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MULTIPLIERS IN WEAK HEYTING ALGEBRAS

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Abstract. In this paper, we introduce the notion of multipliers in weak Heyting algebras and investigate some related properties of them. We obtain the relations between multipliers, closure operators, and homomorphisms in weak Heyting algebras. Relations among image sets and fixed point sets of multipliers in weak Heyting algebras are investigated. Also, we study algebraic structures of the set of all multipliers in weak Heyting algebras. Using multipliers, the left and right m-stabilizers in weak Heyting algebras are introduced, and some related properties are given. Also, we obtain conditions such that the left and right m-stabilizers form two weak Heyting algebras.

Keywords: Weak Heyting algebra, Multiplier, m-stabilizer, Fix point, Closure operator.
2020 MSC: 06D20, 06A15, 03G25.

1. Introduction

In 2005, Celani and Jansana introduced the notion of weak Heyting algebras, or WH -algebras for short, in [5] under the name of weakly Heyting algebras. As they mention, these algebras correspond to the strict implication fragment of the normal modal logic K which is also known as the subintuitionistic local consequence of the class of all Kripke models.

A WH-algebra is a bounded distributive lattice with a binary operation $\rightarrow$ with the properties of the strict implication in the modal logic K. Also, a WH-algebra is a generalization of Heyting algebras. Some examples of WH-algebras appearing in this paper are the Basic algebras introduced by M. Ardeshir and W. Ruitenburg in [1], and self distributive weak Heyting algebras in [13]. The variety of WH-algebras is arithmetical, has equationally definable principal congruences, has the amalgamation property, the congruence extension property and is finitely approximable. Also see [2] and [14].

In 1974, Cornish introduced the concept of multiplier for distributive lattices in [8]. In 1980, Schmid used multipliers in order to give a nonstandard construction of the maximal lattice of quotients for a distributive lattice in [15].

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The notion of multipliers has been extended to commutative semigroup, BE-algebras [11], d-algebras [6] and BL-algebras [16].
The notion of stabilizers is introduced from fixed point set theory. Since stabilizer was successful in several distinct tasks in various branches of mathematics it has been extended to various logical algebras, for example see [3], [17] and [18].
This paper is organized as follows: In Sect. 2, we recall some basic concepts and properties of WH-algebras. In Sect. 3, we introduce the notion of multiplier in WH-algebras and obtain some related results. The relations between multiplier, closure operator and lattice homomorphism are obtained. We study the fixed points of a multiplier and set of all multipliers in a WH-algebra. In Sect. 4, using multipliers, we introduce the notion of left and right m-stabilizers in WH-algebras. In particular, we obtain conditions that left and right m-stabilizers have the same structure as WH-algebras.

2. Preliminaries

In this section, we recall the basic definitions and some properties of WH-algebras that we need in the rest of the paper.

Definition 2.1. ([5]) An algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is called a weak Heyting algebra (or WH-algebra) if $(H, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the following conditions hold for all $x, y, z \in H$:
(WH1) $(x \rightarrow y) \wedge (x \rightarrow z) = x \rightarrow (y \wedge z)$,
(WH2) $(x \rightarrow z) \wedge (y \rightarrow z) = (x \vee y) \rightarrow z$,
(WH3) $(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z$,
(WH4) $x \rightarrow x = 1$.

Remark that every bounded distributive lattice can be seen as a WH-algebra if we define $x \rightarrow y = 1$ for every x, y.
In the rest of this paper, we denote an algebra $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ by $\mathcal{H}$. Also, a homomorphism between two WH -algebras $\mathcal{H}_1$ and $\mathcal{H}_2$ is a bounded lattice homomorphism $h : H_1 \rightarrow H_2$ such that $h(x \rightarrow y) = h(x) \rightarrow h(y)$, for all $x, y \in H_1$.

Proposition 2.2. *([5]) Let $\mathcal{H}$ be a WH-algebra. Then the following hold for all $x, y, z \in H$:*
(W1) if $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,
(W2) if $x \leq y$, then $x \rightarrow y = 1$.

Remark that in a WH-algebra $\mathcal{H}$, we have $x \rightarrow 1 = 1$ for all $x \in H$ by (W2).

Definition 2.3. ([13]) Let $\mathcal{H}$ be a WH-algebra.
(1) $\mathcal{H}$ is a Basic algebra iff satisfies the inequality $x \leq 1 \rightarrow x$,
(2) $\mathcal{H}$ is an RWH-algebra iff satisfies the inequality $x \wedge (x \rightarrow y) \leq y$ (R),
(3) $\mathcal{H}$ is an SDWH-algebra iff $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$ (SD).

A non-empty subset I of a WH-algebra $\mathcal{H}$ is called an ideal if $a \vee b \in I$ and $a \wedge x \in I$ whenever $a, b \in I$ and $x \in H$. A proper ideal I is called prime if $a \wedge b \in I$ implies that $a \in I$ or $b \in I$.

Definition 2.4. ([13]) A subset F of a WH algebra $\mathcal{H}$ is called a filter, if it satisfies the following conditions, for all $x, y \in H$
(F1) If $x, y \in F$, then $x \wedge y \in F$,
(F2) If $x \in F$ and $x \leq y$, then $y \in F$.
A filter F of $\mathcal{H}$ is called open, if it satisfies the following condition,
(OF) If $x \in F$, then $\Box x \in F$ where $\Box x := 1 \to x$.

A proper filter F on $\mathcal{H}$ is called prime if $a \vee b \in F$ implies that $a \in F$ or $b \in F$. The filter (ideal) generated by a set $X \subseteq H$ will be denoted by $[X)$ $((X])$. We will write $[x)$ $((x])$ to refer to the filter (ideal) generated by $\{x\}$.

Definition 2.5. ([13]) A subset D of an SDWH algebra $\mathcal{H}$ is called a deductive system if it satisfies the following conditions, for all $x, y \in H$:
(D1) $1 \in D$,
(D2) $x, x \to y \in D$ imply $y \in D$.

3. Multipliers in WH-algebras

Definition 3.1. Let $\mathcal{H}$ be a WH-algebra. A self map $m : H \to H$ is called a multiplier in H, if it satisfies the following conditions for all $x, y \in H$:
(M1) $m(x \to y) = x \to m(y)$,
(M2) $m(x \vee y) = x \vee m(y)$.

The set of all multipliers in $\mathcal{H}$ is denoted by $M(\mathcal{H})$.

Example 3.2. *(1) The identity mapping id_H and the unit mapping $1_H : H \to H$ defined by $1_H(x) = 1$ for all $x \in H$ are multipliers.*
(2) Let $H = \{0, a, 1\}$ such that $0 < a < 1$. Consider the following binary operation $\to$ on H

$\to$	0	a	1
0	1	1	1
a	1	1	1
1	a	a	1

Then $\mathcal{H} = (H, \wedge, \vee, \to, 0, 1)$ is an SDWH-algebra ([13]). Consider the maps m_i, $1 \leq i \leq 3$, given the table below:

x	0	a	1
$m_1(x)$	1	1	1
$m_2(x)$	0	a	1
$m_3(x)$	a	a	1

Then $M(\mathcal{H}) = \{m_1, m_2, m_3\}$.

Proposition 3.3. *Let m be a multiplier in a WH-algebra $\mathcal{H}$ and $x, y \in H$. Then*

(1) $m(1) = 1$,
(2) $x \leq m(x)$,
(3) $m(m(x)) = m(x)$,
(4) $m(x \vee y) = m(x) \vee m(y)$,
(5) $x \leq y$ *implies* $m(x) \leq m(y)$,
(6) $m(x \wedge y) = m(x) \wedge m(y)$,
(7) $m(\Box x) = \Box m(x)$,
(8) $m(x) \to m(y) \leq x \to m(y) = m(x \to y)$,
(9) if $m(x) = 0$, *then* $x = 0$,
(10) $m(0) = 0$ *iff* $m = id_H$,
(11) $Ker(m) = \{x \in H | m(x) = 1\}$ *is an open filter.*

Proof. (1) We have $m(1) = m(m(1) \to 1) = m(1) \to m(1) = 1$ by (W2).
(2) Using (M1), we obtain $m(x) = m(x \vee x) = x \vee m(x)$. So $x \leq m(x)$.
(3) By (M2), $m(m(x)) = m(m(x \vee x)) = m(x \vee m(x)) = m(x) \vee m(x) = m(x)$.
(4) Using part (3) and (M2), we obtain

$$m(x \vee y) = m(m(x \vee y)) = m(x \vee m(y)) = m(x) \vee m(y).$$

(5) Suppose that $x \leq y$. Then $y = x \vee y$. By (M2) and part (4), we have $m(y) = m(x \vee y) = m(x) \vee m(y)$.
(6) Using distributive property, we get $m(x \wedge y) \vee 0 = m((x \wedge y) \vee 0) = m(0) \vee (x \wedge y) = (m(0) \vee x) \wedge (m(0) \vee y) = m(0 \vee x) \wedge m(0 \vee y) = m(x) \wedge m(y)$.
(7) We have $m(\Box x) = m(1 \to x) = 1 \to m(x) = \Box m(x)$.
(8) Applying part (2) and then (W1), we obtain
$m(x) \to m(y) \leq x \to m(y) = m(x \to y)$.
(9) It follows from part (2).
(10) By (M2), $m(x) = m(x \vee 0) = x \vee m(0) = x$ for all $x \in H$. Hence m is the identity map. The converse is trivial.
(11) It follows from parts (5), (6) and (7). $\Box$

Corollary 3.4. *Let m be a multiplier in a chain WH-algebra $\mathcal{H}$ and $1 \neq a \in H$ such that $m(a) = 1$. Then m is the unite mapping.*

Proof. By (M2), we have $1 = m(a) = m(a \vee 0) = m(0) \vee a$. By assumption $\mathcal{H}$ is a chain and $a \neq 1$, so $m(0) = 1$. By Proposition 3.3 part (5), we obtain $m(x) = 1$, for all $x \in H$. $\Box$

Let P be a poset. Recall that a function $C : P \to P$ is called a closure operator if (i) $x \leq C(x)$,(ii) $x \leq y$ implies $C(x) \leq C(y)$ and (iii) $C(C(x)) = C(x)$, hold for all $x, y \in P$.

Corollary 3.5. *Let m be a multiplier in a WH-algebra $\mathcal{H}$. Then m is a lattice homomorphism and a closure operator.*

Proof. It follows from Proposition 3.3 parts (2), (3) and (5) that m is a closure operator. By Proposition 3.3 parts (4) and (6), we conclude that m is a lattice homomorphism. $\Box$

The converse of Corollary 3.5 may not be true in general. See the following example.

Example 3.6. *Let* $H = \{0, a, 1\}$ *such that* $0 < a < 1$. *Consider the following binary operation* $\rightarrow$*:*

$\rightarrow$	0	a	1
0	1	1	1
a	1	1	1
1	0	0	1

Then $\mathcal{H} = (H, \wedge, \vee, \rightarrow, 0, 1)$ *is a WH-algebra ([13]). Define* $C(0) = 0$ *and* $C(a) = C(1) = 1$. *It is easy to check that* C *is a lattice homomorphism and a closure operator. But it is not a multiplier, because* $C(1 \rightarrow a) = C(0) = 0 \neq 1 = 1 \rightarrow C(a)$.

Proposition 3.7. *Let* m *be a multiplier in an RWH-algebra* $\mathcal{H}$. *If* $\mathcal{H}$ *is an SDWH-algebra, then* $m(x \rightarrow y) = m(x) \rightarrow m(y)$, *for all* $x, y \in H$

Proof. By Proposition 3.3 parts (6), (5) and (R), we get $m(x) \wedge m(x \rightarrow y) = m(x \wedge (x \rightarrow y)) \leq m(y)$. Thus $m(x) \rightarrow (m(x) \wedge m(x \rightarrow y)) \leq m(x) \rightarrow m(y)$ by (W1). Applying (W1), (WH1) and (WH4), we get

$$\begin{aligned} 1 \rightarrow m(x \rightarrow y) &\leq m(x) \rightarrow m(x \rightarrow y) \\ &= m(x) \rightarrow (m(x) \wedge m(x \rightarrow y)) \\ &\leq m(x) \rightarrow m(y). \end{aligned}$$

By (SD), (WH4) and (W2), we obtain $m(x \rightarrow y) \rightarrow (1 \rightarrow m(x \rightarrow y)) = 1$. Using (R), we have

$$m(x \rightarrow y) \wedge (m(x \rightarrow y) \rightarrow (1 \rightarrow m(x \rightarrow y))) \leq 1 \rightarrow m(x \rightarrow y).$$

Hence $m(x \rightarrow y) \leq m(x) \rightarrow m(y)$. Also, we have $m(x) \rightarrow m(y) \leq m(x \rightarrow y)$ by Proposition 3.3 part (8). □

Proposition 3.8. *Let* m *be a self map on a WH-algebra* $\mathcal{H}$ *such that satisfies (M2) and* $m(x \rightarrow y) = m(x) \rightarrow m(y)$, *for all* $x, y \in H$. *Then* m *is a multiplier in* $\mathcal{H}$.

Proof. Suppose that $x, y \in H$ are arbitrary. Using (M2), we have $m(x) = m(x \vee x) = x \vee m(x)$. Hence $x \leq m(x)$. Also, similar to the proof of Proposition 3.3 parts (3) and (5), we can prove that $m(m(x)) = m(x)$ and $x \leq y$ implies $m(x) \leq m(y)$. Using (W1), we obtain $m(x) \rightarrow m(y) \leq x \rightarrow m(y)$. Hence $m(x \rightarrow y) \leq x \rightarrow m(y)$ by assumption. On the other hand, we have $x \rightarrow m(y) \leq m(x \rightarrow m(y)) = m(x) \rightarrow m(m(y)) = m(x) \rightarrow m(y) = m(x \rightarrow y)$. Thus m satisfies (M1) and hence it is a multiplier in $\mathcal{H}$. □

Proposition 3.9. *Let* m *be a multiplier in a WH-algebra* $\mathcal{H}$ *and* $Fix_m(H) = \{x \in H | m(x) = x\}$. *Then*
(1) $Fix_m(H) = Im(m)$,
(2) $Fix_m(H)$ *is closed under* $\wedge$, $\vee$ *and* $\rightarrow$.

(3) $Fix_m(H)$ *is an open filter,*
(4) If m *is onto, then* $m = id_H$.

Proof. (1) Suppose that $y \in Im(m)$. Then there exists $x \in H$ such that $m(x) = y$. Using Proposition 3.3 part (3) , we have $m(y) = m(m(x)) = m(x) = y$. So $y \in Fix_m(H)$. It is obvious that $Fix_m(H) \subseteq Im(m)$.
(2) Let $x, y \in Fix_m(H)$. Then $m(x \to y) = x \to m(y) = x \to y$. Therefore $x \to y \in Fix_m(H)$. By Proposition 3.3 parts (4) and (6), $Fix_m(H)$ is closed under $\wedge$ and $\vee$.
(3) By Proposition 3.3 part (1) we have $m(0), 1 \in Fix_m(H)$. Suppose that $x \leq y$ and $x \in Fix_m(H)$. Then $m(y) = m(y \vee x) = y \vee m(x) = y \vee x = y$. Hence $y \in Fix_m(H)$. By part (2) and Proposition 3.3 part (7), we get that $Fix_m(H)$ is an open filter.
(4) Suppose that $x \in H = Im(m)$. By part (1), we have $x \in Fix_m(H)$. Hence $m(x) = x$, that is $m = id_H$. □

Proposition 3.10. *Let* m_1 *and* m_2 *be two multipliers in a WH-algebra* $\mathcal{H}$*. Then the following hold:*
(1) $m_1 = m_2$ *if and only if* $Fix_{m_1}(H) = Fix_{m_2}(H)$.
(2) $m_1 \circ m_2 \in M(\mathcal{H})$.
(3) $m_1 \circ m_2 = m_2 \circ m_1$.
(4) $(M(\mathcal{H}), \circ, id_H)$ *is an ablian monoid.*

Proof. (1) Suppose that $Fix_{m_1}(H) = Fix_{m_2}(H)$ and let $x \in H$. Thus $m_2(m_1(x)) = m_1(x)$. Using Proposition 3.3 parts (2) and (5), we get $m_2(x) \leq m_2(m_1(x)) = m_1(x)$. Similarly, we can prove that $m_1(x) \leq m_2(x)$. Hence $m_1 = m_2$. The converse is trivial.
(2) The proof is straightforward.
(3) Let $x \in H$ be arbitrary. By Proposition 3.3 parts (2) and (5), we have $m_1(x) \leq m_1(m_2(x))$. Using Proposition 3.3 parts (5), (2) and then part (3), we obtain

$$m_2(m_1(x)) \leq m_2(m_1(m_2(x))) \leq m_1(m_2(m_1(m_2(x)))) = m_1(m_2(x))$$

Similarly, we can show that $m_1(m_2(x)) \leq m_2(m_1(x))$.
(4) It follows from parts (2) and (3). □

Proposition 3.11. *Let* $\mathcal{H}$ *be a WH-algebra and* $m_1, m_2 \in M(\mathcal{H})$*. Define* $m_1 \leq m_2$ *if and only if* $m_1(x) \leq m_2(x)$ *for all* $x \in H$*. Then* $m_1 \leq m_2$ *if and only if* $m_2 \circ m_1 = m_2$.

Proof. Suppose that $m_1 \leq m_2$ and $x \in H$. We have $x \leq m_1(x) = m_1(m_1(x)) \leq m_2(m_1(x))$. Using Proposition 3.3 part (5), we obtain $m_2(x) \leq m_2(m_1(x))$. On the other hand, $m_2(m_1(x)) \leq m_2(m_2(x)) = m_2(x)$. Therefore $m_2 \circ m_1 = m_2$. Conversely, let $m_2 \circ m_1 = m_2$. Then $m_1(x) \leq m_2(m_1(x)) = m_2(x)$. Hence $m_1 \leq m_2$. □

Proposition 3.12. *Let $\mathcal{H}$ be a WH-algebra. Then $(M(\mathcal{H}), \sqcap, \sqcup, id_H, 1_H)$ is a bounded distributive lattice, where $(m_1 \sqcap m_2)(x) = m_1(x) \wedge m_2(x)$ and $(m_1 \sqcup m_2)(x) = m_1(m_2(x))$.*

Proof. Let $m_1, m_2 \in M(\mathcal{H})$ be arbitrary. By (WH1) and distributive property,

$$\begin{aligned}(m_1 \sqcap m_2)(x \to y) =& m_1(x \to y) \wedge m_2(x \to y) = (x \to m_1(y)) \wedge (x \to m_2(y))\\ =& x \to (m_1(y) \wedge m_2(y)) = x \to (m_1 \sqcap m_2)(y),\\ (m_1 \sqcap m_2)(x \vee y) =& m_1(x \vee y) \wedge m_2(x \vee y) = (x \vee m_1(y)) \wedge (x \vee m_2(y))\\ =& x \vee (m_1(y) \wedge m_2(y)) = x \vee (m_1 \sqcap m_2)(y).\end{aligned}$$

Hence $m_1 \sqcap m_2 \in M(\mathcal{H})$. Also, we have

$$\begin{aligned}m_1((m_1 \sqcap m_2)(x)) &= m_1((m_1(x) \wedge m_2(x)) = m_1(m_1(x)) \wedge m_1(m_2(x))\\ &= m_1(x) \wedge m_1(m_2(x)) = m_1(x).\end{aligned}$$

Thus $m_1 \sqcap m_2 \leq m_1$ by Proposition 3.11. Similarly, we can show that $m_1 \sqcap m_2 \leq m_2$. So $m_1 \sqcap m_2$ is a lower bound of m_1 and m_2. Now, suppose that $m \in M(\{H\})$ is such that $m \leq m_1$ and $m \leq m_2$. Thus $m_1 \circ m = m_1$ and $m_2 \circ m = m_2$. Then

$$\begin{aligned}((m_1 \sqcap m_2) \circ m)(x) &= (m_1 \sqcap m_2)(m(x)) = m_1(m(x)) \wedge m_2(m(x))\\ &= m_1(x) \wedge m_2(x) = (m_1 \sqcap m_2)(x)\end{aligned}$$

Hence $m \leq m_1 \sqcap m_2$ and so $m_1 \sqcap m_2$ is the g.l.b. of $\{m_1, m_2\}$.
By Proposition 3.10, we have $m_1 \sqcup m_2 \in M(\mathcal{H})$. Using Proposition 3.11, we can show that $m_1 \sqcup m_2$ is the u.l.b. of $\{m_1, m_2\}$.
It is easily obtain that $id_H \leq m \leq 1_H$ for all $m \in M(\mathcal{H})$. Hence id_H is the the smallest element and 1_H is the greatest element of $M(\mathcal{H})$. Since

$$\begin{aligned}(m_3 \sqcup (m_1 \sqcap m_2))(x) &= m_3(m_1(x) \wedge m_2(x)) = m_3(m_1(x)) \wedge m_3(m_2(x))\\ &= (m_3 \sqcup m_1)(x) \wedge (m_3 \sqcup m_2)(x)\\ &= ((m_3 \sqcup m_1) \sqcap (m_3 \sqcup m_2))(x),\end{aligned}$$

then $(M(\mathcal{H}), \sqcap, \sqcup, id_H, 1_H)$ is a bounded distributive lattice. □

We recall that a lattice L is called complete if every subset S of L has the least upper bound (supremum) and the greatest lower bound (infimum) in L.

Proposition 3.13. *Let $\mathcal{H}$ be a complete WH-algebra. Then $(M(\mathcal{H}), \sqcap, \sqcup, \rightsquigarrow, id_H, 1_H)$ is a complete Heyting algebra, where $m_1 \rightsquigarrow m_2 := \bigsqcup\{m \in M(\mathcal{H}) | m \sqcap m_1 \leq m_2\}$.*

Proof. By Proposition 3.12, $(M(\mathcal{H}), \sqcap, \sqcup, id_H, 1_H)$ is a bounded distributive lattice. Since $\mathcal{H}$ is a complete lattice, then $(M(\mathcal{H}), \sqcap, \sqcup, id_H, 1_H)$ is complete. So $\bigsqcup\{m \in M(\mathcal{H}) | m \sqcap m_1 \leq m_2\}$ exists in $M(\mathcal{H})$. Put $P := \{m \in M(\mathcal{H}) | m \sqcap m_1 \leq m_2\}$. If $m_3 \in M(\mathcal{H})$ such that $m_3 \sqcap m_1 \leq m_2$, then $m_3 \in P$. So $m_3 \leq \bigsqcup P = m_1 \rightsquigarrow m_2$. Conversely, if $m_3 \leq m_1 \rightsquigarrow m_2$, then $m_3 \leq m$ for all

$m \in P$. Hence $m_3 \sqcap m_1 \leq m \sqcap m_1 \leq m_2$. Therefore $(M(\mathcal{H}), \sqcap, \sqcup, \rightsquigarrow, id_H, 1_H)$ is a Heyting algebra. □

Definition 3.14. Let $\mathcal{H}$ be a WH-algebra. A congruence relation θ on $(H, \wedge, \vee)$ is called a weak congruence on $\mathcal{H}$ if $(x, y) \in \theta$ implies that $(a \to x, a \to y) \in \theta$ for any $a \in H$.

Proposition 3.15. *Let m be a multiplier in a WH-algebra $\mathcal{H}$. Define a binary relation θ_m on H as follows for all $x, y \in H$:*

$$(x, y) \in \theta_m \text{ if and only if } m(x) = m(y).$$

Then θ_m is a weak congruence on H.

Proof. It is clear that θ_m is an equivalence relation on $\mathcal{H}$. By Proposition 3.3 part (4) and (6), θ_m is a congruence relation on the lattice reduct $(H, \wedge, \vee)$. Using (M1), one can see that θ_m is a weak congruence on H. □

Proposition 3.16. *Let m be a multiplier in a WH-algebra $\mathcal{H}$. Then $Fix_m(H) \cap [x]_m$ is a singleton set for all $x \in H$, where $[x]_m$ is the congruence class of x with respect to θ_m.*

Proof. Suppose that m is a multiplier in $\mathcal{H}$. Let x be an arbitrary element of H. We have $m(x) = m(m(x))$, so $(x, m(x)) \in \theta_m$. Thus $m(x) \in Fix_m(H)$. Thus $m(x) \in Fix_m(H) \cap [x]_m$. Hence $Fix_m(H) \cap [x]_m$ is non-empty. Now suppose that $a, b \in Fix_m(H) \cap [x]_m$. Then $(a, x), (b, x) \in \theta_m$. So $m(a) = m(b)$. Hence $a = b$. □

Definition 3.17. Let $\mathcal{H}$ be a WH-algebra. A non-empty subset F of H is called a quasi-filter if it satisfies the following conditions for all $x, y \in H$:
(qf1) $x \in F$ implies $y \to x \in F$,
(qf2) if $x \leq y$ and $x \in F$, then $y \in F$.

Example 3.18. *(1) It is clear that $\{1\}$ is a quasi-filter of a WH-algebra $\mathcal{H}$.*
(2) Let $H = \{0, a, b, 1\}$ with $0 < a, b < 1$, such that a, b are not comparable. Consider the following binary operation:

$\to$	*0*	*a*	*b*	*1*
0	*1*	*1*	*1*	*1*
a	*1*	*1*	*1*	*1*
b	*b*	*b*	*1*	*1*
1	*b*	*b*	*1*	*1*

Then $\mathcal{H} = (H, \vee, \wedge, \to, 0, 1)$ is aWH-algebra ([11]). Then filter $F = \{b, 1\}$ is a quasi-filter. Also, $F_1 = \{a, 1\}$ is a filter but it is not a quasi filter because $a \in F_1$ but $b \to a = b \notin F_1$.

Proposition 3.19. *(1) Every filter of a basic algebra is a quasi filter.*
(2) Every deductive system of an SDWH-algebra is a quasi filter.
(3) If m is a multiplier in a WH-algebra $\mathcal{H}$, then $Fix_m(H)$ is a quasi filter.

Proof. (1) Suppose that F is a filter of a basic algebra $\mathcal{H}$ and $x \in F$. Then $x \leq 1 \to x \leq y \to x$. Hence $y \to x \in F$.
(2) Let D be a deductive system of SDWH-algebra $\mathcal{H}$ and $x \in D$. Then $x \to (y \to x) = (x \to y) \to (x \to x) = (x \to y) \to 1 = 1 \in D$. So $y \to x \in D$.
(3) Let m is a multiplier in a WH-algebra $\mathcal{H}$ and $x \in Fix_m(H)$. Then $m(y \to x) = y \to m(x) = y \to x$. Thus $y \to x \in Fix_m(H)$. □

Proposition 3.20. *Let D be a quasi-filter of a WH-algebra $\mathcal{H}$ and θ be a weak congruence on H such that $D \cap [x]_\theta$ is a singleton set for all $x \in H$. Then there exists a multiplier m in H such that $Fix_m(H) = D$.*

Proof. Let x_0 be the single element of $D \cap [x]_\theta$. Define a self-mapping $m : H \to H$ by $m(x) = x_0$ for all $x \in H$. We have $m(x) = x_0 \in D \cap [x]_\theta$. Thus $(m(x), x) \in \theta$, $y \to m(x) \in D$ by (qf1) and $m(x) \vee y \in D$ by (qf2). So $(y \to m(x), y \to x) \in \theta$ and $(y \vee m(x), y \vee x) \in \theta$. Hence $y \to m(x) \in [y \to x]_\theta$ and $y \vee m(x) \in [y \vee x]_\theta$. Thus $y \to m(x) \in D \cap [y \to x]_\theta$ and $y \vee m(x) \in D \cap [y \vee x]_\theta$. Since $m(y \to x) \in D \cap [y \to x]_\theta$ and $D \cap [y \to x]_\theta$ is a singleton set, we obtain $m(y \to x) = y \to m(x)$. Similarly, $m(y \vee x) = y \vee m(x)$. Therefore, m is a multiplier in H. It is easy to prove that $Fix_m(H) = D$. □

4. m-Stabilizer in WH-algebras

In this section, we apply the concept of multiplier in WH-algebras to define m-stabilizers and study them.

Definition 4.1. Let m be a multiplier in a WH-algebra $\mathcal{H}$ and X be a non-empty subset of H. Then the sets

$$L_m(X) = \{a \in H | m(a) \vee x = x, \forall x \in X\}$$

and

$$R_m(X) = \{a \in H | x \vee m(a) = m(a), \forall x \in X\}$$

are called the left and right m-stabilizers of X, respectively.

Remark that $L_m(X) = \{a \in H | m(a) \leq x, \forall x \in X\}$ and $R_m(X) = \{a \in H | x \leq m(a), \forall x \in X\}$.

Example 4.2. *(1) Let $m = id_H$ be identity map on an arbitrary WH-algebra $\mathcal{H}$. Then $L_m(\{x\}) = (x]$ and $R_m(\{x\}) = [x)$ for all $x \in H$.*
(2) Let m be a unit mapping on a WH-algebra $\mathcal{H}$. If $X = \{1\}$, then $L_m(X) = H$, otherwise $L_m(X) = \emptyset$. Also, $R_m(X) = H$ for all $\emptyset \neq X \subseteq H$.
(3) Consider multiplier m_3 in Example 3.2. Let $X = \{a\}$, then $L_{m_3}(X) = \{0, a\}$ and $R_{m_3}(X) = H$.

Proposition 4.3. *Let m be a multiplier in a WH-algebra $\mathcal{H}$, $x \in H$ and X, Y be two non-empty subsets of H. Then*
(1) $x, 1 \in R_m(\{x\})$,
(2) $R_m(\{0\}) = H$ and $R_m(\{1\}) = R_m(H) = Ker(m)$,
(3) if $X \subseteq Y$, then $R_m(Y) \subseteq R_m(X)$ and $L_m(Y) \subseteq L_m(X)$,

(4) if $x \in Fix_m(H)$*, then* $0, x \in L_m(\{x\})$*,*
(5) $L_m(\{1\}) = H$*,*
(6) $L_m(H) \neq \emptyset$ *if and only if* $m = id_H$ *if and only if* $L_m(\{0\}) \neq \emptyset$*.*

Proof. (1) It follows from Proposition 3.3 parts (1) and (2).
(4) By assumption $x \in Fix_m(H)$. Thus $m(0) \vee x = m(0) \vee m(x) = m(0 \vee x) = m(x) = x$. Hence $0 \in L_m(\{x\})$. Also $m(x) \vee x = x \vee x = x$. Thus $x \in L_m(\{x\})$.
(6) Suppose that $L_m(H) \neq \emptyset$. Then there exists $a \in H$ such that $m(a) \vee x = x$, for all $x \in H$. Since $0 \in H$, then $m(a) = 0$. By Proposition 3.3 part (9), we get $a = 0$. Then $m = id_H$ by 3.3 part (10). Conversely, if $m = id_H$, then $m(0) = 0$. Hence $0 \in L_m(H)$, that is $L_m(H) \neq \emptyset$. Similarly, we can prove that $L_m(\{0\}) \neq \emptyset$ if and only if $m = id_H$.
The proofs of the other parts are easy by Proposition 3.3. Hence the details are omitted. □

Proposition 4.4. *Let* m *be a multiplier in a WH-algebra* $\mathcal{H}$*. Then*
(1) $R_m(\{x\})$ *is a filter,*
(2) if H *is a chain, then* $R_m(\{x\})$ *is a prime filter,*
(3) if m *is one to one and* $R_m(\{x\})$ *is a prime filter for all* $x \in H$*, then* H *is a chain.*

Proof. (1) Let $a, b \in R_m(\{x\})$. Then

$$\begin{aligned} x \vee m(a \wedge b) &= x \vee (m(a) \wedge m(b)) = (x \vee m(a)) \wedge (x \vee m(b)) \\ &= m(a) \wedge m(b) = m(a \wedge b). \end{aligned}$$

Hence $a \wedge b \in R_m(\{x\})$. Now, suppose that $a \leq b$ and $a \in R_m(\{x\})$. Then $x \vee m(b) = x \vee m(a \vee b) = x \vee a \vee m(b) = x \vee m(b)$ by (M2) and Proposition 3.3 part (2). Thus $b \in R_m(\{x\})$. Hence $R_m(\{x\})$ is a filter.
(2) The proof is trivial.
(3) We have $a \vee b \in R_m(\{a \vee b\})$ by Proposition 4.3 part (1). By assumption $a \in R_m(\{a \vee b\})$ or $b \in R_m(\{a \vee b\})$. Suppose that $a \in R_m(\{a \vee b\})$. Then $(a \vee b) \vee m(a) = m(a)$. So $b \vee m(a) = m(a)$. We get $m(a \vee b) = m(a)$ by (M2). Since m is one to one, then $a \vee b = a$. Hence $b \leq a$. □

Corollary 4.5. *Let* m *be a multiplier in a complete WH-algebra* $\mathcal{H}$ *and* $x \leq m(0)$*. Then* $R_m(\{x\}) = H$*.*

Proof. Since $x \leq m(0)$, then $m(x) \leq m(0)$. Hence $m(x) = m(0)$. We have $x \vee m(0) = m(x \vee 0) = m(x) = m(0)$. Therefor $0 \in R_m(\{x\})$. Since $R_m(\{x\})$ is a filter, then $R_m(\{x\}) = H$. □

Recall that a completely distributive lattice is a complete lattice in which arbitrary joins distribute over arbitrary meets.

Proposition 4.6. *Let* m *be a multiplier in a completely distributive WH-algebra* $\mathcal{H}$ *and* $x_0 = \bigwedge R_m(\{x\})$*. Then* $(R_m(\{x\}), \wedge, \vee, \rightsquigarrow, x_0, 1)$ *is a WH-algebra where* $a \rightsquigarrow b := (a \rightarrow m(b)) \vee x$ *for all* $a, b \in R_m(\{x\})$*.*

Proof. Since $\mathcal{H}$ is complete, then $x_0 = \bigwedge R_m(\{x\})$ exists in H. We have

$$\begin{aligned} m(x_0) =& m(\bigwedge\{a|a \in R_m(\{x\})\}) \vee 0 = m(\bigwedge\{a|a \in R_m(\{x\})\} \vee 0) \\ =& m(0) \vee \bigwedge\{a|a \in R_m(\{x\})\} = \bigwedge\{m(0) \vee a|a \in R_m(\{x\})\} \\ =& \bigwedge\{m(0 \vee a)|a \in R_m(\{x\})\} = \bigwedge\{m(a)|a \in R_m(\{x\})\}. \end{aligned}$$

Thus

$$\begin{aligned} x \vee m(x_0) =& x \vee \bigwedge\{m(a)|a \in R_m(\{x\})\} = \bigwedge\{x \vee m(a)|a \in R_m(\{x\})\} \\ =& \bigwedge\{m(a)|a \in R_m(\{x\})\} = m(x_0). \end{aligned}$$

Therefore $x_0 \in R_m(\{x\})$. Using Proposition 3.3 parts (4) and (6), we obtain $(R_m\{x\}, \wedge, \vee, x_0, 1)$ is a bounded distributive lattice.
Suppose that $a, b, c \in R_m(\{x\})$ be arbitrary. By Proposition 3.3 part (2)

$$\begin{aligned} x \vee m(a \rightsquigarrow b) =& x \vee (m(x \vee (a \to m(b)))) = x \vee m(x) \vee (a \to m(b)) \\ =& m(x) \vee (a \to m(b)) = m(x \vee (a \to m(b))) = m(a \rightsquigarrow b). \end{aligned}$$

Thus $a \rightsquigarrow b \in R_m(\{x\})$.
(WH1) Using distributivity and Proposition 3.3 part (6), we obtain

$$\begin{aligned} (a \rightsquigarrow b) \wedge (a \rightsquigarrow c) =& ((a \to m(b)) \vee x) \wedge ((a \to m(c)) \vee x) \\ =& (a \to (m(b) \wedge m(c))) \vee x \\ =& (a \to m(b \wedge c)) \vee x = a \rightsquigarrow (b \wedge c). \end{aligned}$$

(WH2) We have

$$\begin{aligned} (a \rightsquigarrow c) \wedge (b \rightsquigarrow c) &= ((a \to m(c)) \vee x) \wedge ((b \to m(c)) \vee x) \\ &= ((a \vee b) \to m(c)) \vee x = (a \vee b) \rightsquigarrow c. \end{aligned}$$

(WH3) Applying (M1) and Proposition 3.3 part (6)

$$\begin{aligned} (a \rightsquigarrow b) \wedge (b \rightsquigarrow c) &= ((a \to m(b)) \vee x) \wedge ((b \to m(c)) \vee x) \\ &= (m(a \to b) \wedge m(b \to c)) \vee x \\ &= m((a \to b) \wedge (b \to c)) \vee x \\ &\leq (m(a \to c) \vee x) \\ &= (a \to m(c)) \vee x = a \rightsquigarrow c. \end{aligned}$$

(WH4) By Proposition 3.3 part (2) and the (W2), we get $a \rightsquigarrow a = (a \to m(a)) \vee x = 1 \vee x = 1$. □

Corollary 4.7. *Let m be a multiplier in a finite WH-algebra $\mathcal{H}$, $x \in H$ and $x_0 = \bigwedge R_m(\{x\})$. Then $(R_m\{x\}, \wedge, \vee, \rightsquigarrow, x_0, 1)$ is a WH-algebra where $a \rightsquigarrow b := (a \to m(b)) \vee x$ for all $a, b \in R_m(\{x\})$.*

Proof. Since H is finite, then it is complete. The result follows from the above proposition. □

Proposition 4.8. *Let m be a multiplier in a WH-algebra $\mathcal{H}$ and $x \in Fix_m(H)$. Then*
(1) $L_m(\{x\})$ is an ideal,
(2) $L_m(\{x\}) = (x]$ where $(x] = \{a \in H | a \leq x\}$,
(3) if $\mathcal{H}$ is a chain, then $L_m(\{x\})$ is a prime ideal,
(4) if $L_m(\{x\})$ is a prime ideal for all $x \in Fix_m(H)$, then $Fix_m(H)$ is a chain.

Proof. (1)Since $x \in Fix_m(H)$, then $L_m(\{x\}) \neq \emptyset$ by Proposition 4.3 part (4). Suppose that $a \leq b$ and $b \in L_m(\{x\})$. Thus $x \vee m(b) = x$. Using Proposition 3.3 part (6), we obtain $x \vee m(a) = x \vee m(a \wedge b) = x \vee (m(a) \wedge m(b) = (x \vee m(a)) \wedge (x \vee m(b)) = (x \vee m(a)) \wedge x = x$. Hence $a \in L_m(\{x\})$. If $a, b \in L_m(\{x\})$, then $x \vee m(a \vee b) = x \vee m(a) \vee m(b) = x \vee m(b) = x$. Therefore $L_m(\{x\})$ is an ideal.
(2) Since $x \in Fix_m(H)$, then $x \in L_m(\{x\})$. Let $a \in (x]$ be arbitrary. Then $a \leq x$. Thus $m(a) \leq m(x) = x$. We get $x \vee m(a) = x$. Hence $x \in L_m\{x\}$. So $(x] \subseteq L_m(\{x\})$. Conversely, let $a \in L_m(\{x\})$. Then $x \vee m(a) = x$. So $m(a) \leq x$. By Proposition 3.3 part (2), we get $a \leq x$. Hence $a \in (x]$. Thus $L_m(\{x\}) \subseteq (a]$.
(3) Suppose that $a \wedge b \in L_m(\{x\})$. Since $\mathcal{H}$ is a chain, then $a \leq b$ or $b \leq a$. So $a \in L_m(\{x\})$ or $b \in L_m(\{x\})$.
(4) Let $a, b \in Fix_m(H)$ be arbitrary. Then $a \wedge b \in Fix_m(H)$ by Proposition 3.3 part (6). So $L_m(\{a \wedge b\})$ is a prime ideal by assumption. Since $a \wedge b \in L_m(\{a \wedge b\})$ by Proposition 4.3 part (4), then $a \in L_m(\{a \wedge b\}) = (a \wedge b]$ or $b \in L_m(\{a \wedge b\}) = (a \wedge b]$. Hence $a \leq b$ or $b \leq a$. □

Proposition 4.9. *Let m be a multiplier in a WH-algebra $\mathcal{H}$ and $x \in Fix_m(H)$. Then $(L_m(\{x\}), \wedge, \vee, \rightsquigarrow, 0, x)$ is a WH-algebra where $a \rightsquigarrow b := (a \rightarrow b) \wedge x$ for all $a, b \in L_m(\{x\})$.*

Proof. By Proposition 3.3 parts (4) and (6) , it is easy to show that $L_m\{x\}$ is closed under $\wedge$ and $\vee$. Let $a \in L_m(\{x\})$ be arbitrary. Then $m(a) \vee x = x$. Thus $m(a) \leq x$. By Proposition 3.3 part (2), we get $a \leq x$. Also, $0 \in L_m(\{x\})$ by Proposition 4.3 part (4). Therefore $(L_m(\{x\}), \wedge, \vee, 0, x)$ is a bounded distributive lattice.
Let $a, b \in L_m(\{x\})$. We will prove that $a \rightsquigarrow b \in L_m(\{x\})$. We have
$x \vee m(a \rightsquigarrow b) = x \vee (m(a \rightarrow b) \wedge m(x)) = x \vee (m(a \rightarrow b) \wedge x) = x$.
It is easy to see that it satisfies (WH1)-(WH3). Also, we have $a \rightsquigarrow a = (a \rightarrow a) \wedge x = x$. Hence $(L_m(\{x\}), \wedge, \vee, \rightsquigarrow, 0, x)$ is a WH-algebra. □

5. Conclusion and future research

We have studied multipliers in WH-algebras and proved the set of all multipliers in a WH-algebra is a bounded distributive lattice. Also, if a WH-algebra is complete, then the set of all its multipliers is a Heyting algebra and so is a WH-algebra. By a multiplier, we have introduced m-stabilizers and investigated their properties.

The notion of F-multipliers has been investigated and studied in Heyting algebras [9] and some other algebraic structures such as Bl-algebras [4] and hoop-algebras [10]. In the future, we will define the notion of F-multipliers in WH-algebras and will obtain the relationship between F-multipliers in WH-algebras and multipliers are defined in this paper. Also, since any Heyting algebra is a WH-algebra, we will study the relationship between the F-multipliers in WH-algebras and F-multipliers in Heyting algebras. We will develop the localization theory for WH-algebras by F-multipliers. Then using the notions of F-multipliers and the prime spectrum of a WH-algebra, we will define a sheaf on the prime spectrum of WH-algebras and we will try to obtain a Grothendieck-like sheaf duality for WH-algebras.

6. Data Availability Statement

"Not applicable".

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8. Conflict of interest

The author declares no conflict of interest.

References

[1] Ardeshir, M., & Ruitenburg, W. (1998). Basic propositional calculus I. Mathematical Logic Quarterly, 44(3), 317-343. http://dx.doi.org/10.1002/malq.19980440304

[2] Alizadeh, M., & Joharizadeh, N. (2015). Counting weak Heyting algebras on finite distributive lattices. Logic Journal of the IGPL, 23(2), 247-258.http://dx.doi.org/10.1093/jigpal/jzu033

[3] Borzooei, R. A.,& Paad, A. (2012). Some new types of stabilizers in BL-algebras and their applications. Indian Journal of science and Technology, 5(1), 1910-1915.http://dx.doi.org/10.17485/ijst/2012/v5i1.29

[4] Buşneag, D., & Piciu, D. (2005). BL-algebra of fractions and maximal BL-algebra of quotients. Soft Computing, 9, 544-555.http://dx.doi.org/10.1007/s00500-004-0372-9

[5] Celani, S., & Jansana, R. (2005). Bounded distributive lattices with strict implication. Mathematical Logic Quarterly, 51(3), 219-246. http://dx.doi.org/10.1002/malq.200410022

[6] Chaudhry, M. A., & Ali, F. (2012). Multipliers in d-algebras. World Applied Sciences Journal, 18(11), 1649-1653.

[7] Cirulis, J. (1986). Multipliers in implicative algebras. Bulletin of the Section of Logic, 15(4), 152-157.

[8] Cornish, W. H. (1974). The multiplier extension of a distributive lattice. Journal of Algebra, 32(2), 339-355.http://dx.doi.org/10.1016/0021-8693(74)90143-4

[9] Dan, C. (1997). F-multipliers and the localization of Heyting algebras. An. Univ. Craiova Ser. Mat. Inform, 24, 98-109.

[10] Ghorbani, S. (2013). Localization of hoop-algebras. J. Adv. Res. Pure Math, 5(3), 1-13.http://dx.doi.org/10.5373/jarpm.1378.032812

[11] Kim, K. H. (2011). Multipliers in BE-algebras. In International Mathematical Forum, 6(17), 815-820.

[12] Kim, K. H., & Lim, H. J. (2013). On Multipliers of BCC-algebras. Honam Mathematical Journal, 35(2), 201-210.http://dx.doi.org/10.5831/HMJ.2013.35.2.201

[13] Nourany, M., Ghorbani, S., & Saeid, A. B. (2023). On self-distributive weak Heyting algebras. Mathematical Logic Quarterly, 69(2), 192-206.http://dx.doi.org/10.1002/malq.202200073

[14] San Martín, H. J. (2016). Principal congruences in weak Heyting algebras. Algebra universalis, 75, 405-418.http://dx.doi.org/10.1007/s00012-016-0381-4

[15] Schmid, J. (1980). Multipliers on distributive lattices and rings of quotients. Houston Journal of Mathematics, 6(3), 401-425.

[16] Tayebi Khorami, R., & Borumand Saeid, A. (2014). Multiplier in BL-algebras. Iranian Journal of Science, 38(2), 95-103.

[17] Wang, J. T., He, P. F., & Borumand Saeid, A. (2018). Stabilizers in MTL-algebras. Journal of Intelligent & Fuzzy Systems, 35(1), 717-727.http://dx.doi.org/10.3233/JIFS-171105

[18] Zhu, K., Wang, J., & Yang, Y. (2019). On two new classes of stabilizers in residuated lattices. Soft Computing, 23(23), 12209-12219.

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