Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

A NOVEL METHOD FOR SOLVING FUZZY PARABOLIC PDE BY USING THE SG-HUKUHARA DIFFERENTIABILITY

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Special issue dedicated to Professor Esfandiar Eslami
Article type: Research Article
(Received: 02 December 2023, Received in revised form 25 February 2024)
(Accepted: 28 April 2024, Published Online: 22 May 2024)

Abstract. In this paper, the method of Crank-Nicolson is proposed for approximating the solution of a fuzzy parabolic PDE by applying the subject of SG-Hukuhara differentiability where the initial and boundary conditions are fuzzy numbers. The consistency and stability of this method are investigated and finally, a non-trivial example is given by this method.

Keywords: Generalized Hukuhara derivative, Fuzzy parabolic PDE, Crank-Nicolson solution
2020 MSC: 34A07, 18A20, 35R13

1. Introduction

Since a partial differential equation (PDE) may not have an analytical solution and the initial and boundary values be random such that the base structure does not admit probabilistic then the using of fuzzy numbers is very helpful. If the underlying structure is appropriate to use fuzzy numbers, then the concept of the fuzzy derivative is useful [1, 2, 4, 8–10].
The strongly generalized differentiability introduced by Hukuhara and considered as SG-Hukuhara derivatives [12]. The advantage of SG-Hukuhara derivatives of functions in comparison to other derivatives is that, other fuzzy differential equations solving methods may have not a unique solution. Recently, a SG-Hukuhara derivative defining on interval valued functions was studied by Bede and Stefanini [6]. They examined, effective methods for solution of fuzzy PDE's and gave some useful examples.

Furthermore, Allahviranloo applied some numerical methods for the solving of fuzzy PDE, which used Seikkala derivative method [4, 7]. Also, a fuzzy finite derivative scheme for solving fuzzy heat equations which are not homogeneous has been studied in [13]. Motivated by the above methods, we introduce the fuzzy parabolic PDE by using the SG-Hukuhara differentiability method and we apply the Crank-Nicolson way for finding the solution of a parabolic PDE.

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https://doi.org/10.22103/jmmr.2024.22610.1545
Publisher: Shahid Bahonar University of Kerman
How to cite: P. Moatamed, M.B. Ahmadi, *A novel method for solving fuzzy parabolic PDE by using the SG-Hukuhara differentiability*, J. Mahani Math. Res. 2024; 13(4): 47 - 62.

The Sections of this paper are organized in the following way: In Section 2, the basic and important results and definitions on fuzzy number systems are presented. Also, definition of a SG-Hukuhara differentiability is given. In Section 3, we define the fuzzy parabolic PDE by applying the SG-Hukuhara differentiability method and we use the Crank-Nicolson way for finding the solution of a parabolic PDE. We investigate the consistency and stability of this method. We solve the fuzzy parabolic equation with this scheme in Section 4. Finally, a non-trivial example is given by this method.

2. **Basic concepts**

First of all, we briefly give some basic and important concepts and results about the Hukuhara and generalized Hukuhara differentiability. We show the set of fuzzy numbers by $\mathbb{R}_{\mathcal{F}}$. As usual, we denote for interval $\beta \in (0,1]$, the following sets on $\mathbb{R}^n$:

$$[a]^{\beta} = \{y \in |a(y) \geq \beta\},$$

and

$$a^0 = cl\{y \in \mathbb{R}^n | a(y) > 0\} = supp(a).$$

One can write $[a]^{\beta} = [\overset{-}{a}(\beta), \overset{+}{a}(\beta)]$. For any $a, b \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, the definition of summation and scalar product are as $[a+b]^{\beta} = [a]^{\beta} + [b]^{\beta}, [ka]^{\beta} = k[a]^{\beta}$.

In addition, Hausdorff distance of fuzzy numbers is defined by

$$\rho : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}^{+} \cup \{0\}$$

$$\rho(a,b) = sup_{\beta \in [0,1]} d([a]^{\beta}, [b]^{\beta}) = sup_{\beta \in [0,1]} \max\{|\overset{-}{a}(\beta) - \overset{-}{b}(\beta)|, |\overset{+}{a}(\beta) - \overset{+}{b}(\beta)|\},$$

in which d is denoted as the Hausdorff metric. It is well known that $(\mathbb{R}_{\mathcal{F}}, \rho)$ is a complete metric space, such that satisfies the below properties:

(1) $\rho(a \oplus c, b \oplus c) = \rho(a, b), \qquad \forall a, b, c \in \mathbb{R}_{\mathcal{F}}$;
(2) $\rho(\mu a, \mu b) = |\mu| \rho(a, b), \qquad \forall \mu \in \mathbb{R}, a, b, \in \mathbb{R}_{\mathcal{F}}$;
(3) $\rho(a \oplus c, b \oplus c) \leq \rho(a, c) + \rho(b, z), \qquad \forall a, b, c, z \in \mathbb{R}_{\mathcal{F}}$;
(4) $\rho(a \ominus b, c \ominus z) \leq \rho(a, c) + \rho(b, z)$ as long as $a \ominus b$ and $c \ominus z$ exist, where $a, b, z \in \mathbb{R}_{\mathcal{F}}$,

Here, $\ominus$ stands as Hukuhara difference and we have $c \ominus b = a$ iff $a \oplus b = c$.

Now, we recall the basic definitions about fuzzy derivative.

Definition 2.1. [5] Let a function ω be a fuzzy value and $\omega : (a, b) \to \mathbb{R}_{\mathcal{F}}$. Then at the point $y_0 \in (a, b)$, we define the generalized Hukuhara derivative (gH-derivative) as follows

$$\omega'_{gH}(y_0) = \lim_{h \to 0} \frac{\omega(y_0 + h) \ominus_{gH} \omega(y_0)}{h}. \tag{1}$$

Definition 2.2. [5] Suppose a fuzzy valued function $\omega : [a, b] \to \mathbb{R}_{\mathcal{F}}$ is differentiable at y_0. We define

- ω at point y_0 is named $[(j) - gH]-$diff if

(2) $$\omega'_{i.gH}(y_0;\beta) = [(\omega^-)'(y_0:\beta), (\omega^+)'(y_0:\beta)],\ \beta \in [0,1],$$

- ω at point y_0 is called $[(jj) - gH]-$diff if

(3) $$\omega'_{ij,gH}(y_0;\beta) = [(\omega^+)'(y_0:\beta), (\omega^-)'(y_0:\beta)],\ \beta \in [0,1].$$

Definition 2.3. [5] We say that a point $y_0 \in (a, b)$, is a switching point for the differentiability of ω, if in any neighborhood V of y_0 there exist points $y_1 < y_2 < y_3$ such that

type(I) at y_1, (2) holds while (3) does not hold and at y_2, (3) holds and (2) does not hold, or

type(II) at y_1, (3) holds while (2) does not hold and at y_2, (2) holds and (3) does not hold.

Example 2.4. *Assume that we have the following initial value problem:*

$$w''(y) + w'(y) + w(y) = y, w(0) = 0, w'(0) = 1.$$

It is easy to see that the solution of the equation is equal to

$$w(y) = e^{-(\frac{y}{2})}(cos(\frac{\sqrt{3}}{2}y) + \frac{1}{\sqrt{3}}sin(\frac{\sqrt{3}}{2}y)) + y - 1.$$

One can find the Fuzzy solution of this differential equation in [11]. By analyzing the solution function w and its derivatives we have $w(y).w'(y) > 0$ and $w'(y).w''(y) < 0$ for any $y \in [0, 1.204]$. So, w and w' have $[(j) - gH]-diff$ and $[(jj) - gH]-diff$, respectively. Also, their types switch for any $y \in [1.204, 5]$. Thus $y = 1.204$ is a switching point.

In this paper we assume that there is not any switching point at the intervals of computations.

Moreover, the second order gH-differentiability is defined as

$$\omega''_{gH}(y_0) = \lim_{h\to 0} \frac{\omega'_{gH}(y_0 + h) \ominus_{gH} \omega'_{gH}(y_0)}{h}$$

if $\omega'_{gH}(y_0 + h) \ominus_{gH} \omega'_{gH}(y_0) \in \mathbb{R}_{\mathcal{F}}$.

Also, we have

- $\omega'_{gH}(y)$ is called $[(j)-gH]$-diff if gH-differentiability of $\omega(y)$ and $\omega'_{gH}(y)$ are from the same type and:
$$\omega''_{i.gH}(y_0;\beta) = [(\omega^-)''(y_0;\beta), (\omega^+)''(y_0;\beta)],\ \beta \in [0,1]$$
- $\omega'_{gH}(y)$ is called $[(jj) - gH]$-diff if gH-differentiability of $\omega(y)$ and $\omega'_{gH}(y)$ are from different type and:
$$\omega''_{ii.gH}(y_0;\beta) = [(\omega^+)''(y_0;\beta), (\omega^-)''(y_0;\beta)],\ \beta \in [0,1]$$

The number c of an open interval $J \subseteq \mathbb{R}$ is called a critical point [12] of ω if gH-derivative of ω vanishes at c (i.e. $\omega'_{gH}(c) = 0$) or ω at the point c is not differentiable.

Now, we recall some properties of fuzzy partial derivatives.

For a fixed $a \in \mathbb{R}_{\mathcal{F}}$ and a crisp differentiable function $q(y,s)$, [gH-p]-derivative with respect to s from $\gamma(y,s) = q(y,s) \odot u$ is as follows [12]: $\partial_{s_{gH}}\gamma(y,s) = \partial_s q(s,y) \odot u$. Also the equality

$\gamma(y,s+k) \ominus_{gH} \gamma(s,y) = q(y,s+k) \odot u \ominus_{gH} q(y,s) \odot u = (q(y,s+k) - q(s,y)) \odot u$

holds. Moreover, by using Hausdorff distance we obtain

$$\begin{aligned}
&\lim_{k\to 0}\rho(\frac{\gamma(y,s+k)\ominus_{gH}\gamma(s,y)}{k}\partial_s q(s,y)\odot a) \\
&\qquad = \lim_{k\to 0}\rho(\frac{q(y,s+k)-q(s,y))\odot u}{k}, \partial_s q(s,y)\odot a) \\
&\qquad = \lim_{k\to 0}\rho(\frac{q(y,s+k)-q(s,y)}{k}\odot u, \partial_s q(s,y)\odot a) \\
&\qquad = 0. \qquad (4)
\end{aligned}$$

This means [gH-p]-derivative act as $\partial_{s_{gH}}\gamma(s,y) = \partial_s q(s,y) \odot u$.

Definition 2.5. [12] Suppose $\omega(s,y) : J \to \mathbb{R}_{\mathcal{F}}$ is a function such that $\omega^-(y,s;\beta)$, $\omega^+(y,s;\beta)$ are partial differentiable with respect to y then

- $\omega(s,y)$ is called $[(j)-p]-$diff with respect to y at (y_0,s_0) if

$$\partial_{y_{i.gH}}\omega(y_0,s_0;\beta) = [\partial_y\omega^-(y_0,s_0;\beta), \partial_y\omega^+(y_0,s_0;\beta)] \tag{5}$$

- $\omega(s,y)$ is called $[(jj)-p]-$diff with respect to y at (y_0,s_0) if

$$\partial_{y_{ii.gH}}\omega(y_0,s_0;\beta) = [\partial_y\omega^+(y_0,s_0;\beta), \partial_y\omega^-(y_0,s_0;\beta)] \tag{6}$$

If both of $\omega(y,s)$ and $\partial_y\omega(y,s)$ are [gH-p]-diff at $(y_0,s_0) \in J$, the second order derivative in these senses are defined as the following cases:

- $\partial_{yy_{gH}}\omega(y,s)$is $[(j)-p]$-diff with respect to x if $\omega(y,s)$ and $\partial_{y_{gH}}\omega(y,s)$ have the same type of $[gH-p]$-differentiability and:
$$\partial_{yy_{i.gH}}\omega(y_0,s_0;\beta) = [\partial_{yx}\omega^-(y_0,s_0;\beta), \partial_{yx}\omega^+(y_0,s_0;\beta)].$$
- $\partial_{yxgH}\omega(y,s)$ is $[(jj)-p]$-diff with respect to x if $\omega(y,s)$ and $\partial_{ygH}\omega(y,s)$ have different $[gH-p]$-differentiability type and:
$$\partial_{yy_{ii.gH}}\omega(y_0,s_0;\beta) = [\partial_{yx}\omega^+(y_0,s_0;\beta), \partial_{yx}\omega^-(y_0,s_0;\beta)].$$

3. **Solution of fuzzy parabolic equations by using the method of Crank-Nicolson**

We consider fuzzy linear parabolic equations of second order by using the generalized Hukuhara differentiable [4]:

$$\partial_{s_{gH}}\omega(s,y) \ominus_{gH} \beta\partial_{y^2_{gH}}\omega(s,y) = G(s,y) \tag{7}$$

$$G(s,y) = 0$$

with the fuzzy initial condition

$$\omega(0,y)=\widetilde{\omega}_0\in E,\quad s=0,y\in[0,l],$$

and the fuzzy boundary conditions

$$\omega(s,0)=\widetilde{\omega}_1\in E,\quad y=0,s>0,$$

$$\omega(s,l)=\widetilde{\omega}_2\in E,\quad y=l,s>0.$$

In terms of the definition of generalized Hukuhara differentiable we have:

$$\partial_{s_{gH}}\omega(s,y)\ominus_{gH}\beta\partial_{y^2_{gH}}\omega(s,y)=G(s,y) \tag{8}$$

$$\Leftrightarrow\begin{cases}(j)\partial_{s_{gH}}\omega(s,y)=\beta\partial_{y^2_{gH}}\omega(s,y)+G(s,y),\\ or,\\ (jj)\beta\partial_{y^2_{gH}}\omega(s,y)=\partial_{s_{gH}}\omega(s,y)+(-1)G(s,y).\end{cases} \tag{9}$$

Let $G(s,y)=0$, then we have :

$$\partial_{s_{gH}}\omega(s,y)=\beta\partial_{y^2_{gH}}\omega(s,y).$$

By using Definition (2.2), suppose $\omega(s,y)=:J\to\mathbb{R}_{\mathcal{F}},(s_0,y_0)\in J$ and $\overset{-}{\omega},\overset{+}{\omega}$ are partial differentiable with respect to y then
$\partial_{s_{gH}}\omega(s_0,y_0)$ is called (j-gH)-diff:

$$\partial_{s_{i.gH}}\omega(s_0,y_0)=\Bigg[\partial_{s_{gH}}\underline{\omega}(s_0,y_0,l),\partial_{s_{gH}}\overline{\omega}(s_0,y_0,l)\Bigg].$$

$\partial_{s_{gH}}\omega(s_0,y_0)$ is called (j-gH)-diff:

$$\partial_{s_{ii.gH}}\omega(s_0,y_0)=\Bigg[\partial_{s_{gH}}\overline{\omega}(s_0,y_0,l),\partial_{s_{gH}}\underline{\omega}(s_0,y_0,l)\Bigg].$$

Also, let $\omega(s,y):J\to\mathbb{R}_{\mathcal{F}},\partial\omega_{s_{gH}}(s,y)$ be gH-diff at (s_0,y_0), then $\partial_{{y^2}_{gH}}\omega(s,y)$ is called $\Bigg[(j)-gH\Bigg]$-diff if $\omega(s,y)$ and $\partial_{y_{gH}}\omega(s,y)$ have same type of gH-differentiability and

$$\partial_{{y^2}_{i.gH}}\omega(s_0,y_0)=\Bigg[\partial_{y^2_{gH}}\overline{\omega}(s_0,y_0,l),\partial_{y^2_{gH}}\underline{\omega}(s_0,y_0,l)\Bigg]\quad l\in[0,1]$$

$\partial_{{y^2}_{gH}}\omega(s,y)$ is named $\Bigg[(jj)-gH\Bigg]$-diff if $\omega(s,y)$ and $\partial_{y_{gH}}\omega(s,y)$ have different type of gH-differentiability and

$$\partial_{{y^2}_{ii.gH}}\omega(s_0,y_0)=\Bigg[\partial_{{y^2}_{gH}}\overline{\omega}(s_0,y_0,l),\partial_{{y^2}_{gH}}\underline{\omega}(s_0,y_0,l)\Bigg]\quad l\in[0,1].$$

Theorem 3.1. *The solution of the fuzzy parabolic PDE* (7) *by SG-Hukuhara differentiability method is as follows:*
Case(1): *If both* $\partial_{s_{gH}}\omega(s,y)$ *and* $\partial_{{y^2}_{gH}}\omega(s,y)$ *be considered (j-gH)-differentiability, or* $\partial_{s_{gH}}\omega(s,y)$ *and* $\partial_{{y^2}_{gH}}\omega(s,y)$ *be considered (jj-gH)-differentiability, then*

(10) $\overline{\partial_{s_{gH}}}\omega(s,y,l)=\beta\overline{\partial_{y^2}}f(s,y,l)$ *and* $\underline{\partial_{s_{gH}}}\omega(s,y,l)=\beta\underline{\partial_{{y^2}_{gH}}}\omega(s,y,l).$

Case(2): *If* $\partial_{s_{gH}}\omega(s,y)$ *and* $\partial_{{y^2}_{gH}}\omega(s,y)$ *have different type of gH-differentiability:*

(11) $\underline{\partial_{s_{gH}}\omega}(s,y,l)=\beta\overline{\partial_{{y^2}_{gH}}\omega}(s,y,l)$ *and* $\overline{\partial_{s_{gH}}\omega}(s,y,l)=\beta\underline{\partial_{{y^2}_{gH}}}\omega(s,y,l),$

such that initial conditions are

$$\underline{\omega}(0,y,l)=\underline{\omega}_0(l) \quad \text{and} \quad \overline{\omega}(0,y,l)=\overline{\omega}_0(l),$$

and boundary conditions are

$$\underline{\omega}(s,0,l)=\underline{\omega}_1(l) \quad \text{and} \quad \overline{\omega}(s,0,l)=\overline{\omega}_1(l)$$
$$\underline{\omega}(s,l,l)=\underline{\omega}_2(l) \quad \text{and} \quad \overline{\omega}(s,l,l)=\overline{\omega}_2(l)$$
$$\underline{\omega}(0,y,l)=\underline{\omega}_0(l) \quad \text{and} \quad \overline{\omega}(0,y,l)=\overline{\omega}_0(l),$$

where $\underline{\partial_{s_{gH}}}\omega(s,y,l)=\min\{\partial_{s_{gH}}\underline{\omega}(s,y,l),\partial_{s_{gH}}\bar{\omega}(s,y,l)\}$ *and* $\overline{\partial_{s_{gH}}}\omega(s,y,l)=\max\{\partial_{s_{gH}}\underline{\omega}(s,y,l),\partial_{s_{gH}}\bar{\omega}(s,y,l)\}$.

Proof. Since the parametric form of the fuzzy number $\omega(m,j)$ is as follows:

$$\omega(m,j)=(\underline{\omega}(m,j,l),\overline{\omega}(m,j,l)),$$
$$\partial_{s_{gH}}\omega(s,y)=\beta\partial_{{y^2}_{gH}}\omega(s,y),$$

fuzzy parabolic PDE in grid point (s_n,y_i) is designated as:

$$[\partial_{s_{gH}}\omega]_{(s_{m+\frac{1}{2}},y_i)}=[\beta\partial_{{y^2}_{gH}}\omega]_{(s_{m+\frac{1}{2}},y_i)} \tag{12}$$

$$\partial_{s_{gH}}\omega(m+\frac{1}{2},j,l)\cong\frac{\omega(m+1,j,l)\ominus_{gH}\omega(m,j,l)}{\Delta s}.$$

Also, the variable y can be kept constant when we compute the partial derivative of the function ω w.r.t. the variable s, so:

$$\partial_{s_{gH}}\omega(m,j)=[\partial_{s_{gH}}\underline{\omega}(m,j,l),\partial_{s_{gH}}\overline{\omega}(m,j,l)]$$

or

$$\partial_{s_{gH}}\omega(m,j)=[\partial_{s_{gH}}\overline{\omega}(m,j,l),\partial_{s_{gH}}\underline{\omega}(m,j,l)].$$

Thus we can state the following cases in different types of integrability.

Case I : If $\partial_{s_{gH}}\omega(m+\frac{1}{2},j)$ is considered as $(j-gH)$-diff and we get:

$$\partial_{s_{gH}}\omega(m+\frac{1}{2},j,l) \simeq \frac{\omega(m+1,j,l)\ominus_{gH}\omega(m,j,l)}{\Delta t},$$

(13)
$$\partial_{s_{gH}}\omega(m+\frac{1}{2},j,l) \simeq (\frac{\underline{\omega}(m+1,j,l)-\underline{\omega}(m,j,l)}{\Delta t},\frac{\overline{\omega}(m+1,j,l)-\overline{\omega}(m,j,l)}{\Delta t}).$$

Case II : If we consider $\partial_{s_{gH}}\omega(m+\frac{1}{2},j)$ is differentiable in the first form $(jj-gH)$-diff and so we can write

$$\partial_{s_{gH}}\omega(m+\frac{1}{2},j,l) \simeq \frac{\omega(m,j,l)-\ominus_{gH}\omega(m+1,j,l)}{\Delta t}$$

(14)
$$\partial_{s_{gH}}\omega(m+\frac{1}{2},j,l) \simeq (\frac{\overline{\omega}(m+1,j,l)-\overline{\omega}(m,j,l)}{\Delta t},\frac{\underline{\omega}(m+1,j,l)-\underline{\omega}(m,j,l)}{\Delta t}).$$

The variables can be kept constant when we compute the second-order partial derivative of the function ω w.r.t. the variable y. It should be noted that, the counter m is used for the s-direction derivative and j for the y-direction derivative.

Case I : If both of $\partial_{y_{gH}}\omega(s,y)$ and $\partial_{y^2{}_{gH}}\omega(s,y)$ are considered as (j-gH)-diff or (jj-gH)-diff, then

$$\partial_{y^2{}_{gH}}\omega(m+\frac{1}{2},j)$$

(15)
$$\simeq \frac{1}{2}\begin{bmatrix} (\frac{\underline{\omega}(m,j-1,l)-2\underline{\omega}(m,j,l)+\underline{\omega}(m,j+1,l)}{(\Delta y)^2}+\frac{\underline{\omega}(m+1,j-1,l)-2\underline{\omega}(m+1,j,l)+\underline{\omega}(m+1,j+1,l)}{(\Delta y)^2} \\ \frac{\overline{\omega}(m,j-1,l)-2\overline{\omega}(m,j,l)+\overline{\omega}(m,j+1,l)}{(\Delta y)^2}+\frac{\overline{\omega}(m+1,j-1,l)-2\overline{\omega}(m+1,j,l)+\overline{\omega}(m+1,j+1,l)}{(\Delta y)^2}) \end{bmatrix}$$

Case II : If both of $\partial_{y_{gH}}\omega(s,y)$ and $\partial_{y^2{}_{gH}}\omega(s,y)$ are considered as different form of (gH)-differentiability, then we put:

$$\partial_{y^2{}_{gH}}\omega(m+\frac{1}{2},j)$$

(16)
$$\simeq \frac{1}{2}\begin{bmatrix} \frac{\overline{\omega}(m,j-1,l)-2\overline{\omega}(m,j,l)+\overline{\omega}(m,j+1,l)}{(\Delta y)^2}+\frac{\overline{\omega}(m+1,j-1,l)-2\overline{\omega}(m+1,j,l)+\overline{\omega}(m+1,j+1,l)}{(\Delta y)^2}) \\ (\frac{\underline{\omega}(m,j-1,l)-2\underline{\omega}(m,j,l)+\underline{\omega}(m,j+1,l)}{(\Delta y)^2}+\frac{\underline{\omega}(m+1,j-1,l)-2\underline{\omega}(m+1,j,l)+\underline{\omega}(m+1,j+1,l)}{(\Delta y)^2} \end{bmatrix}$$

For the solution of fuzzy parabolic PDE's by applying the explicit method, we replace each first-order and second-order partial derivative in (12) by its approximation in forward and central differences and by (13), (14),(15) and (16) we obtain the following recursive equations.

Case(I):

$$\frac{\underline{\omega}(m+1,j,l)-\underline{\omega}(m,j,l)}{\Delta t}=\frac{1}{2}\beta(\frac{\underline{\omega}(m,j-1,l)-2\underline{\omega}(m,j,l)+\underline{\omega}(m,j+1,l)}{(\Delta y)^2}$$
$$+\frac{\underline{\omega}(m+1,j-1,l)-2\underline{\omega}(m+1,j,l)+\underline{\omega}(m+1,j+1,l)}{(\Delta y)^2}$$

(17)
$$\frac{\overline{\omega}(m+1,j,l)-\overline{\omega}(m,j,l)}{\Delta t}=\frac{1}{2}\beta\frac{\overline{\omega}(m,j-1,l)-2\overline{\omega}(m,j,l)+\overline{\omega}(m,j+1,l)}{(\Delta y)^2}$$
$$+\frac{\overline{\omega}(m+1,j-1,l)-2\overline{\omega}(m+1,j,l)+\overline{\omega}(m+1,j+1,l)}{(\Delta y)^2}$$

Case(II):

$$\frac{\underline{\omega}(m+1,j,l)-\underline{\omega}(m,j,l)}{\Delta t}=\frac{1}{2}\beta\frac{\overline{\omega}(m,j-1,l)-2\overline{\omega}(m,j,l)+\overline{\omega}(m,j+1,l)}{(\Delta y)^2}$$
$$+\frac{\overline{\omega}(m+1,j-1,l)-2\overline{\omega}(m+1,j,l)+\overline{\omega}(m+1,j+1,l)}{(\Delta y)^2}$$

(18)
$$\frac{\overline{\omega}(m+1,j,l)-\overline{\omega}(m,j,l)}{\Delta t}=\frac{1}{2}\beta(\frac{\underline{\omega}(m,j-1,l)-2\underline{\omega}(m,j,l)+\underline{\omega}(m,j+1,l)}{(\Delta y)^2}$$
$$+\frac{\underline{\omega}(m+1,j-1,l)-2\underline{\omega}(m+1,j,l)+\underline{\omega}(m+1,j+1,l)}{(\Delta y)^2})$$

where, they leads to a recursive equation expressed as follows:

Case(I):

$$(2+2d)\underline{\omega}(m+1,j,l)-d\underline{\omega}(m+1,j+1,l)-d\underline{\omega}(m+1,j-1,l)$$
$$=(2-2d)\underline{\omega}(m,j,l)+d\underline{\omega}(m,j+1,l)+d\underline{\omega}(m,j-1,l)$$

(19)
$$(2+2d)\overline{\omega}(m+1,j,l)-d\overline{\omega}(m+1,j+1,l)-d\overline{\omega}(m+1,j-1,l)$$
$$=(2-2d)\overline{\omega}(m,j,l)+d\overline{\omega}(m,j+1,l)+d\overline{\omega}(m,j-1,l)$$

Case(II):

$$2d\underline{\omega}(m+1,j,l)-d\underline{\omega}(m+1,j+1,l)-d\underline{\omega}(m+1,j-1,l)+2\overline{\omega}(m+1,j,l)$$
$$=-2d\underline{\omega}(m,j,l)+d\underline{\omega}(m,j+1,l)+d\underline{\omega}(m,j-1,l)+2\overline{\omega}(m,j,l)$$

(20)
$$2d\overline{\omega}(m+1,j,l)-d\overline{\omega}(m+1,j+1,l)-d\overline{\omega}(m+1,j-1,l)+2\underline{\omega}(m+1,j,l)$$
$$=-2d\overline{\omega}(m,j,l)+d\overline{\omega}(m,j+1,l)+d\overline{\omega}(m,j-1,l)+2\underline{\omega}(m,j,l)$$

where $d = \frac{\beta \Delta s}{(\Delta y)^2}$, $l \in [0,1]$, $m = 1, ..., N-1$ and $j = 1, ..., m-1$,

and also $\underline{\omega}(m,j,l)$ and $\overline{\omega}(m,j,l)$ at $j = 1$ and $j = m$, for $m > 1$ and $\underline{\omega}(m,j,l)$ and $\overline{\omega}(m,j,l)$ at $m = 1$, for $1 < j < m$, are known.
So, the proof is completed. □

4. **On the stability of numerical methods**

Here, the Von-Neumann method for the stability analysis of numerical methods are considered. In this method, solutions of Eq.(19) and (20) are written as follows:

$$\underline{\omega}(m,j,l) = \underline{U}^m(l)e^{Ip\Delta x(j)}, \qquad \underline{\omega}(m,j,l) = \overline{U}^m(l)e^{Ip\Delta x(j)}$$

where $I = \sqrt{-1}$ and p is the wave number in the x-direction. If the phase angle is $\Omega = p\Delta x$, then we have

$$\underline{\omega}(m,j,l) = \underline{U}^m(l)e^{I\Omega(j)}, \qquad \overline{\omega}(m,j,l) = \overline{U}^m(l)e^{I\Omega(j)}$$
$$\underline{\omega}(m+1,j,l) = \underline{U}^{m+1}(l)e^{I\Omega(j)}, \qquad \overline{\omega}(m+1,j,l) = \overline{U}^{m+1}(l)e^{I\Omega(j)}$$
$$\underline{\omega}(m,i\pm 1,l) = \underline{U}^m(l)e^{I\Omega(j\pm 1)}, \qquad \overline{\omega}(m,i\pm 1,l) = \overline{U}^m(l)e^{I\Omega(j\pm 1)}$$

First, we replace each relation above in Eq. (19) to obtain $(m, 1:l, l)$:

$$\underline{U}(l) = \frac{1 - 2d\sin^2(\frac{\Omega}{2})}{1 + 2d\sin^2(\frac{\Omega}{2})}$$

$$\overline{U}(l) = \frac{1 - 2d\sin^2(\frac{\Omega}{2})}{1 + 2d\sin^2(\frac{\Omega}{2})}.$$

for proving the stability of the method in case (I), it is sufficient to show the absolute value of $\overset{-}{U}(l)$ and $\overset{+}{U}(l)$ is less than or equal to one.
Therefore Equation (19) will be stable when

$$|\ \underline{U}(l)\ | \leq 1 \quad and \quad |\ \overline{U}(l)\ | \leq 1$$

i.e.,

$$\frac{1 - 2d\sin^2(\Omega/2)}{1 + 2d\sin^2(\frac{\Omega}{2})} \leq 1$$

Then the method is stable.
Or we replace each relation above in Eq. (20) to obtain:

(21)
$$\begin{cases} 2d(1-\cos(\Omega))\underline{U}^{m+1}(m+1,l) + 2\overline{U}^{m+1}(l) = 2d(\cos(\Omega)-1)\underline{U}^m(m,l) + 2\overline{U}^n(l) \\ 2d(1-\cos(\Omega))\overline{U}^{m+1}(m+1,l) + 2\underline{U}^{m+1}(l) = 2d(\cos(\Omega)-1)\overline{U}^m(l) + 2\underline{U}^m(l). \end{cases}$$

from (21) we have:
(22)

$$\begin{bmatrix} d(1-\cos\Omega) & 1 \\ 1 & d(1-\cos\Omega) \end{bmatrix}\begin{bmatrix} \underline{U}^{m+1}(l) \\ \overline{U}^{m+1}(l) \end{bmatrix}=\begin{bmatrix} d(\cos\Omega-1) & 1 \\ 1 & d(\cos\Omega-1) \end{bmatrix}\times\begin{bmatrix} \underline{U}^{m}(l) \\ \overline{U}^{m}(l) \end{bmatrix}$$

we denote $a = \begin{bmatrix} d(1-\cos\Omega) & 1 \\ 1 & d(1-\cos\Omega) \end{bmatrix}$ and $b = \begin{bmatrix} d(\cos\Omega-1) & 1 \\ 1 & d(\cos\Omega-1) \end{bmatrix}$.
and $G = a^{-1}b$

For proving the stability of the method, it is sufficient to show the absolute value of the eigenvector of matrix G is less than or equal to one.
G is a (2×2) matrix whose eigenvalues μ_1, μ_2 are given by

$$\mu_1 = \frac{1-2d\sin^2(\frac{\Omega}{2})}{1+2d\sin^2(\frac{\Omega}{2})}$$

$$\mu_2 = \frac{1+2d\sin^2(\frac{\Omega}{2})}{1-2d\sin^2(\frac{\Omega}{2})}$$

Therefore the Equation (20) will be stable when $\sin^2(\frac{\Omega}{2}) = 0$ then the method is stability is for
$\sin(\frac{\Omega}{2}) = 0 = \sin k\pi, k \in \mathbb{Z} \quad and \quad \Omega = 2k\pi$

4.1. **Investigating the consistency.** For determining the consistency and scheme order of fuzzy partial differential equation (FPDE), $\underline{\omega}$ and $\overline{\omega}$ in (19) and (20) are replaced with exact solutions $\underline{G}$ and $\overline{G}$ of (7) to give truncation errors.
The local truncation errors are defined as operators which map the actual solutions of the FPDE to the corrections required to make them satisfy the scheme at any time step:
Case(I):

$$\begin{aligned}(2+2d)\underline{\omega}(m+1,j,l) - d\underline{\omega}(m+1,j+1,l) - d\underline{\omega}(m+1,j-1,l) \\ = (2-2d)\underline{\omega}(m,j,l) - d\underline{\omega}(m,j+1,l) - d\underline{\omega}(m,j-1,l)\end{aligned}$$

$$\begin{aligned}(2+2d)\overline{\omega}(m+1,j,l) - d\overline{\omega}(m+1,j+1,l) - d\overline{\omega}(m+1,j-1,l) \\ = (2-2d)\overline{\omega}(m,j,l) - d\overline{\omega}(m,j+1,l) - d\overline{\omega}(m,j-1,l)\end{aligned}$$

where $d = \frac{r\Delta t}{(\Delta y)^2}$

$$\begin{aligned}\underline{E}(m,j,l) = (2+2d)\underline{G}(m+1,j,l) - d\underline{G}(m+1,j+1,l) - d\underline{G}(m+1,j-1,l) \\ - (2-2d)\underline{G}(m,j,l) + d\underline{G}(m,j+1,l) + d\underline{G}(m,j-1,l)\end{aligned}$$

$$\begin{aligned}\overline{E}(m,j,l) = (2+2d)\overline{G}(m+1,j,l) + d\overline{G}(m+1,j+1,l) + d\overline{G}(m+1,j-1,l) \\ - (2-2d)\overline{G}(m,j,l) - d\overline{G}(m,j+1,l) - d\overline{G}(m,j-1,l)\end{aligned}$$

Case(II):

$$
\begin{aligned}
2d\underline{\omega}(m+1,j,l) - d\underline{\omega}(m+1,j+1,l) - d\underline{\omega}(m+1,j-1,l) + 2\overline{\omega}(m+1,j,l)\\
= -2d\underline{\omega}(m,j,l) + d\underline{\omega}(m,j+1,l) + d\underline{\omega}(m,j-1,l)
\end{aligned}
$$

$$
\begin{aligned}
2d\overline{\omega}(m+1,j,l) - d\overline{\omega}(m+1,j+1,l) - d\overline{\omega}(m+1,j-1,l) + 2\underline{\omega}(m+1,j,l)\\
= -2d\overline{\omega}(m,j,l) + d\overline{\omega}(m,j+1,l) + d\overline{\omega}(m,j-1,l) + 2\underline{\omega}(m,j,l)
\end{aligned}
$$

where $d = \frac{r\Delta t}{(\Delta y)^2}$

$$
\begin{aligned}
\underline{E}(m,j,l) = 2d\underline{G}(m{+}1,j,l) - d\underline{G}(m{+}1,j{+}1,l) - d\underline{G}(m{+}1,j{-}1,l) + 2\overline{G}(m{+}1,j,l)\\
+ 2d\underline{G}(m,j,l) + d\underline{G}(m,j+1,l) - d\underline{G}(m,j-1,l) + d\overline{G}(m,j,l)
\end{aligned}
$$

$$
\begin{aligned}
\overline{E}(m,j,l) = 2d\overline{G}(m{+}1,j,l) - d\overline{G}(m{+}1,j{+}1,l) - d\overline{G}(m{+}1,j{-}1,l) + 2\underline{G}(m{+}1,j,l)\\
+ 2d\overline{G}(m,j,l) + d\overline{G}(m,j+1,l) + d\overline{G}(m,j-1,l) + d\underline{G}(m,j,l)
\end{aligned}
$$

We begin by taking the Taylor expansion of functions $\underline{G}$ and $\overline{G}$ in s and y about (s_n, y_i).

$$
\underline{G}(m{+}1,j,l) = \underline{G}(m,j,l) + k\partial_{s_{gH}}\underline{G}(m,j,l) + \frac{k^2}{2}\partial_{t^2_{gH}}\underline{G}(m,j,l) + \frac{k^3}{6}\partial_{t^3_{gH}}\underline{\omega}(m,j,l) + \cdots
$$

$$
\overline{G}(m{+}1,j,l) = \overline{G}(m,j,l) + k\partial_{s_{gH}}\overline{G}(m,j,l) + \frac{k^2}{2}\partial_{t^2_{gH}}\overline{G}(m,j,l) + \frac{k^3}{6}\partial_{t^3_{gH}}\overline{\omega}(m,j,l) + \cdots
$$

$$
\underline{G}(m,i{\mp}1,l) = \underline{G}(m,j,l) \mp h\partial_{y_{gH}}\underline{\omega}(m,j,l) + \frac{h^2}{2}\partial_{y^2_{gH}}\underline{\omega}(m,j,l) \mp \frac{h^3}{6}\partial_{y^3_{gH}}\underline{\omega}(m,j,l) + \cdots
$$

$$
\overline{G}(m,i{\mp}1,l) = \overline{G}(m,j,l) \mp h\partial_{y_{gH}}\overline{\omega}(m,j,l) + \frac{h^2}{2}\partial_{y^2_{gH}}\overline{\omega}(m,j,l) \mp \frac{h^3}{6}\partial_{y^3_{gH}}\overline{\omega}(m,j,l) + \cdots
$$

$$
\begin{aligned}
\underline{G}(m+1,i\mp 1,l) = {} & \underline{G}(m,j,l) \mp h\partial_{y_{gH}}\underline{\omega}(m,j,l) + k\partial_{s_{gH}}(m,j,l)\\
& + \frac{1}{2}(\frac{h^2}{2}\partial_{y^2_{gH}}\underline{\omega}(m,j,l) \mp 2hk\partial_{y_{gH}}\partial_{s_{gH}}\underline{G}(m,j,l)\\
& + k^2(\frac{h^2}{2}\partial_{t^2_{gH}}\underline{\omega}(m,j,l)) + \frac{1}{6}(\frac{h^3}{6}\partial_{y^3_{gH}}\underline{\omega}(m,j,l)\\
& + 3h^2k\frac{h^3}{6}\partial_{y^2_{gH}}\partial_{s_{gH}}\underline{G}(m,j,l)\\
& \mp 3hk^2\partial_{y_{gH}}\partial_{t^2_{gH}}\underline{G}(m,j,l) + k^3\partial_{t^3_{gH}}\underline{\omega}(m,j,l)) + \cdots
\end{aligned}
$$

$$\begin{aligned}\overline{G}(m+1,i\mp 1,l) &= \overline{G}(m,j,l) \mp h\partial_{y_{gH}}\overline{\omega}(m,j,l) + k\partial_{s_{gH}}\overline{G}(m,j,l)\\ &+ \frac{1}{2}(\frac{h^2}{2}\partial_{y^2_{gH}}\overline{\omega}(m,j,l) \mp 2hk\partial_{y_{gH}}\partial_{s_{gH}}\overline{G}(m,j,l)\\ &+ k^2\frac{h^2}{2}\partial_{t^2_{gH}}\overline{G}(m,j,l) + \frac{1}{6}(\frac{h^3}{6}\partial_{y^3_{gH}}\overline{\omega}(m,j,l)\\ &+ 3h^2k\frac{h^3}{6}\partial_{y^2_{gH}}\partial_{s_{gH}}\overline{G}(m,j,l) \mp 3hk^2\frac{\partial^3\overline{G}}{\partial x\partial t^2}(m,j,l)\\ &+ k^3\partial_{t^3_{gH}}\overline{G}(m,j,l) + \cdots\end{aligned}$$

So, by replacing (11) and (12) by their Taylor expansion about the (m,j), implies:

case(I)

$$\begin{aligned}\underline{E}(m,j,l) &= 2k(\partial_{s_{gH}}\underline{G}(m,j,l) - \partial_{y^2_{gH}}\underline{G}(m,j,l)) + k^2\partial_{t^2_{gH}}\underline{G}(m,j,l) + \frac{k^3}{3}\partial_{t^3_{gH}} + \frac{k^4}{12}\\ &+ \partial_{t^4_{gH}}\underline{G}(m,j,l) - k^2\partial_{y^2{}_{gH}}\partial_{s_{gH}}\underline{G}(m,j,l) - \frac{1}{2}k^3\partial_{y^2{}_{gH}}\partial_{s^2_{gH}}\underline{G}(m,j,l)\\ &- \frac{1}{6}h^2k\partial_{s^4_{gH}}\underline{G}(m,j,l)\end{aligned}$$

$$\begin{aligned}\overline{E}(m,j,l) &= 2k(\partial_{s_{gH}}\overline{G}(m,j,l) - \partial_{y^2_{gH}}\overline{G}(m,j,l)) + k^2\partial_{s^2_{gH}}\overline{G}(m,j,l)\\ &+ \frac{k^3}{3}\partial_{s^3_{gH}} + \frac{k^4}{12} + \partial_{s^4_{gH}}\overline{G}(m,j,l) - k^2\partial_{y^2{}_{gH}}\partial_{t_{gH}}\overline{G}(m,j,l)\\ &- \frac{1}{2}k^3\partial_{y^2{}_{gH}}\partial_{s^2_{gH}}\overline{G}(m,j,l) - \frac{1}{6}h^2k\partial_{s^4_{gH}}\overline{G}(m,j,l)\\ &\lim_{(h,k)\to 0}\underline{E}(m,j,l) = 0 \qquad \lim_{(h,k)\to 0}\overline{E}(m,j,l) = 0\end{aligned}$$

case(II)

$$\begin{aligned}\underline{E}(m,j,l) &= 2k(\partial_{s_{gH}}\overline{G}(m,j,l) - \partial_{y^2_{gH}}\underline{G}(m,j,l)) - k^2\partial_{y^2_{gH}}\partial_{s_{gH}}\underline{G}(m,j,l)\\ &- \frac{1}{6}h^2k\partial_{y^4_{gH}}\underline{G}(m,j,l) - \frac{1}{2}k^3\partial_{y^2{}_{gH}}\overline{G}(m,j,l)\partial_{t^2_{gH}}\underline{G}(m,j,l)\\ &+ k^3\partial_{s^3_{gH}}\overline{G}(m,j,l) + k^4\partial_{s^4_{gH}}\overline{G}(m,j,l)\end{aligned}$$

$$\begin{aligned}\overline{E}(m,j,l) &= 2k(\partial_{s_{gH}}\underline{G}(m,j,l) - \partial_{y^2_{gH}}\overline{G}(m,j,l)) - k^2\partial_{y^2_{gH}}\partial_{s_{gH}}\overline{G}(m,j,l)\\ &- \frac{1}{6}h^2k\partial_{y^4_{gH}}\overline{G}(m,j,l) - \frac{1}{2}k^3\partial_{y^2{}_{gH}}\underline{G}(m,j,l)\partial_{t^2_{gH}}\overline{G}(m,j,l)\\ &+ k^3\partial_{s^3_{gH}}\underline{G}(m,j,l) + k^4\partial_{s^4_{gH}}\underline{G}(m,j,l)\\ &\lim_{(h,k)\to 0}\underline{E}(m,j,l) = 0 \qquad \lim_{(h,k)\to 0}\overline{E}(m,j,l) = 0\end{aligned}$$

So, we declare the method accuracy to be $O(k^2 + kh^2)$.
Thus, $\underline{E}(m,j,l) \to 0$ and $\overline{E}(m,j,l) \to 0$ as $(k,h) \to 0$

5. Numerical example

Example 5.1. *We consider the heat equation such that its initial and boundary conditions are fuzzy valued and :*

$$\partial_{s_{gH}}\omega(s,y) = \beta\partial_{y^2_{gH}}\omega(s,y)$$

$T(0,y) = \tilde{1}\sin(\pi y)$ *at* $s = 0, 0 < y < 1$ *(initial condition)*
$T(s,0) = \frac{\tilde{1}}{2}$ *at* $y = 0, s > 0$
$T(s,l) = \frac{\tilde{1}}{2}$ *at* $y = l, s > 0$ *(boundary condition)*
where $\underline{\omega}(m,m,l) = \frac{1}{2}r, \overline{\omega}(m,m,l) = 1 - \frac{1}{2}r, \underline{\omega}(m,0,l) = \frac{1}{2}r, \overline{\omega}(m,0,l) = 1 - \frac{1}{2}, \underline{\omega}(0,j,l) = rsin(\pi y)$ *and*$\overline{\omega}(0,j,l) = (2-l)sin(\pi y)$ *the following cases are possible:*
Case(I):

$$(2+2d)\underline{\omega}(m+1,j,l) - d\underline{\omega}(m+1,j+1,l) - d\underline{\omega}(m+1,j-1,l) = (2-2d)\underline{\omega}(m,j,l) + d\underline{\omega}(m,j+1,l) + d\underline{\omega}(m,j-1,l)$$

$$(2+2d)\overline{\omega}(m+1,j,l) - d\overline{\omega}(m+1,j+1,l) - d\overline{\omega}(m+1,j-1,l) = (2-2d)\overline{\omega}(m,j,l) + d\overline{\omega}(m,j+1,l) + d\overline{\omega}(m,j-1,l)$$

where $y \in [0,1]$, $\Delta s = \frac{1}{36}$, $\beta = 1$, $\Delta y = \frac{1}{3}$, $d = \frac{1}{4}$ *and* $l \in [0,1]$, $m = 0, ..., N-1$ *and* $j = 0, ..., m-1$,

and also $\underline{\omega}(m,j,l)$ *and* $\overline{\omega}(m,j,l)$ *at* $j = 0$ *and* $j = m$, *for* $m > 0$ *and* $\underline{\omega}(m,j,l)$ *and* $\overline{\omega}(m,j,l)$ *at* $m = 0$, *for* $0 < j < m$, *are known.*

The approximated solutions are compared for $l = 0, 0.1, ..., 1$ *in the following Tables 1, 2, 3.*

j	$\underline{\omega}(3,j,0)$	$\overline{\omega}(3,j,0)$	$\underline{\omega}(3,j,0.1)$	$\overline{\omega}(3,j,0.1)$	$\underline{t}(3,j,0.2)$	$\overline{\omega}(3,j,0.2)$	$\underline{\omega}(3,j,0.3)$	$\overline{\omega}(3,j,0.3)$
0	0	1	0.05	0.95	0.1	0.9	0.15	0.85
1	0	1.2156	0.06078	1.1548	0.1256	1.094	0.1823	1.0333
2	0	1.2156	0.06078	1.1548	0.1256	1.094	0.1823	1.0333
3	0	1	0.05	0.95	0.1	0.9	0.15	0.85

Table (1).

j	$\underline{\omega}(3,j,0.4)$	$\overline{\omega}(3,j,0.4)$	$\underline{\omega}(3,j,0.5)$	$\overline{\omega}(3,j,0.5)$	$\underline{t}(3,j,0.6)$	$\overline{\omega}(3,j,0.6)$	$\underline{\omega}(3,j,0.7)$	$\overline{\omega}(3,j,0.7)$
0	0.2	0.8	0.25	0.75	0.3	0.7	0.35	0.65
1	0.24312	0.9725	0.3039	0.9117	0.36468	0.8509	0.4255	0.7901
2	0.24312	0.9725	0.3039	0.9117	0.36468	0.8509	0.4255	0.7901
3	0.2	0.8	0.25	0.75	0.3	0.7	0.35	0.65

Table (2).

j	$\underline{\omega}(3,j,0.8)$	$\overline{\omega}(3,j,0.8)$	$\underline{\omega}(3,j,0.9)$	$\overline{\omega}(3,j,0.9)$	$\underline{t}(3,j,1)$	$\overline{\omega}(3,j,1)$
0	*0.4*	*0.6*	*0.45*	*0.55*	*0.5*	*0.5*
1	*0.4862*	*0.7293*	*0.547*	*0.6686*	*0.6078*	*0.6078*
2	*0.4862*	*0.7293*	*0.547*	*0.6686*	*0.6078*	*0.6078*
3	*0.4*	*0.6*	*0.45*	*0.55*	*0.5*	*0.5*

Table (3).

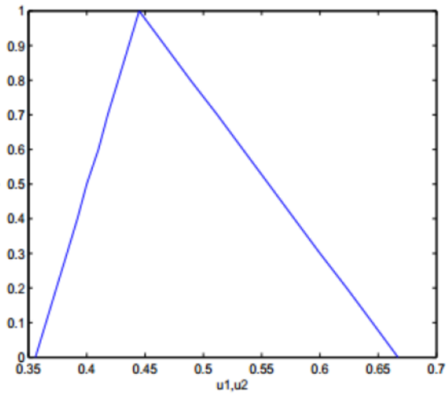

FIGURE 1. Fuzzy PDE answer at $t = 2/3$ and $y = 1$

$$s = 0.08, \ y = \frac{2}{3}$$

Case(II):

$$2d\underline{\omega}(m+1,j,l) - d\underline{\omega}(m+1,j+1,l) - d\underline{\omega}(m+1,j-1,l) + 2\overline{\omega}(m+1,j,l)$$
$$= -2d\underline{\omega}(m,j,l) + d\underline{\omega}(m,j+1,l) + d\underline{\omega}(m,j-1,l) + 2\overline{\omega}(m,j,l)$$

$$2d\overline{\omega}(m+1,j,l) - d\overline{\omega}(m+1,j+1,l) - d\overline{\omega}(m+1,j-1,l) + 2\underline{\omega}(m+1,j,l)$$
$$= -2d\overline{\omega}(m,j,l) + d\overline{\omega}(m,j+1,l) + d\overline{\omega}(m,j-1,l) + 2\underline{\omega}(m,j,l)$$

where $y \in [0,1]$, $\Delta s = \frac{1}{36}$, $\beta = 1$, $\Delta y = \frac{1}{3}$, $d = \frac{1}{4}$ *and* $l \in [0,1]$, $m = 0, ..., N-1$ *and* $j = 0, ..., m-1$, *and also* $\underline{\omega}(m,j,l)$ *and* $\overline{\omega}(m,j,l)$ *at* $j = 0$ *and* $j = m$, *for* $m > 0$ *and* $\underline{\omega}(m,j,l)$ *and* $\overline{\omega}(m,j,l)$ *at* $m = 0$, *for* $0 < j < m$, *are known.*

The approximated solutions are compared for $r = 0, 0.1, ..., 1$ *in the following Tables 4, 5, 6.*

j	$\underline{\omega}(3,j,0)$	$\overline{\varpi}(3,j,0)$	$\underline{\omega}(3,j,0.1)$	$\overline{\varpi}(3,j,0.1)$	$\underline{t}(3,j,0.2)$	$\overline{\varpi}(3,j,0.2)$	$\underline{\omega}(3,j,0.3)$	$\overline{\varpi}(3,j,0.3)$
0	*0*	*1*	*0.05*	*0.95*	*0.1*	*0.9*	*0.15*	*0.85*
1	*-1.5255*	*2.5544*	*-1.3122*	*2.3597*	*-1.0988*	*2.1651*	*-0.8855*	*1.9704*
2	*-1.5255*	*2.5544*	*-1.3122*	*2.3597*	*-1.0988*	*2.1651*	*-0.8855*	*1.9704*
3	*0*	*1*	*0.05*	*0.95*	*0.1*	*0.9*	*0.15*	*0.85*

Table (4).

j	$\underline{\omega}(3,j,0.4)$	$\overline{\varpi}(3,j,0.4)$	$\underline{\omega}(3,j,0.5)$	$\overline{\varpi}(3,j,0.5)$	$\underline{t}(3,j,0.6)$	$\overline{\varpi}(3,j,0.6)$	$\underline{\omega}(3,j,0.7)$	$\overline{\varpi}(3,j,0.7)$
0	*0.2*	*0.8*	*0.25*	*0.75*	*0.3*	*0.7*	*0.35*	*0.65*
1	*-0.6721*	*1.7757*	*-0.4588*	*1.581*	*0.2455*	*1.3864*	*-0.0321*	*1.1917*
2	*-0.6721*	*1.7757*	*-0.4588*	*1.581*	*0.2455*	*1.3864*	*-0.0321*	*1.1917*
3	*0.2*	*0.8*	*0.25*	*0.75*	*0.3*	*0.7*	*0.35*	*0.65*

Table (5).

j	$\underline{\omega}(3,j,0.8)$	$\overline{\varpi}(3,j,0.8)$	$\underline{\omega}(3,j,0.9)$	$\overline{\varpi}(3,j,0.9)$	$\underline{t}(3,j,1)$	$\overline{\varpi}(3,j,1)$
0	*0.4*	*0.6*	*0.45*	*0.55*	*0.5*	*0.5*
1	*0.1812*	*0.997*	*0.3946*	*0.8024*	*0.6078*	*0.6078*
2	*0.1812*	*0.997*	*0.3946*	*0.8024*	*0.6078*	*0.6078*
3	*0.4*	*0.6*	*0.45*	*0.55*	*0.5*	*0.5*

Table (6).

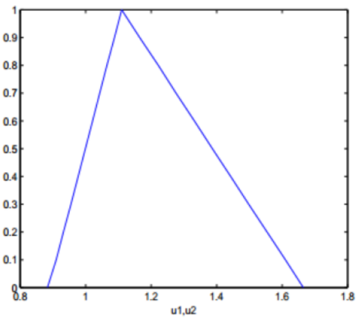

FIGURE 2. Solution of fuzzy PDE at the point $y = 1, s = 0.08$

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