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J-HYPERIDEALS AND RELATED GENERALIZATIONS IN MULTIPLICATIVE HYPERRINGS

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Abstract. In this paper, we define the concept of J-hyperideals which is a generalization of n-hyperideals. A proper hyperideal I of a multiplicative hyperring R is said to be a J-hyperideal if $x, y \in R$ such that $x \circ y \subseteq I$, then either $x \in J(R)$ or $y \in I$. We study and investigate the behavior of the J-hyperideals to introduce several results. Moreover, we extend the notion of J-hyperideals to quasi J-hyperideals and 2-absorbing J-hyperideals. Various characterizations of them are provided.

Keywords: J-hyperideal, Quasi J-hyperideal, 2-absorbing J-hyperideal.
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1. Introduction

The importance of the prime ideal in commutative rings has encouraged several authors to expand this notion. In [22] Tekir et al. defined the concept of n-ideals and they investigated many properties of the new class of ideals with similar prime ideals. Afterward, Hani et al. presented the notion of J-ideals as an extension of n-ideals in [15]. A proper ideal I in a ring R is said to be a J-ideal if $xy \in I$ for $x, y \in R$ such that x is not in the intersection of all maximal ideals of R, then $y \in I$.

In 1934, at the 8th Congress of Scandinavian Mathematicians, a new theory was introduced about algebraic systems by Marty [16]. He defined the hypergroups and began to investigate their properties with applications to groups, algebraic functions and rational fractions. Later on, many researchers have worked on this new field of modern algebra and developed it. The multiplicative hyperring, as an important class of hyperrings, was introduced by Rota in 1982 [20]. In this hyperring, the multiplication is a hyperoperation, while the addition is an operation. In 1990, the strongly distributive multiplicative hyperrings were characterized by Rota [19]. The polynomials over multiplicative hyperrings were studied by Procesi and Rota in [17]. Ameri and Kordi introduced the notions of clean multiplicative hyperring and regular multiplicative hyperring, as two generalizations of classical rings, in [3] and [4]. The concept of derivation on multiplicative hyperrings was introduced by Ardekani and Davvaz

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in [14]. Ameri et al. [5] have studied the notion of hyperring of fractions generated by a multiplicative hyperring. Soltani et al. introduced zero-divisor graphs of a commutative multiplicative hyperring, as a generalization of commutative rings [21]. The codes over multiplicative hyperrings were studied by Akbiyik [1]. The notion of primeness of hyperideal in a multiplicative hyperring was conceptualized by Procesi and Rota in [18]. The notions of prime and primary hyperideals in multiplicative hyperrings were fully studied by Dasgupta in [9]. The concept of S-prime hyperideals in multiplicative hyperrings as a generalization of prime hyperideals was studied in [13]. Ghiasvand introduced the concept of 2-absorbing hyperideals in a multiplicative hyperring which is a generalisation of prime hyperideals [12]. In [23], Ulucak defined the notion of δ-primary hyperideals in multiplicative hyperrings, which unifies the prime and primary hyperideals under one frame. Recently, we introduced the notions of n-hyperideals and r-hyperideals in a multiplicative hyperring [8]. Let R be a multiplicative hyperring with identity 1. A hyperideal I of R refers to an n-hyperideal if $x, y \in R$ and $x \circ y \subseteq I$ imply either x is in the intersection of all prime hyperideals of R or y is in I. The intersection of all maximal hyperideals of R is denoted by $J(R)$. By Proposition 2.18 in [9], all maximal hyperideals are prime. Thus the intersection of all prime hyperideals of R is contained in $J(R)$. Our aim in this paper is to introduce and study the concept of J-hyperideals which is a generalization of n-hyperideals. This generalization offers a broader perspective on hyperideal structures within multiplicative hyperrings, providing a new context for investigating hyperideal properties beyond n-hyperideals. Furthermore, we define two generalizations of J-hyperideals in a multiplicative hyperring. The paper is organized as follows. In Section 2, we give some basic definitions and results of multiplicative hyperrings which we need to develop our paper. In Section 3, we introduce the concept of J-hyperideals and discuss their relations with some other types of hyperideals. Moreover, we investigate the behavior of J-hyperideals under a good homomorphism. In Section 4, we study a generalization of the J-hyperideals which is called quasi J-hyperideals. We present a characterization of local multiplicative hyperrings in terms of quasi J-hyperideals. In Section 5, we extend the notion of J-hyperideals to 2-absorbing J-hyperideals and give some properties of them.

2. **Preliminaries**

We first recall the basic terms and definitions from the hyperring theory [11]. A hyperoperation on a non-empty set G is a map $\circ : G \times G \longrightarrow P^*(G)$, where $P^*(G)$ is the set of all the non-empty subsets of G. An algebraic system $(G, \circ)$ is called a hypergroupoid. A hypergroupoid $(G, \circ)$ is called a hypergroup if it satisfies the following:

(1) $a \circ (b \circ c) = (a \circ b) \circ c$, for all $a, b, c \in G$.

(2) $a \circ G = G \circ a = G$, for all $a \in G$.

A hypergroupoid with the associative hyperoperation is called a semihypergroup [11]. A non-empty set R with an operation $+$ and a hyperoperation $\circ$ is called a *multiplicative hyperring* if it satisfies the following:

(1)$(R,+)$ is an abelian group;

(2)$(R,\circ)$ is a semihypergroup;

(3) for all $a,b,c \in R$, we have $a\circ(b+c) \subseteq a\circ b+a\circ c$ and $(b+c)\circ a \subseteq b\circ a+c\circ a$;

(4) for all $a,b \in R$, we have $a \circ (-b) = (-a) \circ b = -(a \circ b)$.

Let $(R,+,\circ)$ be a multiplicative hyperring. If $a \circ b = b \circ a$ for all $a,b \in R$, then R is said to be commutative [9]. An element 1 of R is called identity if $a \in a \circ 1$ for all $a \in R$ [5]. Throughout this paper all multiplicative hyperings are commutative with identity 1. Let A and B be non-empty subsets of R and $r \in R$. Then we define

$$A \circ B = \bigcup_{x\in A,\ y\in B} x \circ y, \qquad A \circ r = A \circ \{r\}$$

[9] A non-empty subset I of a hyperring R is a *hyperideal* if

(i) If $a,b \in I$, then $a-b \in I$;

(iii) If $x \in I$ and $r \in R$, then $r \circ x \subseteq I$ and $x \circ r \subseteq I$.

Definition 2.1. [11] Let $(R_1,+_1,\circ_1)$ and $(R_2,+_2,\circ_2)$ be two hyperrings. A mapping from R_1 into R_2 is said to be a *good homomorphism* if for all $x,y \in R_1$, $\phi(x +_1 y) = \phi(x) +_2 \phi(y)$ and $\phi(x \circ_1 y) = \phi(x) \circ_2 \phi(y)$.

Definition 2.2. [9] A proper hyperideal P of R is called *prime* if $x\circ y \subseteq P$ for $x,y \in R$ implies that $x \in P$ or $y \in P$. The intersection of all prime hyperideals of R containing I is called the prime radical of I, being denoted by $\sqrt{I}$. If the hyperring R does not have any prime hyperideal containing I, we define $\sqrt{I} = R$.

Definition 2.3. [5] A proper hyperideal I of R is *maximal* if for any hyperideal J of R with $I \subseteq J \subseteq R$ then $J = I$ or $J = R$. Also, we say that R is a local hyperring if it has just one maximal hyperideal. For a hyperring R we define the *Jacobson radical* $J(R)$ as the intersection of all maximal hyperideals of R. Moreover, if I is a proper hyperideal of R, then the Jacobson radical $J(I)$ is defined as the intersection of all maximal hyperideals of R containing I.

Let $\mathbf{C}$ be the class of all finite products of elements of R i.e. $\mathbf{C} = \{r_1 \circ r_2 \circ ... \circ r_n \ : \ r_i \in R, n \in \mathbb{N}\} \subseteq P^*(R)$. A hyperideal I of R is said to be a $\mathbf{C}$-*hyperideal* of R if for any $A \in \mathbf{C}, A \cap I \neq \varnothing$ implies $A \subseteq I$. Let I be a hyperideal of R. Then, $D \subseteq \sqrt{I}$ where $D = \{r \in R : r^n \subseteq I \text{ for some } n \in \mathbb{N}\}$. The equality holds when I is a $\mathbf{C}$-hyperideal of R (see[9, Proposition 3.2]).

Definition 2.4. [5] An element $x \in R$ is called a *unit*, if there exists $y \in R$ such that $1 \in x \circ y$. Denote the set of all unit elements in R by $U(R)$.

Definition 2.5. [2] An element $x \in R$ is said to be a *zero divisor*, if there exists $0 \neq y \in R$ such that $\{0\} = x \circ y$. The set of all zero divisors in R is denoted by $Z(R)$.

Definition 2.6. [5] Let I and J be hyperideals of R and $x \in R$. Then define:

$$(I : a) = \{r \in R : r \circ a \subseteq I\}$$

$$ann(x) = \{y \in R : x \circ y = \{0\}\}$$

3. J-hyperideals

In this section, we introduce and study the concept of J-hyperideals.

Definition 3.1. Let I be a proper hyperideal of a hyperring R. We call I a *J-hyperideal* if $x, y \in R$ with $x \circ y \subseteq I$, then either $x \in J(R)$ or $y \in I$.

Example 3.2. *Let $\mathbb{Z}$ be the ring of integers. For all $a, b \in \mathbb{Z}$, we define the hyperoperation $a \circ b = \{axb \mid x \in A\}$ where $A = \{2, 3\}$. Then $(\mathbb{Z}_A, +, \circ)$ is a hyperring in which $\mathbb{Z}_A = \mathbb{Z}$. In the hyperring, every principal hyperideal generated by a prime integer is a J-hyperideal. Now, let $A = \{14, 21\}$. Then the principal hyperideal $\langle 7 \rangle$ is not a J-hyperideal. Because, $1 \circ 1 = \{14, 21\} \subseteq \langle 7 \rangle$, but neither $1 \in \langle 7 \rangle$ nor $1 \in J(\mathbb{Z})$.*

Example 3.3. *Consider the ring of integers $(\mathbb{Z}, +, \cdot)$. Let x be an indeterminate. Assume that $R = \mathbb{Z} + 3x\mathbb{Z}[x]$. For all $\alpha, \beta \in \mathbb{Z}$, we define the hyperoperation $\alpha \circ \beta = \{2\alpha\beta, 4\alpha\beta\}$. Consider the hyperideal $I = 3x\mathbb{Z}[x]$. Then I is a J-hyperideal of R.*

Proposition 3.4. *If I is a J-hyperideal of R, then I is contained in $J(R)$.*

Proof. Assume that I is a J-hyperideal of R such that it is not contained in $J(R)$. Let $x \in I$ but $x \notin J(R)$. Since I is a hyperideal of R, we have $x \circ 1 \subseteq I$. Since I is a J-hyperideal of R and $x \notin J(R)$, we get $1 \in I$ which is a contradiction. Thus, I is contained in $J(R)$. □

The converse of Proposition 3.4 may not be always true as it is shown in the following example.

Example 3.5. *Assume that M_1 and M_2 are the only maximal hyperideals of R. Then there are $x \in M_1$ and $y \in M_2$ such that $x \notin M_2$ and $y \notin M_1$. Thus we conclude that $x \circ y \subseteq M_1 \cap M_2$ and so $\langle x \circ y \rangle \subseteq M_1 \cap M_2$. Since $x, y \notin \langle x \circ y \rangle$ and $x, y \notin J(R)$, $\langle x \circ y \rangle$ is not a J-hyperideal of R.*

Theorem 3.6. *For a hyperring R, the following statements are equivalent:*
 (i) *R is a local hyperring.*
 (ii) *Every proper hyperideal of R is a J-hyperideal.*
 (iii) *Every proper principal hyperideal of R is a J-hyperideal.*

Proof. (i) $\Longrightarrow$ (ii) Suppose that I is a proper hyperideal of R and M is the only maximal hyperideal of R. So $J(R) = M$. Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin M$. Therefore, $x \in U(R)$. Hence, we have $y \in 1 \circ y \subseteq (x^{-1} \circ x) \circ y = x^{-1} \circ (x \circ y) \subseteq I$. Thus I is a J-hyperideal of R.

(ii)$\Longrightarrow$ (iii) Obvious.

(iii)$\Longrightarrow$ (i) Let every proper principal hyperideal of R be a J-hyperideal. Suppose that the hyperideal M of R is maximal. Let $x \in M$. Since the principal hyperideal $\langle x\rangle$ is a J-hyperideal and $x \circ 1 \subseteq \langle x\rangle$, we get $x \in J(R)$ or $1 \in \langle x\rangle$. In the second case, we have a contradiction. Then $x \in J(R)$ and so $J(R) = M$. Thus, R is a local hyperring. □

Recall from [8] that a proper hyperideal I of R is said to be an *n-hyperideal* if $x \circ y \subseteq I$, then $x \in \sqrt{0}$ or $y \in I$ for any $x, y \in R$.

Proposition 3.7. *Let R be a hyperring. If I is an n-hyperideal of R, then it is a J-hyperideal.*

Proof. Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin J(R)$. By Proposision 2.18 in [9], we have $\sqrt{0} \subseteq J(R)$. Therefore, $x \notin \sqrt{0}$. Since I is an n-hyperideal of R, we obtain $y \in I$. Thus, I is a J-hyperideal of R. □

Theorem 3.8. *Let R be a local hyperring such that $\sqrt{0} \subsetneqq J(R)$. Then $J(R)$ is a J-hyperideal of R which is not an n-hyperideal.*

Proof. Let R be a local hyperring. By Theorem 3.6, the hyperideal $J(R)$ of R is a J-hyperideal. Let $x \in J(R) - \sqrt{0}$. Then we get $x \circ 1 \subseteq J(R)$ such that $x \notin \sqrt{0}$ and $1 \notin J(R)$. This implies that $J(R)$ is not an n-hyperideal of R. □

Recall from [8] that a proper hyperideal I of R is said to be a *r-hyperideal* if for all $x, y \in R$, $x \circ y \subseteq I$ and $ann(x) = \{0\}$, then $y \in I$.

Theorem 3.9. *Let R be a hyperring with $Z(R) \subseteq J(R)$. If I is a r-hyperideal of R, then I is a J-hyperideal.*

Proof. Let I be a r-hyperideal of R. Suppose that $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin J(R)$. Since $Z(R) \subseteq J(R)$, then $x \notin Z(R)$ which implies $ann(x) = \{0\}$. Since I is a r-hyperideal of R, we get $y \in I$. Thus, I is a J-hyperideal of R. □

The proof of the following proposition is easy.

Proposition 3.10. *Let $\{I_i\}_{i\in\Delta}$ be a non-empty set of J-hyperideals of R. Then $\bigcap_{i\in\Delta} I_i$ is a J-hyperideal of R.*

Theorem 3.11. *Let I be a proper hyperideal of R. Then the following statements are equivalent:*

(i) *I is a J-hyperideal of R.*

(ii) *$I = (I : x)$ for every $x \notin J(R)$.*

(iii) *$I_1 \circ I_2 \subseteq I$ for some hyperideals I_1 and I_2 of R implies that $I_1 \subseteq J(R)$ or $I_2 \subseteq I$.*

Proof. (i) $\Longrightarrow$ (ii) Let I be a J-hyperideal of R. It is clear that $I \subseteq (I : x)$ for all $x \in R$. Assume that $y \in (I : x)$ such that $x \notin J(R)$. This means $x \circ y \subseteq I$. Since I is a J-hyperideal of R and $x \notin J(R)$, we get $y \in I$. Thus, we have $I = (I : x)$.

(ii) $\Longrightarrow$ (iii) Let $I_1 \circ I_2 \subseteq I$ for some hyperideals I_1 and I_2 of R such that $I_1 \nsubseteq J(R)$. Therefore, we get $x \in I_1$ such that $x \notin J(R)$. Hence, $x \circ I_2 \subseteq I$ and so $I_2 \subseteq (I : x)$. Since $I = (I : x)$, we obtain $I_2 \subseteq I$.

(iii) $\Longrightarrow$ (i) Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin J(R)$. By Proposition 2.15 in [9], we have $\langle x\rangle \circ \langle y\rangle \subseteq \langle x \circ y\rangle \subseteq I$ but $\langle x\rangle \nsubseteq J(R)$. Then we get $\langle y\rangle \subseteq I$ which implies $y \in I$. Thus, I is a J-hyperideal of R. $\square$

Theorem 3.12. *Let I be a proper hyperideal of R. Then I is a J-hyperideal of R if and only if $(I : y) \subseteq J(R)$ for every $y \notin I$.*

Proof. $\Longrightarrow$ Let $x \in (I : y)$ such that $y \notin I$. So, $x \circ y \subseteq I$. Since I is a J-hyperideal of R, then $x \in J(R)$.

$\Longleftarrow$ Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin J(R)$. If $y \notin I$, then $x \in (I : y) \subseteq J(R)$, by the hypothesis. This is a contradiction. Therefore, $y \in I$. Thus, I is a J-hyperideal of R $\square$

Lemma 3.13. *Let I be a hyperideal of R and let T be a non-empty subset of R such that $T \nsubseteq I$. If I is a J-hyperideal of R, then $(I : T)$ is a J-hyperideal of R.*

Proof. Let $(I : T) = R$. Then $1 \in (I : T)$ and so $T \subseteq I$. This is a contradiction. Hence, $(I : T)$ is a proper hyperideal of R. Suppose that $x \circ y \subseteq (I : T)$ for some $x, y \in R$ such that $x \notin J(R)$. This implies that $x \circ y \circ t \subseteq I$ for all $t \in T$. Then we get $y \circ t \subseteq I$ for all $t \in T$ as I is a J-hyperideal of R. Thus $y \in (I : T)$. $\square$

Theorem 3.14. *Suppose that I is a J-hyperideal of R such that there is no J-hyperideal which contains I properly. Then I is a prime hyperideal.*

Proof. Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin I$. By Lemma 3.13, $(I : x)$ is a J-hyperideal of R. If $y \notin (I : x)$, then $y \circ x \nsubseteq I$ which is a contradiction with the assumption $x \circ y \subseteq I$. Thus, I is a prime hyperideal. $\square$

The next theorem shows that the converse of Theorem 3.14 is true if $I = J(R)$.

Theorem 3.15. *Let the hyperideal $J(R)$ of R be prime. Then $J(R)$ is a J-hyperideal of R such that there is no J-hyperideal which contains $J(R)$ properly.*

Proof. Suppose that $I = J(R)$. Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin J(R)$. Since I is a prime hyperideal of R, then $y \in I = J(R)$ and so the hyperideal $J(R)$ of R is a J-hyperideal. By Proposition 3.4, we conclude that there is no J-hyperideal which contains I properly. $\square$

Proposition 3.16. *Let H be a hyperideal of R such that $H \nsubseteq J(R)$. Then*

(i) *If A_1 and A_2 of R are J-hyperideals such that $A_1 \circ H = A_2 \circ H$, then $A_1 = A_2$.*

(ii) *If $I \circ H$ is a J-hyperideal for some hyperideal I of R, then $I \circ H = I$.*

Proof. (i) It is clear that $A_1 \circ H = A_2 \circ H \subseteq A_2$. By Theorem 3.11, we obtain $A_1 \subseteq A_2$ as the hyperideal A_2 is a J-hyperideal. By a similar argument we get $A_2 \subseteq A_1$. Thus $A_1 = A_2$. (ii) Since I is a hyperideal of R, then $I \circ H \subseteq I$. Let the hyperideal $I \circ H$ of R be a J-hyperideal. Since $I \circ H \subseteq I \circ H$ and $H \nsubseteq J(R)$, we have $I \subseteq I \circ H$, by Theorem 3.11. Thus, $I \circ H = I$. □

Theorem 3.17. *Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two hyperrings and $\phi : R_1 \longrightarrow R_2$ be a good epimorphism.*

(i) *If I_2 is a J-hyperideal of R_2 such that $Ker\phi \subseteq J(R_1)$, then $\phi^{-1}(I_2)$ is a J-hyperideal of R_1.*

(ii) *If the $\mathbf{C}$-hyperideal I_1 is a J-hyperideal of R_1 such that $Ker\phi \subseteq I_1$, then $\phi(I_1)$ is a J-hyperideal of R_2.*

Proof. (i) Let I_2 is a J-hyperideal of R_2. Suppose that $x \circ_1 y \subseteq \phi^{-1}(I_2)$ for some $x, y \in R_1$ such that $x \notin J(R_1)$. This implies that $\phi(x) \circ_2 \phi(y) = \phi(x \circ_1 y) \subseteq I_2$. Let M be a maximal hyperideal of R_1 and $\phi(x) \in J(R_2)$. Then $\phi(M)$ is a maximal hyperideal of R_2 which implies $\phi(x) \in \phi(M)$. Since $Ker\phi \subseteq M$, then we have $x \in M$ and so $x \in J(R_1)$, a contradiction. Therefore $\phi(x) \notin J(R_2)$. Now, we have $\phi(y) \in I_2$ as I_2 is a J-hyperideal of R_2. Then we conclude that $y \in \phi^{-1}(I_2)$. Thus, $\phi^{-1}(I_2)$ is a J-hyperideal of R_1.

(ii) Let $x_2 \circ_2 y_2 \subseteq \phi(I_1)$ for some $x_2, y_2 \in R_2$ such that $x_2 \notin J(R_2)$. Then for some $x_1, y_1 \in R_1$ we have $\phi(x_1) = x_2$ and $\phi(y_1) = y_2$. So $\phi(x_1) \circ_2 \phi(y_1) = \phi(x_1 \circ_1 y_1) \subseteq \phi(I_1)$. Now, take any $u \in x_1 \circ y_1$. Then $\phi(u) \in \phi(x_1 \circ y_1) \subseteq \phi(I_1)$ and so there exists $w \in I_1$ such that $\phi(u) = \phi(w)$. This means $\phi(u - w) = 0$, that is, $u - w \in Ker\phi \subseteq I_1$ and then $u \in I_1$. Since I_1 is a $\mathbf{C}$-hyperideal of R_1, then we get $x_1 \circ_1 y_1 \subseteq I_1$. Since $\phi(J(R_1)) \subseteq J(R_2)$, then $x_1 \notin J(R_1)$. Hence, we have $y_1 \in I_1$ as I_1 is a J-hyperideal of R_1. Thus, $y_2 = \phi(y_1) \in \phi(I_1)$. It follows that $\phi(I_1)$ is a J-hyperideal of R_2. □

Corollary 3.18. *Let H and I be $\mathbf{C}$-hyperideals of R such that $H \subseteq I$.*

(i) *If I is a J-hyperideal of R, then the hyperideal I/H of R/H is a J-hyperideal.*

(ii) *If the hyperideal I/H of R/H is a J-hyperideal such that $H \subseteq J(R)$, then I is a J-hyperideal of R.*

(iii) *If H is a J-hyperideal of R and the hyperideal I/H of R/H is a J-hyperideal, then I is a J-hyperideal of R.*

Proof. (i) Define the natural epimorphism $\pi : R \longrightarrow R/H$ by $\pi(x) = x + H$. Since $Ker\pi \subseteq I$, we conclude that $\pi(I) = I/H$ is a J-hyperideal of R/H, by Theorem 3.17 (ii).

(ii) Let us consider the natural epimorphism mentioned in (i). Since $H \subseteq J(R)$, we conclude that $\pi^{-1}(I/H)$ is a J-hyperideal of R by Theorem 3.17 (i). Consequently, I is a J-hyperideal of R.

(iii) This can be proved by using (ii) and Proposition 3.4. □

Definition 3.19. A non-empty subset S of R containing $R - J(R)$ is called a J-multiplicatively closed subset if $x \circ y \subseteq S$ for every $x \in R - J(R)$ and every $y \in S$.

Example 3.20. *Suppose that $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$. Then $(\mathbb{Z}_4, +, \circ)$ is a multiplicative hyperring with "$+$" and "$\circ$" defined by*

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

$\circ$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\{\bar{0}\}$	$\{\bar{0}\}$	$\{\bar{0}\}$	$\{\bar{0}\}$
$\bar{1}$	$\{\bar{0}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{2}\}$	$\{\bar{1}, \bar{3}\}$
$\bar{2}$	$\{\bar{0}\}$	$\{\bar{2}\}$	$\{\bar{0}\}$	$\{\bar{2}\}$
$\bar{3}$	$\{\bar{0}\}$	$\{\bar{1}, \bar{3}\}$	$\{\bar{2}\}$	$\{\bar{1}, \bar{3}\}$

In the hyperring, $S = \{\bar{1}, \bar{3}\}$ is a J-multiplicatively closed subset.

Proposition 3.21. *Let I be a proper hyperideal of R. Then I is a J-hyperideal if and only if $R - I$ is a J-multiplicatively closed subset of R.*

Proof. $\Longrightarrow$ Let I be a proper hyperideal of R. By Proposition 3.4, we conclude that $R - J(R) \subseteq R - I$. Suppose that $x \in R - J(R)$ and $y \in R - I$. To establish the claim, suppose, on the contrary, that $x \circ y \subseteq I$. From $x \in R - J(R)$ it follows that $y \in I$ as I is a J-hyperideal of R. This is a contradiction. Hence $x \circ y \subseteq R - I$ as needed.

$\Longleftarrow$ Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin J(R)$. If $y \notin I$, then we conclude that $x \circ y \subseteq R - I$. Thus, we arrive at a contradiction. Therefore $y \in I$ and so I is a J-hyperideal. $\square$

Theorem 3.22. *Let S be a J-multiplicatively closed subset of R and I be a hyperideal of R disjoint from S. Then there exists a hyperideal Q which is maximal in the set of all hyperideals of R disjoint from S, containing I. Any such hyperideal Q is a J-hyperideal of R.*

Proof. Let Υ be the set of all hyperideals of R disjoint from S, containing I. From $I \in \Upsilon$ it follows that $\Upsilon \neq \varnothing$. Υ is a partially ordered set with respect to set inclusion relation. Assume that $I_1 \subseteq I_2 \subseteq \cdots$ is some chain in Υ. Clearly, $\cup_{i=1}^{\infty} I_i$ is a hyperideal of R such that $(\cup_{i=1}^{\infty} I_i) \cap S = \varnothing$. Thus $\cup_{i=1}^{\infty} I_i$ is an upper bound of the mentioned chain. By Zorn's lemma, there is a hyperideal Q which is maximal in Υ. Suppose that Q is not a J-hyperideal of R. Let $x \circ y \subseteq Q$ such that $x \in R - J(R)$ and $y \in R - Q$. Therefore we have $Q \subsetneqq (Q : x)$. Since Q is a maximal element of Υ, then $(Q : x) \cap S \neq \varnothing$. Let $t \in (Q : x) \cap S$. Then $t \circ x \subseteq Q \cap S$ and so $Q \cap S \neq \varnothing$. Thus we arrive at a contradiction. Consequently, Q is a J-hyperideal of R. $\square$

Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two hyperrings with nonzero identity. Recall $(R_1 \times R_2, +, \circ)$ is a hyperring with the operation $+$ and the hyperoperation $\circ$ are defined respectively as $(x_1, x_2) + (y_1, y_2) = (x_1 +_1 y_1, x_2 +_2 y_2)$ and $(x_1, x_2) \circ (y_1, y_2) = (x_1 \circ_1 y_1) \times (x_2 \circ_2 y_2)$ [23].

Theorem 3.23. *Let R_1 and R_2 be two hyperrings with nonzero identity. Then $R_1 \times R_2$ has no J-hyperideals.*

Proof. Let $I_1 \times I_2$ is a J-hyperideal of $R_1 \times R_2$ for some hyperideals I_1, I_2 of R_1, R_2. Since $(0,0) \in (1,0) \circ (0,1) \cap I_1 \times I_2$, then $(1,0) \circ (0,1) \subseteq I_1 \times I_2$. Let $(1,0), (0,1) \in J(R_1 \times R_2)$. Then $(1,0) = (1,1) - (0,1)$ and $(0,1) = (1,1) - (1,0)$ are unit elements of $R_1 \times R_2$. Therefore we conclude that $(1,0), (0,1)$ are not in $J(R_1 \times R_2)$. Thus, $(1,0), (0,1)$ are in $I_1 \times I_2$. So, $(1,1) = (1,0) + (0,1) \in I_1 \times I_2$ and so $I_1 \times I_2 = R_1 \times R_2$. □

Definition 3.24. A proper hyperideal I of R is said to be *J-primary* if $x, y \in R$ and $x \circ y \subseteq I$, then $x \in J(I)$ or $y \in I$.

Example 3.25. *Consider the set of integers $\mathbb{Z}$ with ordinary addition " $+$ " and hyperoperation " $\circ$ " defined by $a \circ b = \{2ab, 3ab\}$. Then $(\mathbb{Z}, +, \circ)$ is a multiplicative hyperring. $\langle 2\rangle$ and $\langle 3\rangle$ are J-primary hyperideals of $\mathbb{Z}$. The hyperideal $\langle 6\rangle$ is not J-primary. In fact, $2 \circ 3 \subseteq \langle 6\rangle$ but $2, 3 \notin \langle 6\rangle$ and $2, 3 \notin J(\langle 6\rangle)$.*

Theorem 3.26. *Let I be a hyperideal of R with $I \subseteq J(R)$. Then I is a J-hyperideal if and only if I is J-primary.*

Proof. $\Longrightarrow$ Let I be a J-hyperideal of R. Suppose that $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin J(I)$. This means $x \notin J(R)$ as $J(R) \subseteq J(I)$. Since I is a J-hyperideal of R and $x \notin J(R)$, we get $y \in I$. Consequently, the hyperideal I is J-primary.

$\Longleftarrow$ Suppose that the hyperideal I of R is J-primary. Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin J(R)$. By the hypothesis, we have $J(I) \subseteq J(R)$. This implies that $x \notin J(I)$. Since I is a J-primary hyperideal of R and $x \notin J(I)$, then we get $y \in I$. Thus, I is a J-hyperideal. □

4. quasi J-hyperideals

In this section, we define the concept of quasi J-hyperideals as a generalization of J-hyperideals.

Definition 4.1. A proper hyperideal I of R is called a *quasi J-hyperideal* if $\sqrt{I}$ is a J-hyperideal.

Example 4.2. *In Example 3.2, the hyperideals $\langle 2\rangle$ and $\langle 3\rangle$ are quasi J-hyperideals. The hyperideal $\langle 12\rangle$ is not a quasi J-hyperideal. In fact, $3 \circ 4 = \{24, 36\} \subseteq \langle 12\rangle$, but $3, 4 \notin \sqrt{\langle 12\rangle} = \langle 2\rangle \cap \langle 3\rangle$ and $3, 4 \notin J(\mathbb{Z})$.*

Proposition 4.3. *Let I be a quasi J-hyperideal of R. If $x \circ H \subseteq I$ for some hyperideal H of R and some element $x \in R$, then $x \in J(R)$ or $H \subseteq \sqrt{I}$.*

Proof. Let I be a quasi J-hyperideal of R. Suppose that $x \circ H \subseteq I$ for some hyperideal H of R and some element $a \in R$ such that $x \notin J(R)$. By Theorem 3.11, we get $\sqrt{I} = (\sqrt{I} : x)$. Since $H \subseteq (I : x) \subseteq (\sqrt{I} : x)$, then $H \subseteq \sqrt{I}$. □

Theorem 4.4. *Let R be a hyperring and I be a proper hyperideal of R. Then the following are equivalent:*

(i) *I is a quasi J-hyperideal of R.*

(ii) *If $H \circ T \subseteq I$ for some hyperideals H and T of R, then $H \subseteq J(R)$ or $T \subseteq \sqrt{I}$.*

(iii) *If $x \circ y \subseteq I$ for some $x, y \in R$, then $x \in J(R)$ or $y \in \sqrt{I}$.*

Proof. (i)$\Longrightarrow$ (ii) Let $H \circ T \subseteq I$ for some hyperideals H and T of R such that $H \nsubseteq J(R)$. Take $x \in H$ such that $x \notin J(R)$. Clearly, $x \circ T \subseteq I$. Since I is a quasi J-hyperideal of R and $x \notin J(R)$, then $T \subseteq \sqrt{I}$, by Theorem 4.3.

(ii) $\Longrightarrow$ (iii) Let $x \circ y \subseteq I$ for some $x, y \in R$. Put $H = \langle x \rangle$ and $T = \langle y \rangle$. Hence, $\langle x \rangle \circ \langle y \rangle \subseteq \langle x \circ y \rangle \subseteq I$. By the assumption, we get $x \in \langle x \rangle \subseteq J(R)$ or $y \in \langle y \rangle \subseteq I \subseteq \sqrt{I}$.

(iii) $\Longrightarrow$ (i) Let $x \circ y \subseteq \sqrt{I}$ for some $x, y \in R$ such that $x \notin J(R)$. This means we have $(x \circ y)^n = x^n \circ y^n \subseteq I$ for some $n \in \mathbb{N}$. Since $x \notin J(R)$, we obtain $x^n \nsubseteq J(R)$. Take $t \in x^n - J(R)$. If $y \notin \sqrt{I}$, then $y^n \nsubseteq I$ for all $n \in \mathbb{N}$. Assume that $s \in y^n - I$. Since $t \circ s \subseteq I$ and $s \notin \sqrt{I}$, we conclude that $t \in J(R)$ which is a contradiction. Thus $y \in \sqrt{I}$ as needed. □

Theorem 4.5. *Let I be a proper hyperideal of R. Then I is a quasi J-hyperideal of R if and only if $I \subseteq J(R)$ and for $x, y \in R$, $x \circ y \subseteq I$ implies that $x \in J(I)$ or $y \in \sqrt{I}$.*

Proof. $\Longrightarrow$ Let $\sqrt{I}$ be a J-hyperideal of R. By Proposition 3.4, we have $\sqrt{I} \subseteq J(R)$. Since $I \subseteq \sqrt{I}$, we get $I \subseteq J(R)$. Let $x \circ y \subseteq I$ for some $x, y \in R$. Then we have $x \circ y \subseteq \sqrt{I}$. Since $\sqrt{I}$ is a J-hyperideal of R, we obtain $x \in J(R) \subseteq J(I)$ or $y \in \sqrt{I}$.

$\Longleftarrow$ Let $x \circ y \subseteq I$ such that $x \notin J(R)$. By the hypothesis, we get $J(I) \subseteq J(R)$ which implies $x \notin J(I)$. Hence, we have $y \in \sqrt{I}$. We conclude that the hyperideal I of R is a quasi J-hyperideal. □

Lemma 4.6. *Let I be a hyperideal of R and S be a subset of R with $S \nsubseteq J(R)$. If I is a quasi J-hyperideal of R, then $(\sqrt{I} : S) \subseteq \sqrt{(I : S)}$.*

Proof. Let $x \in (\sqrt{I} : S)$. Then $x \circ S \subseteq \sqrt{I}$. Since $x \circ S = \bigcup_{y \in S} x \circ y$, we have $x \circ y \subseteq \sqrt{I}$ for every $y \in S$. Since $S \nsubseteq J(R)$, then there exists $z \in S - J(R)$ such that $x \circ z \subseteq \sqrt{I}$. Since $\sqrt{I}$ is a J-hyperideal of R, we get $x \in \sqrt{I}$. Therefore for some $n \in \mathbb{N}$, $x^n \subseteq I$. This implies that $x^n \subseteq (I : S)$ and so $x \in \sqrt{(I : S)}$. Consequently, $(\sqrt{I} : S) \subseteq \sqrt{(I : S)}$. □

Theorem 4.7. *Let I be a* **C***-hyperideal of R and S be a subset of R such that $S \nsubseteq J(R)$. If the hyperideal I of R is a quasi J-hyperideal, then so is $(I : S)$.*

Proof. It is clear that $(I : S)$ is a proper hyperideal of R. Let $x \circ y \subseteq (I : S)$ for some $x, y \in R$ such that $x \notin J(R)$. Then $x \circ y \circ S \subseteq I$. Now we have $x \circ r \subseteq I$ for all $r \in y \circ S$. By Theorem 4.4, we get $r \in \sqrt{I}$. Since $y \circ S \cap \sqrt{I} \neq \varnothing$ and I is a **C**-hyperideal, then we conclude that $y \circ S \subseteq \sqrt{I}$. This means $y \in (\sqrt{I} : S)$ which implies $y \in \sqrt{(I : S)}$, by Lemma 4.6. Thus, $(I : S)$ is a quasi J-hyperideal of R. □

Theorem 4.8. *Let I be a* **C***-hyperideal of R. Let I be a quasi J-hyperideal such that there is no quasi J-hyperideal which contains I properly. Then the hyperideal I of R is a J-hyperideal.*

Proof. Let $x \circ y \subseteq I$ for some $x, y \in R$ such that $x \notin J(R)$. By Theorem 4.7, we conclude that $(I : x)$ is a quasi J-hyperideal of R. Since $I \subseteq (I : x)$, then $I = (I : x)$, by the hypothesis. Thus, we get $y \in I$. Consequently, the hyperideal I of R is a J-hyperideal. □

Corollary 4.9. *Let $J(R)$ be a* **C***-hyperideal of R. $J(R)$ is a J-hyperideal if and only if $J(R)$ is a quasi J-hyperideal.*

Proposition 4.10. *Let M be a maximal hyperideal of R. If $\langle a \circ b \rangle$ is a quasi J-hyperideal of R for each $a, b \in R$, then so is M.*

Proof. Straightforward. □

Theorem 4.11. *Let R be a hyperring. Then every maximal hyperideal of R is a quasi J-hyperideal if and only if R is a local hyperring.*

Proof. $\Longrightarrow$ Let the maximal hyperideal M of R be a quasi J-hyperideal. Then $\sqrt{M}$ is a J-hyperideal of R. By Proposition 3.4, we get $\sqrt{M} \subseteq J(R)$. Since M is maximal, then M is a prime hyperideal of R by Proposition 2.18 in [9]. Hence $M = \sqrt{M}$ and so $M \subseteq J(R)$. Since we have $J(R) \subseteq M$, the assertion follows.

$\Longleftarrow$ Let R be a local hyperring. By Theorem 3.6, every proper hyperideal of R is a J-hyperideal and so every proper hyperideal of R is a quasi J-hyperideal. Now, the claim follows by Proposition 4.10. □

Recall from [7] that a proper hyperideal I of R is said to be a quasi primary hyperideal if $\sqrt{I}$ is prime.

Theorem 4.12. *Let R be a hyperring such that every prime hyperideal of R is maximal. If the hyperideal I of R is a quasi J-hyperideal and $I \subseteq J(R)$, then I is a quasi primary hyperideal of R.*

Proof. We show that $\sqrt{I}$ is prime. Let $x \circ y \subseteq \sqrt{I}$ for some $x, y \in R$ such that $x \notin \sqrt{I}$. Hence we have $x^n \circ y^n \subseteq I$ for some $n \in \mathbb{N}$. By the hypothesis, we conclude that $\sqrt{I} = J(R)$. Since $x^n \notin J(R)$ and the hyperideal I of R is a quasi J-hyperideal, then $y^n \subseteq \sqrt{I}$, by Theorem 4.4. This means $y \in \sqrt{I}$. Thus I is a quasi primary hyperideal of R. □

Theorem 4.13. *Let R be a hyperring. Let every prime hyperideal of R be maximal and I be a hyperideal of R such that $I \subseteq J(R)$. Then I is a quasi J-hyperideal if and only if R is local with the maximal hyperideal $\sqrt{I}$.*

Proof. $\Longrightarrow$ Let I be a quasi J-hyperideal of R. Hence $\sqrt{I}$ is a prime hyperideal of R, by Theorem 4.12. By the hypothesis, $\sqrt{I}$ is a maximal hyperideal of R. Then there exists some prime hyperideal Q such that $I = Q^n$ for some $n \in \mathbb{N}$. This means $\sqrt{I} = Q$ is maximal and so $J(R) \subseteq \sqrt{I} = Q$. Since $I \subseteq J(R)$, then we conclude that $\sqrt{I} = J(R)$. Thus the hyperring R is local with the maximal hyperideal $\sqrt{I}$.

$\Longleftarrow$ The claim follows by Theorem 4.11. □

Proposition 4.14. (i) *If* $\mathbf{C}$*-hyperideal* I_i *is a quasi* J*-hyperideal of* R *for each* $1 \leq i \leq n$*, then is so* $\bigcap_{i=1}^n I_i$.

(ii) *If* I_i *is a quasi* J*-hyperideal of* R *for each* $1 \leq i \leq n$*, then is so* $\Pi_{i=1}^n I_i$.

Proof. (i) By Proposition 3.3. in [9], we have $\sqrt{\bigcap_{i=1}^n I_i} = \bigcap_{i=1}^n \sqrt{I_i}$. Since $\sqrt{I_i}$ is a J-hyperideal of R for each $1 \leq i \leq n$, we conclude that $\bigcap_{i=1}^n \sqrt{I_i}$ is a J-hyperideal of R, by Theorem 3.10. Thus, $\sqrt{\bigcap_{i=1}^n I_i}$ is a J-hyperideal of R and so $\bigcap_{i=1}^n I_i$ is a quasi J-hyperideal of R.

(ii) Let $x \circ y \subseteq \Pi_{i=1}^n I_i$ for some $x, y \in R$ such that $x \notin J(R)$. Since I_i is a quasi J-hyperideal of R for each $1 \leq i \leq n$ and $\Pi_{i=1}^n I_i \subseteq \bigcap_{i=1}^n I_i$, we conclude that $y \in \sqrt{I_i}$ for each $1 \leq i \leq n$. This means $y^{t_i} \subseteq I_i$ for each $1 \leq i \leq n$. Put $t = t_1 + t_2 + ... + t_n$. Therefore, $y^t = y^{t_1} \circ y^{t_2} \circ ... \circ y^{t_n} \subseteq \Pi_{i=1}^n I_i$ which implies $y \in \sqrt{\Pi_{i=1}^n I_i}$. Consequently, the hyperideal $\Pi_{i=1}^n I_i$ of R is a quasi J-hyperideal. □

5. **2-absorbing J-hyperideals**

In this section , we extend the notion of J-hyperideals to 2-absorbing J-hyperideals and give some properties of them.

Definition 5.1. Let I be a proper hyperideal of R. I is called a *2-absorbing J-hyperideal* of R if $x, y, z \in R$ with $x \circ y \circ z \subseteq I$, then $x \circ y \subseteq I$ or $x \circ z \subseteq J(R)$ or $y \circ z \subseteq J(R)$.

Example 5.2. *(1) In Example 3.2, let* $A = \{2, 4\}$*. Then the principal hyperideal* $\langle 15 \rangle$ *is a 2-absorbing* J*-hyperideal.*

(2) Let $\mathbb{Z}_6$ *be the ring* $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ *under addition and multiplication modulo* 6*. We define the hyperoperation* $\bar{x} \odot \bar{y} = \{\bar{xy}, 2\bar{xy}, 3\bar{xy}, 4\bar{xy}, 5\bar{xy}\}$*, for all* $\bar{x}, \bar{y} \in \mathbb{Z}_6$*. In the commutative multiplicative hyperring* $(\mathbb{Z}, +, \odot)$*, hyperideal* $\{\bar{0}\}$ *is a 2-absorbing* J*-hyperideal.*

Example 5.3. *Consider the hyperring* R *in Example 3.3. The hyperideal* I^2 *is a 2-absorbing* J*-hyperideal of* R*.*

Theorem 5.4. *If* I *is a* J*-hyperideal of* R *and* $J(R)$ *is a* $\mathbf{C}$*-hyperideal of* R*, then* I *is a 2-absorbing* J*-hyperideal of* R*.*

Proof. Let I be a J-hyperideal of R. Suppose that $x \circ y \circ z \subseteq I$ for some $x, y, z \in R$. Choose $u \in x \circ z$. Since $y \circ u \subseteq I$ and I is a J-hyperideal of R, we get $y \in I$ or $u \in J(R)$. In the former case, we have $y \circ z \subseteq I$ and so

$y \circ z \subseteq J(R)$, by Proposition 3.4. In the second case, we obtain $x \circ z \subseteq J(R)$ as $J(R)$ is a **C**-hyperideal of R. □

Proposition 5.5. *If I is a 2-absorbing J-hyperideal of R, then $I \subseteq J(I)$.*

Proof. Let I be a 2-absorbing J-hyperideal of R. We suppose that $I \nsubseteq J(I)$ and look for a contradiction. From $I \nsubseteq J(I)$ it follows that there exists $x \in I$ such that $x \notin J(R)$. Since $1 \circ 1 \circ x \subseteq I$ and I is a 2-absorbing J-hyperideal of R, we have $1 \circ 1 \subseteq I$, a contradiction or $x \in 1 \circ x \subseteq J(R)$, a contradiction. Thus, $I \subseteq J(I)$. □

Recall from [6] that a proper hyperideal I of R is said to be an 2-absorbing primary hyperideal if $x \circ y \circ z \subseteq I$ implies $x \circ y \subseteq I$ or $x \circ z \subseteq \sqrt{I}$ or $y \circ z \subseteq \sqrt{I}$ for any $x, y, z \in R$.
The next theorem shows that the inverse of Proposition 5.5 is true if I is a 2-absorbing primary hyperideal of R

Theorem 5.6. *Let I be a 2-absorbing primary hyperideal of R and $I \subseteq J(R)$. Then I is a 2-absorbing J-hyperideal of R.*

Proof. Suppose that I is a 2-absorbing primary hyperideal of R such that $I \subseteq J(R)$. Let $x \circ y \circ z \subseteq I$ for some $x, y, z \in R$ such that $x \circ z, y \circ z \nsubseteq J(R)$. Then we conclude that $x \circ z, y \circ z \nsubseteq \sqrt{I}$. This implies that $x \circ y \subseteq I$ as I is a 2-absorbing primary hyperideal of R. Consequently, I is a 2-absorbing J-hyperideal of R. □

Theorem 5.7. *Let the hyperring R has at most two maximal hyperideals. Suppose that the hyperideal I of R is 2-absorbing primary that is not quasi primary. Then I is a 2-absorbing J-hyperideal of R.*

Proof. Let the hyperideal I of R be 2-absorbing primary. Then $\sqrt{I}$ is prime or $\sqrt{I} = P_1 \cap P_2$ for some prime hyperideals P_1 and P_2 of R, by Theorem 4.5 in [6]. In the former case, we have a contradiction since I is not a quasi primary hyperideal of R. Therefore, we obtain $I \subseteq \sqrt{I} \subseteq J(R)$ as the hyperring R has at most two maximal hyperideals. Thus, I is a 2-absorbing J-hyperideal of R, by Theorem 5.6. □

Corollary 5.8. *Let R_1 and R_2 be two local hyperrings. Then every 2-absorbing primary hyperideal of $R_1 \times R_2$ that is not quasi primary is a 2-absorbing J-hyperideal.*

Theorem 5.9. *Let I be a 2-absorbing J-hyperideal of R. Then for each $a, b \in I$ with $a \circ b \nsubseteq I$, $(I : a \circ b) \subseteq (J(R) : a)$ or $(I : a \circ b) \subseteq (J(R) : b)$.*

Proof. Let I be a 2-absorbing J-hyperideal of R. Let $c \in (I : a \circ b)$ for some $a, b \in R$ with $a \circ b \nsubseteq I$. This means $a \circ b \circ c \subseteq I$. Since the hyperideal I of R is 2-absorbing J-hyperideal and $a \circ b \nsubseteq I$, then $a \circ c \subseteq J(R)$ of $b \circ c \subseteq J(R)$ and so $c \in (J(R) : a)$ or $c \in (J(R) : b)$. This implies that $(I : a \circ b) \subseteq (J(R) : a)$ or $(I : a \circ b) \subseteq (J(R) : b)$. □

Recall from [10] that a hyperideal I of R is called a strong $\mathbf{C}$-hyperideal if for any $E \in \mathfrak{U}$, $E \cap I \neq \varnothing$, then $E \subseteq I$, where $\mathfrak{U} = \{\sum_{i=1}^{n} A_i \ : \ A_i \in \mathbf{C}, n \in \mathbb{N}\}$ and $\mathbf{C} = \{r_1 \circ r_2 \circ ... \circ r_n \ : \ r_i \in R, n \in \mathbb{N}\}$.

Theorem 5.10. *Let I be a proper strong $\mathbf{C}$-hyperideal of R. Then the following are equivalent:*

(i) *I is a 2-absorbing J-hyperideal of R.*

(ii) *If $a \circ b \circ H \subseteq I$ for some $a, b \in R$ and some hyperideal H of R, then $a \circ b \subseteq I$ or $a \circ H \subseteq J(R)$ or $b \circ H \subseteq J(R)$.*

(iii) *If $a \circ H \circ T \subseteq I$ for some $a \in R$ and some hyperideals H, T of R, then $a \circ H \subseteq I$ or $a \circ T \subseteq J(R)$ or $H \circ T \subseteq J(R)$.*

(iv) *If $K \circ H \circ T \subseteq I$ for some hyperideals K, H, T of R, then $K \circ H \subseteq I$ or $K \circ T \subseteq J(R)$ or $H \circ T \subseteq J(R)$.*

Proof. $(i) \Longrightarrow (ii)$ Let $a \circ b \circ H \subseteq I$ for some $a, b \in R$ and some hyperideal H of R such that $a \circ b \nsubseteq I$. Hence, we get $H \subseteq (J(R) : a)$ or $H \subseteq (J(R) : b)$, by Theorem 5.9. This implies that $a \circ H \subseteq J(R)$ or $b \circ H \subseteq J(R)$.

$(ii) \Longrightarrow (iii)$ Let $a \circ H \circ T \subseteq I$ for some $a \in R$ and some hyperideals H, T of R such that $a \circ H \nsubseteq I$ and $a \circ T \nsubseteq J(R)$. Therefore, we get $a \circ h_1 \nsubseteq I$ and $a \circ t_1 \nsubseteq J(R)$ for some $h_1 \in H$ and $t_1 \in T$. Since $a \circ h_1 \circ T \subseteq I$, then $h_1 \circ T \subseteq J(R)$, by (i). Let $h_2 \in H$. Suppose that $h_2 \circ T \nsubseteq J(R)$. By (i), we get $a \circ h_2 \subseteq I$ as $a \circ h_2 \circ T \subseteq I$. Hence $a \circ (h_1 + h_2) \nsubseteq I$. Again by (i), we obtain $(h_1 + h_2) \circ T \subseteq J(R)$ as $a \circ (h_1 + h_2) \circ T \subseteq I$. As $h_1 \circ T \subseteq J(R)$, $h_2 \circ T \subseteq J(R)$ and so $H \circ T \subseteq J(R)$.

$(iii) \Longrightarrow (iv)$ Let $K \circ H \circ T \subseteq I$ for some hyperideals K, H, T of R such that $K \circ H \nsubseteq I$ and $H \circ T \nsubseteq J(R)$. Then we have $k \circ H \nsubseteq I$ for some $k \in K$. It is clear that $k \circ H \circ T \subseteq I$. Hence we have $k \circ T \subseteq J(R)$, by (iii). Assume that $x \in K$. By (iii), we conclude that $x \circ H \subseteq I$ or $x \circ T \subseteq J(R)$ as $x \circ H \circ T \subseteq I$. If $x \circ H \subseteq I$, then $(k + x) \circ H \subseteq (k \circ H) + (x \circ H) \nsubseteq I$. Clearly, $(k + x) \circ H \circ T \subseteq I$. Then $(k + x) \circ T \subseteq J(R)$. Since I is a strong $\mathbf{C}$-hyperideal of R, then $(k \circ T) + (x \circ T) \subseteq J(R)$. Hence, $x \circ T \subseteq J(R)$ as $k \circ T \subseteq J(R)$. Now, let $x \circ H \nsubseteq I$. It is obvious that $x \circ H \circ T \subseteq I$. Therefore, $x \circ T \subseteq J(R)$, by (iii). Consequently, $K \circ T \subseteq J(R)$.

$(iv) \Longrightarrow (i)$ Let $x \circ y \circ z \subseteq I$ for some $x, y, z \in R$ such that $x \circ z \nsubseteq J(R)$ and $y \circ z \nsubseteq J(R)$. We consider $K = \langle x \rangle$, $H = \langle y \rangle$ and $T = \langle z \rangle$. Thus, we conclude that $K \circ T = \langle x \rangle \circ \langle z \rangle \subseteq \langle x \circ z \rangle \nsubseteq J(R)$ and $H \circ T = \langle y \rangle \circ \langle z \rangle \subseteq \langle y \circ z \rangle \nsubseteq J(R)$. Since $K \circ H \circ T \subseteq I$, by (iv), we get $K \circ H \subseteq I$ and so $x \circ y \subseteq I$ and that completes the proof. $\square$

Theorem 5.11. *If every proper hyperideal of R is a 2-absorbing J-hyperideal, then R is a local hyperring.*

Proof. Let every proper hyperideal of R be a 2-absorbing J-hyperideal. Suppose that M is a maximal hyperideal of R. We show that $M \subseteq J(R)$. Let $a \in M$ and $I = \langle a \rangle$. Since I is a 2-absorbing J-hyperideal of R, then $I \subseteq J(R)$

by Proposition 5.5. Therefore, $a \in J(R)$ and so $M \subseteq J(R)$. Thus $J(R) = M$ as $J(R) \subseteq M$. Consequently, R is a local hyperring. □

Proposition 5.12. *Let $\{I_i\}_{i\in\Delta}$ be a non-empty set of 2-absorbing J-hyperideals of R. Then $\bigcap_{i\in\Delta} I_i$ is a 2-absorbing J-hyperideal of R.*

Proof. Let $x \circ y \circ z \subseteq \bigcap_{i\in\Delta} I_i$ for some $x, y, z \in R$ such that $x \circ z \nsubseteq J(R)$ and $y \circ z \nsubseteq J(R)$. This implies that $x \circ y \circ z \subseteq I_i$ for every $i \in \Delta$. Since I_i is a 2-absorbing J-hyperideal of R for every $i \in \Delta$, we get the result that $x \circ y \subseteq I_i$ and so $x \circ y \subseteq \bigcap_{i\in\Delta} I_i$. Thus $\bigcap_{i\in\Delta} I_i$ is a 2-absorbing J-hyperideal of R. □

Definition 5.13. A proper hyperideal I of R is called *2-absorbing J-primary* if elements $x, y, z \in R$ and $x \circ y \circ z \subseteq I$, then $x \circ y \subseteq I$ or $x \circ z \in J(I)$ or $y \circ z \in J(I)$.

Example 5.14. *In Example 3.2, let $A = \{3, 4\}$. Then the hyperideal $\langle 8\rangle$ is a 2-absorbing J-primary of $\mathbb{Z}_A$.*

Theorem 5.15. *Let I be a hyperideal of R . Then I is a 2 -absorbing J-hyperideal of R if and only if I is a 2-absorbing J-primary hyperideal of R with $J(I) = J(R)$.*

Proof. $\Longrightarrow$ Let I be a 2 -absorbing J-hyperideal of R. Suppose that $x, y, z \subseteq I$ for some $x, y, z \in R$. This implies that $x \circ y \subseteq I$ or $x \circ z \subseteq J(R)$ or $y \circ z \subseteq J(R)$ as I is a 2 -absorbing J-hyperideal of R. Since $J(R) \subseteq J(I)$, we have $x \circ y \subseteq I$ or $x \circ z \subseteq J(I)$ or $y \circ z \subseteq J(I)$. Thus I is a 2-absorbing J-primary hyperideal of R. It is clear that $J(R) \subseteq J(I)$. For the reverse inclusion, we have $I \subseteq J(R)$ by Proposition 5.5. Thus $J(I) \subseteq J(R)$ and so $J(I) = J(R)$.

$\Longleftarrow$ Let I be a 2-absorbing J-primary hyperideal of R such that $J(I) = J(R)$. Suppose that $x, y, z \subseteq I$ for some $x, y, z \in R$. This implies that $x \circ y \subseteq I$ or $x \circ z \subseteq J(I)$ or $y \circ z \subseteq J(I)$ as I is a 2-absorbing J-primary hyperideal of R. Since $J(I) = J(R)$, we get $x \circ y \subseteq I$ or $x \circ z \subseteq J(R)$ or $y \circ z \subseteq J(R)$. This means I is a 2 -absorbing J-hyperideal of R. □

Theorem 5.16. *Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two hyperrings and let I_1 and I_2 be hyperideals of R_1 and R_2, respectively. Then $I_1 \times I_2$ is a 2-absorbing J-hyperideal of $R_1 \times R_2$ if and only if I_1 is a J-hyperideal of R_1 and I_2 is a J-hyperideal of R_2.*

Proof. $\Longrightarrow$ Let $I_1 \times I_2$ is a 2-absorbing J-hyperideal of $R_1 \times R_2$. Suppose that $a \circ_1 b \subseteq I_1$ such that $b \notin J(R_1)$. Therefore $(a, 1) \circ (1, 0) \circ (b, 1) = \{(x, y) \mid x \in a \circ_1 1 \circ_1 b, y \in 1 \circ_2 0 \circ_2 1\} \subseteq I_1 \times I_2$, $(a, 1) \circ (b, 1) = \{(x', y') \mid x' \in a \circ_1 b, y' \in 1 \circ_2 1\} \nsubseteq J(R_1 \times R_2)$ and $(b, 0) \in (1, 0) \circ (b, 1) \nsubseteq J(R_1 \times R_2)$. Hence we get $(a, 0) \in (a, 1) \circ (1, 0) \subseteq I_1 \times I_2$ and so $a \in I_1$. Consequently, I_1 is a J-hyperideal of R_1. By a similar argument, we can prove that I_2 is a J-hyperideal of R_2.

$\Longleftarrow$ Let I_1 be a J-hyperideal of R_1 and I_2 is a J-hyperideal of R_2. Suppose that $(a_1, b_1) \circ (a_2, b_2) \circ (a_3, b_3) \subseteq I_1 \times I_2$ such that $(a_2, b_2) \circ (a_3, b_3) \nsubseteq J(R_1 \times R_2)$. Take $a \in a_2 \circ_1 a_3$. Then $a_1 \circ_1 a \subseteq a_1 \circ_1 a_2 \circ_1 a_3 \subseteq I_1$. We assume that

$a_2 \circ_1 a_3 \nsubseteq J(R_1)$. So $a \notin J(R_1)$. Since I_1 is a J-hyperideal of R_1, $a_1 \in I_1$ and so $a_1 \circ_1 a_2 \subseteq I_1$. On the other hand, if $b_1 \circ_2 b_3 \subseteq J(R_2)$, then $(a_1, b_1) \circ (a_3, b_3) \subseteq I_1 \times J(R_2) \subseteq J(R_1 \times R_2)$, as needed. Then we may assume $b_1 \circ b_3 \nsubseteq J(R_2)$. Take $b \in b_1 \circ_2 b_3$. Then $b_2 \circ_2 b \subseteq I_2$. Since I_2 is a J-hyperideal of R_2 and $b \notin J(R_2)$, we have $b_2 \in I_2$ and so $b_1 \circ_2 b_2 \subseteq I_2$. This $(a_1, b_1) \circ (a_2, b_2) \subseteq I_1 \times I_2$. This implies that $I_1 \times I_2$ is a 2-absorbing J-hyperideal of $R_1 \times R_2$. □

Theorem 5.17. *Let $(R_1, +_1, \circ_1)$, $(R_2, +_2, \circ_2)$ and $(R_3, +_3, \circ_3)$ be three hyperrings with nonzero identity. Then, $R_1 \times R_2 \times R_3$ has no 2-absorbing J-hyperideals.*

Proof. Let $I_1 \times I_2 \times I_3$ is a 2-absorbing J-hyperideal of $R_1 \times R_2 \times R_2$ for some hyperideals I_1, I_2 and I_3 of R_1, R_2 and R_3, respectively. Since $(0, 0, 0) \in (1, 1, 0) \circ (0, 1, 1) \circ (1, 0, 1) \cap I_1 \times I_2 \times I_3$, then $(1, 1, 0) \circ (0, 1, 1) \circ (1, 0, 1) \subseteq I_1 \times I_2 \times I_3$. Since $(1, 1, 0) \circ (0, 1, 1) \nsubseteq J(R_1 \times R_2 \times R_3)$ and $(1, 1, 0) \circ (1, 0, 1) \nsubseteq J(R_1 \times R_2 \times R_3)$, then $(0, 0, 1) \in (0, 0, 1) \circ (1, 0, 1) \subseteq I_1 \times I_2 \times I_3$. Moreover, we can get $(0, 1, 0), (1, 0, 0) \in I_1 \times I_2 \times I_3$. Thus $(1, 1, 1) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1) \in I_1 \times I_2 \times I_3$ which is a contradiction. □

6. Conclusion

In this paper, we defined the notion of J-hyperideals as a generalization of n-hyperideals and studied the relations between J-hyperideals and other classical hyperideals such as n-hyperideals, r-hyperideals, prime and maximal hyperideals. Moreover, we extended this concept to quasi J-hyperideals and 2-absorbing J-hyperideals. Some main results and examples are given to explain the structures of these concepts.

7. future work

Definition 7.1. Let P be a hyperideal of R. P refers to a strongly quasi J-hyperideal if $x, y \in R$ and $x \circ y \subseteq P$ imply $x^2 \subseteq P$ or $y \in J(R)$.

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9. Conflict of interest

The authors declare no conflict of interest.

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