Proof. (1) It is obvious.

(2) Let $t > 0$ and $a \in S_1$. So $t = \|a\| + |\psi(a)|$. Set $a_n = \frac{t-\frac{t}{2n}}{\|a\|+|\psi(a)|}a$ for all $n \in \mathbb{N}$. Obviously $a_n \longrightarrow a$ and so $\psi(a_n) \longrightarrow \psi(a)$, as $n \longrightarrow \infty$. Also

$$\begin{aligned}\|a_n\| &= \frac{t-\frac{t}{2n}}{\|a\|+|\psi(a)|}\|a\| \\ &< \frac{t}{\|a\|+|\psi(a)|}\|a\| \\ &= \|a\|, \quad n \in \mathbb{N},\end{aligned}$$

and

$$\begin{aligned}|\psi(a_n)| &= \frac{t-\frac{t}{2n}}{\|a\|+|\psi(a)|}|\psi(a)| \\ &\leq \frac{t}{\|a\|+|\psi(a)|}|\psi(a)| \\ &= |\psi(a)|, \quad n \in \mathbb{N}.\end{aligned}$$

So $\|a_n\| + |\psi(a_n)| <\|a\| + |\psi(a)| = t$ for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}\lim_{n\to\infty} N_\psi^{(1)}(a_n,t) &= \lim_{n\to\infty} \frac{t}{t+\|a_n\|+|\psi(a_n)|} \\ &= \frac{t}{t+t} \\ &= \frac{1}{2} \\ &\neq N_\psi^{(1)}(a,t) \\ &= 0.\end{aligned}$$

This shows that $N_\psi^{(1)}(\cdot,t)$ is discontinuous at every $a \in S_1$. Now let $a \in A \setminus S_1$ and $\{z_n\}_{n=1}^{\infty}$ be a sequence such that $z_n \longrightarrow a$, as $n \longrightarrow \infty$. So $t >\|a\| + |\psi(a)|$ or $t <\|a\| + |\psi(a)|$. If $t >\|a\| + |\psi(a)|$, then there exists an $n_0 \in \mathbb{N}$ such that $t >\|z_n\| + |\psi(z_n)|$ for all $n \geq n_0$. Hence,

$$\begin{aligned}\lim_{n\to\infty} N_\psi^{(1)}(z_n,t) &= \lim_{n\to\infty} \frac{t}{t+\|z_n\|+|\psi(z_n)|} \\ &= \frac{t}{t+\|a\|+|\psi(a)|} \\ &= N_\psi^{(1)}(a,t).\end{aligned}$$

If $t <\|a\| + |\psi(a)|$, then there exists an $n_0 \in \mathbb{N}$ such that $t <\|z_n\| + |\psi(z_n)|$ for all $n \geq n_0$. So

$$\lim_{n\to\infty} N_\psi^{(1)}(z_n,t) = \lim_{n\to\infty} 0 = 0 = N_\psi^{(1)}(a,t).$$

This shows that $N_\psi^{(1)}(\cdot,t)$ is continuous at every $a \in A \setminus S_1$.

(3) Let $a \in S_2$. So $t = |\psi(a)|$, $a \neq 0$ and $N_\psi^{(2)}(a,t) = N_\psi^{(3)}(a,t) = 0$. Set $a_n = (1-\frac{1}{2n})a$ for all $n \in \mathbb{N}$. Clearly $a_n \longrightarrow a$, $\psi(a_n) \longrightarrow \psi(a)$, as $n \longrightarrow \infty$. Also $|\psi(a_n)| = (1-\frac{1}{2n})|\psi(a)| < |\psi(a)| = t$ for all $n \in \mathbb{N}$.

Hence,

$$\begin{aligned}\lim_{n\to\infty} N_\psi^{(2)}(a_n,t) &= \lim_{n\to\infty}\frac{t}{t+\|a_n\|+|\psi(a_n)|}\\ &= \frac{t}{t+\|a\|+t}\\ &= \frac{t}{2t+\|a\|}\\ &\neq 0\\ &= N_\psi^{(2)}(a,t).\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} N_\psi^{(3)}(a_n,t) &= \lim_{n\to\infty}\frac{t}{t+|\psi(a_n)|}\\ &= \frac{t}{t+t}\\ &= \frac{1}{2}\\ &\neq 0\\ &= N_\psi^{(3)}(a,t).\end{aligned}$$

Hence, $N_\psi^{(2)}(\cdot,t)$ and $N_\psi^{(3)}(\cdot,t)$ are discontinuous at every $a\in S_2$.
Now let $a\notin S_2$ and $z_n\longrightarrow a$, as $n\longrightarrow\infty$. Then $t<|\psi(a)|$ or $t>|\psi(a)|$. If $t<|\psi(a)|$, then there exists an $n_0\in\mathbb{N}$ such that $t<|\psi(z_n)|$ for all $n\geq n_0$. Hence,

$$\lim_{n\to\infty} N_\psi^{(2)}(z_n,t)=0=N_\psi^{(2)}(a,t),$$

and

$$\lim_{n\to\infty} N_\psi^{(3)}(z_n,t)=0=N_\psi^{(3)}(a,t).$$

If $t>|\psi(a)|$, then there exists an $n_0\in\mathbb{N}$ such that
$t>|\psi(z_n)|$ for all $n\geq n_0$. So

$$\begin{aligned}\lim_{n\to\infty} N_\psi^{(2)}(z_n,t) &= \lim_{n\to\infty}\frac{t}{t+\|z_n\|+|\psi(z_n)|}\\ &= \frac{t}{t+\|a\|+|\psi(a)|}\\ &= N_\psi^{(2)}(a,t),\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} N_\psi^{(3)}(z_n,t) &= \lim_{n\to\infty} \frac{t}{t+|\psi(z_n)|}\\ &= \frac{t}{t+|\psi(a)|}\\ &= N_\psi^{(3)}(a,t).\end{aligned}$$

Hence, $N_\psi^{(2)}(\cdot,t)$ and $N_\psi^{(3)}(\cdot,t)$ are continuous at every $a \in A \setminus S_2$.

(4) Let $t \in T_1$. Then $t = \|a\| + |\psi(a)|$ and $N_\psi^{(1)}(a,t) = 0$. Set $t_n = \|a\| + |\psi(a)| + \frac{1}{n}$ for all $n \in \mathbb{N}$. Clearly $t_n > \|a\| + |\psi(a)|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} t_n = t$. So

$$\begin{aligned}\lim_{n\to\infty} N_\psi^{(1)}(a,t_n) &= \lim_{n\to\infty} \frac{t_n}{t_n + \|a\| + |\psi(a)|}\\ &= \begin{cases} 1, & a = 0\\ \frac{t}{t+t} = \frac{1}{2}, & a \neq 0.\end{cases}\end{aligned}$$

It follows that $\lim_{n\to\infty} N_\psi^{(1)}(a,t_n) \neq N_\psi^{(1)}(a,t) = 0$. This shows that $N_\psi^{(1)}(a,\cdot)$ is discontinuous at every $t \in T_1$. One can easily verify that if $t \in \mathbb{R} \setminus T_1$, then $N_\psi^{(1)}(a,\cdot)$ is continuous at t.

(5) Let $a \neq 0$. If $t \in T_2$, then $t = |\psi(a)| > 0$ and $N_\psi^{(2)}(a,t) = 0$. Set $t_n = (1+\frac{1}{n})|\psi(a)|$ for all $n \in \mathbb{N}$. Clearly $t_n \longrightarrow t$, as $n \longrightarrow \infty$, and $t_n > |\psi(a)|$ for all $n \in \mathbb{N}$. So

$$\begin{aligned}\lim_{n\to\infty} N_\psi^{(2)}(a,t_n) &= \lim_{n\to\infty} \frac{t_n}{t_n + \|a\| + |\psi(a)|}\\ &= \frac{t}{2t + \|a\|}\\ &\neq 0\\ &= N_\psi^{(2)}(a,t).\end{aligned}$$

This shows that $N_\psi^{(2)}(a,\cdot)$ is discontinuous at every $t \in T_2$.
Let $t \in \mathbb{R}\setminus T_2$. Then $t \leq 0$ or $t \neq |\psi(a)|$. for the case $t < 0$ or $t \neq |\psi(a)|$, the continuity of $N_\psi^{(2)}(a,\cdot)$ at t, can be obviously verified. If $t = 0$ and $t_n \longrightarrow 0$, as $n \longrightarrow \infty$, then for all subsequences $\{t_{n_k}\}_{k=1}^\infty \subseteq \{t_n\}_{n=1}^\infty$ satisfying $t_{n_k} \leq |\psi(a)|$ we have, $\lim_{k\longrightarrow\infty} N_\psi^{(2)}(a,t_{n_k}) = 0 = N_\psi^{(2)}(a,0)$. Also for all subsequences $\{t_{n_k}\}_{k=1}^\infty \subseteq \{t_n\}_{n=1}^\infty$ satisfying $t_{n_k} > |\psi(a)|$ we have,

$$\lim_{k\to\infty} N_\psi^{(2)}(a,t_{n_k}) = \lim_{k\to\infty} \frac{t_{n_k}}{t_{n_k + \|a\| + |\psi(a)|}} = \frac{0}{0 + \|a\| + |\psi(a)|} = 0 = N_\psi^{(2)}(a,0).$$

So $N_\psi^{(2)}(a,\cdot)$ is continuous at $t=0$.

Clearly the map $N_\psi^{(2)}(0,\cdot)$ is continuous at every $t\in\mathbb{R}$ except $t=0$.

(6) Inspired by part (5), the proof is obvious.

□

Theorem 3.4. *Let A be a normed algebra and $\eta : A \longrightarrow \mathbb{F}$ be a continuous homomorphism. If*
$N_\eta^{(1)} : A\times\mathbb{R}\longrightarrow[0,1]$ *is defined by*

$$N_\eta^{(1)}(a,t)=\begin{cases}0, & t\leq\|a\|+|\eta(a)|\\ \frac{t-\|a\|-|\eta(a)|}{t}, & t>\|a\|+|\eta(a)|,\end{cases}$$

and for the case $\ker\eta=\{0\}$, $N_\eta^{(2)} : A\times\mathbb{R}\longrightarrow[0,1]$ *is defined by*

$$N_\eta^{(2)}(a,t)=\begin{cases}0, & t\leq|\eta(a)|\\ \frac{t-|\eta(a)|}{t}, & t>|\eta(a)|,\end{cases}$$

then

(1) *the map $N_\eta^{(1)}(\cdot,t)$ is continuous for all $t\in\mathbb{R}$,*
(2) *the map $N_\eta^{(1)}(a,\cdot)$ is continuous for all $a\in A$ except $a=0$,*
(3) *the map $N_\eta^{(2)}(\cdot,t)$ is continuous for all $t\in\mathbb{R}$,*
(4) *the map $N_\eta^{(2)}(a,\cdot)$ is continuous for all $a\in A$ except $a=0$.*

Proof. The theorem can be established following the technique applied in the previous theorem. □

4. Conclusion

Generating algebra fuzzy norms by the described method introduces a large class of algebra fuzzy norms which is very important. Such algebra fuzzy norms can be used as a very useful source of examples and counterexamples in the field of fuzzy normed algebras. It is obvious that these algebra fuzzy norms are different in each category and in certain conditions these algebra fuzzy norms will be equivalent. Investigation of their topological features and other properties will be investigated in future research.

References

[1] Bag, T., & Samanta, S. K. (2003). Finite dimensional fuzzy normed linear spaces. J. Fuzzy Math., 11, (3), 687-705.
[2] Bag, T., & Samanta, S. K. (2013). Finite dimensional fuzzy normed linear spaces. Ann. Fuzzy Math. Inform., 6, (2), 271-283.
[3] Bag, T., & Samanta, S. K. (2005). Fuzzy bounded linear operators. Fuzzy Sets and Syst., 151, 513-547. https://doi.org/10.1016/j.fss.2004.05.004.
[4] Bag, T., & Samanta, S. K. (2006). Fixed point theorems on fuzzy normed linear spaces. Inf. Sci., 176, 2910-2931. https://doi.org/10.1016/j.ins.2005.07.013.
[5] Bag, T., & Samanta, S. K. (2007). Some fixed point theorems on fuzzy normed linear spaces. Inf. Sci., 177, 3271-3289. https://doi.org/10.1016/j.ins.2007.01.027.

[6] Cheng, S. C., & Mordeson, J. N. (1994). Fuzzy linear operators and fuzzy normed linear spaces. Bull. Cal. Math. Soc., 86, 429-436.

[7] Felbin, C. (1992). Finite dimensional fuzzy normed linear spaces. Fuzzy Sets and Syst., 48, 239-248. https://doi.org/10.1016/0165-0114(92)90338-5.

[8] Golet, Ioan. (2010). On generalized fuzzy normed spaces and coincidence point theorems. Fuzzy Sets and Syst., 161, 1138-1144. https://doi.org/10.1016/j.fss.2009.10.004.

[9] Hasankhani, A., Nazari, A., & Saheli, M. (2010). Some properties of fuzzy Hilbert spaces and norm of operators. Iran. J. Fuzzy Syst., 7, (3), 129-157. https://doi.org/10.22111/ijfs.2010.196.

[10] Kaleva, O., & Seikkala, S. (1984). On fuzzy metric spaces. Fuzzy Sets and Syst., 12, 215-229. https://doi.org/10.1016/0165-0114(84)90069-1.

[11] Katsaras, A. K. (1984). Fuzzy topological vector spaces. Fuzzy Sets and Syst., 12, 143-154. https://doi.org/10.1016/0165-0114(84)90034-4.

[12] Kramosil, A. K., & Michalek, J. (1975). Fuzzy metric and statistical metric spaces. Kybernetika, 11, 326-334.

[13] Mirmostafaee, A. K. (2012). Perturbation of generalized derivations in fuzzy Menger normed algebras. Fuzzy Sets and Syst., 195, 109-117. https://doi.org/10.1016/j.fss.2011.10.015.

[14] Tripathy, B.C., & Debnath, S. (2013). On generalized difference sequence spaces of fuzzy numbers. Acta Sci. Technol., 35, (1), 117-121. https://doi.org/10.4025/actascitechnol.v35i1.15566.

[15] Tripathy, B.C., & Sen, M. (2013). On fuzzy I-convergent difference sequence spaces. J. Intell. Fuzzy Syst., 25, (3), 643-647. https://doi.org/10.3233/IFS-120671.

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