Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

A NEW ENTROPY ESTIMATOR AND ITS APPLICATION TO GOODNESS OF FIT TEST FOR WEIBULL DISTRIBUTION

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Article type: Research Article
(Received: 25 January 2024, Received in revised form 01 April 2024)
(Accepted: 22 May 2024, Published Online: 22 May 2024)

Abstract. In this article, we introduce a new estimator of entropy of continuous random variables. Bias, variance and the mean squared error of the new estimator are obtained and compared with the other existing estimators. The results show that the proposed estimator has a lower mean squared error than its competitors. Then, we propose some goodness of fit tests for Weibull distribution based on the entropy estimators. To assess the effectiveness of the proposed tests, we utilize Monte Carlo simulation to evaluate their power against eighteen different alternatives with varying sample sizes. The results show that the tests are powerful and we can use them in practice. Finally, two real datasets are considered and modeled by the Weibull distribution.

Keywords: Entropy estimator, Weibull distribution, Monte Carlo simulation, Critical points, Power of the test.
2020 MSC: 62G10, 62G30

1. Introduction

Goodness-of-fit (GOF) tests are a class of nonparametric statistical tests that measure how an observed dataset fits a theoretical or expected distribution. It is used to determine whether the observed data are significantly different from the expected values under a given hypothesis, (see Cirrone et al. [1]). The application of the GOF test is in various fields such as economics, finance, engineering, and medicine. For more information and details about the GOF tests, please refer to D'Agostino and Stephens [2]. Also, the GOF tests have been widely used in genetic, quality control, epidemiology, psychology, social sciences, and marketing (see Huber Carol et al. [3]).
Entropy is a frequently employed concept for quantifying the uncertainty or randomness within a dataset. It provides a numerical measurement of the information or disorder presenting within a probability distribution or random variable. Initially introduced as a foundational concept in thermodynamics,

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https://doi.org/10.22103/jmmr.2024.22859.1573
Publisher: Shahid Bahonar University of Kerman
How to cite: S.Q. Alavi, H. Alizadeh Noughabi, S. Jomhoori, *A new entropy estimator and its application to goodness of fit test for Weibull distribution*, J. Mahani Math. Res. 2025; 14(2): 1 - 19.

entropy was used to characterize the level of disorder or randomness in a physical system. Nevertheless, it has subsequently been adopted and applied in numerous disciplines, such as statistics Cover and Thomas [4].
The concept of entropy has been wildly applied in different areas such as physics, probability and statistics, communication theory, economics, signal processing and machine learning. Specifically, in information theory, entropy is a measure of the uncertainty linked to the random variable. Shannon [5], first introduced this concept as follows

$$H(f) = -\int_{-\infty}^{\infty} f(x) \log f(x) dx\,, \tag{1}$$

where f represents the probability density function (pdf) of the random variable X.
Many authors have considered the issue of estimating $H(f)$. For absolutely continuous random variables, Vasicek [6], Joe [7], Hall and Morton [8], Van Es [9], Correa [10] and Alizadeh [11] have studied this problem. Among the numerous entropy estimators available, Vasicek's [6] sample entropy has been the most extensively used in the development of statistical methods based on entropy. His estimate is founded on the basis of equation 1 which can be written as

$$H(f) = \int_{0}^{1} \log\left\{\frac{d}{dp}F^{-1}(p)\right\} dp. \tag{2}$$

In order to estimate $H(f)$, one can substitute the distribution function F by its empirical distribution function F_n and use the difference instead of the differential operator. Hence, given a sample $X_1, ..., X_n$ the estimator $H(f)$ is given by

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\}, \tag{3}$$

where m is a positive integer, $m \leq n/2$ and $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the order statistics and $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$. Vasicek [6] proved that $HV_{mn} \xrightarrow{\text{Pr.}} H(f)$ as $n \to \infty$, $m \to \infty$, $m/n \to 0$. In the following estimators, m is a positive integer, $m \leq n/2$. An estimator for Shannon entropy proposed by Van Es [9] is as follows.

$$HVE_{mn} = \frac{1}{n-m}\sum_{i=1}^{n-m}\left(\frac{n+1}{m}\left(X_{(i+m)} - X_{(i)}\right)\right) + \sum_{k=m}^{n}\frac{1}{k} + \log(m) - \log(n+1),$$

Correa [10] presented a variation of the Vasicek entropy estimator that yields a reduced mean squared error (MSE). The estimator is given by

$$HC_{mn} = -\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{\sum_{j=i-m}^{i+m}\left(X_{(j)}-\bar{X}_{(i)}\right)(j-i)}{n\sum_{j=i-m}^{i+m}\left(X_{(j)}-\bar{X}_{(i)}\right)}\right),$$

where

$$\bar{X}_{(i)} = \frac{1}{2m+1}\sum_{j=i-m}^{i+m} X_{(j)}.$$

Alizadeh [11] introduced a kernel method for estimating entropy. The estimator is given by

$$HA_{mn} = -\frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{\hat{f}_y\left(X_{(i+m)}\right)+\hat{f}_y\left(X_{(i-m)}\right)}{2}\right\},$$

where

$$\hat{f}\left(X_i\right) = \frac{1}{nh}\sum_{j=1}^{n} k\left(\frac{X_i - X_j}{h}\right)$$

The function $K(.)$ is called the kernel function, and h is the smoothing parameter or the bandwidth. One of the commonly used kernel functions is the standard normal density while its optimal smoothing bandwidth is $h = 1.06\, sn^{-\frac{1}{5}}$,where s is the sample standard deviation. Alizadeh [11] discovered that $HA_{mn} \xrightarrow{\text{Pr.}} H(f)$ as $n \to \infty$, $m \to \infty$, $m/n \to 0$.
Entropy estimators are used in hypothesis testing to assess the randomness or information gain when comparing different groups or variables. It helps determine if there are significant differences between data distributions, aiding in making statistical inferences. So, we propose a new entropy estimator and compare it with other existing entropy estimators.
In Section 2, we propose a new entropy estimator and compare it with the other estimators in terms of RMSE. In Section 3, we suggest some goodness of fit tests for Weibull distribution based on the entropy estimators. In Section 4, through a Monte Carlo simulation, we compute the critical values, type I error and power of the tests. In Section 5, the applicability of the tests in real data are shown. Finally, a brief conclusion is given.

2. The new entropy estimator and RMSE comparison

2.1. **The proposed estimator.** Suppose $X_1, ..., X_n$ is a random sample from an unknown continuous distribution F with a probability density function $f(x)$. Suppose $X_{(1)} \leq X_{(2)} \leq .... \leq X_{(n)}$ are the order statistics of the sample. We

want to estimate the entropy $H(f)$.
We can write

$$H\left(f\right)=-E\left(\log\left(f\left(x\right)\right)\right)$$

and consequently an estimator is as

$$\hat{H}\left(f\right)=-\frac{1}{n}\sum_{i=1}^{n}\log\left(f\left(x_i\right)\right)$$

Now we use the equality [12]

$$\log x=2\left[\left(\frac{x-1}{x+1}\right)+\frac{1}{3}\left(\frac{x-1}{x+1}\right)^3+\frac{1}{5}\left(\frac{x-1}{x+1}\right)^5+...\right]$$

and propose the following estimator for $H(f)$.

$$\begin{aligned}HS_n&=-\frac{1}{n}\sum_{i=1}^{n}2\left[\left(\frac{\hat{f}\left(x_i\right)-1}{\hat{f}\left(x_i\right)+1}\right)+\frac{1}{3}\left(\frac{\hat{f}\left(x_i\right)-1}{\hat{f}\left(x_i\right)+1}\right)^3+....\right]\\&=-\frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{1}{(2j-1)}\left(\frac{\hat{f}\left(x_i\right)-1}{\hat{f}\left(x_i\right)+1}\right)^{2j-1}\end{aligned}$$

where $\hat{f}$ is the kernel density estimator and can be described by

$$\hat{f}\left(x_i\right)=\frac{1}{nh}\sum_{j=1}^{n}k\left(\frac{x_i-x_j}{h}\right),$$

where h represents the bandwidth, and k signifies a kernel function which satisfies

$$\int_{-\infty}^{\infty}k\left(x\right)dx=1$$

In most cases, k is expected to be a probability density function that is symmetric. Here, we assume that it is the normal density function.

2.2. **Comparison of the entropy estimators.** To analyze the performance of the new entropy estimator in comparison with the other existing estimators a simulation study is conducted. This study includes comparisons between HS_n and several other estimators, namely, estimators of Vasicek [6], Correa [10], Van Es [9] and Alizadeh [11]. To facilitate this analysis, 50,000 samples are generated for each sample size and the estimators are computed along with their bias, variance and root mean squared error (RMSE). In our simulation study, we consider three distributions; Normal, Exponential and Uniform, which are the same distributions considered in Correa [10] and Alizadeh [11] works.
Tables 1-3 display the values of the absolute bias, variance and RMSE for the five entropy estimators. These values are provided at different sample sizes for each three considered distributions.

Table 1. Absolute bias, variance and root of mean squared error of the estimators in estimating $H(f)$ for the standard normal distribution.

n	HV_{mn}	HVE_{mn}	HC_{mn}	HA_{mn}	HS_n
			Absolute bias		
10	0.5599	0.2318	0.3807	0.1003	0.1280
20	0.3292	0.2080	0.1947	0.0716	0.0786
30	0.2440	0.1931	0.1281	0.0596	0.0603
50	0.1652	0.1812	0.0735	0.0458	0.0447
			Variance		
10	0.0711	0.0797	0.0717	0.0659	0.0644
20	0.0319	0.0344	0.0330	0.0301	0.0287
30	0.0206	0.0214	0.0212	0.0196	0.0183
50	0.0119	0.0122	0.0123	0.0115	0.0106
			RMSE		
10	0.6202	0.3652	0.4654	0.2757	0.2842
20	0.3745	0.2787	0.2662	0.1876	0.1866
30	0.2830	0.2423	0.1939	0.1521	0.1481
50	0.1981	0.2123	0.1330	0.1168	0.1122

Table 2. Absolute bias, variance and root of mean squared error of the estimators in estimating $H(f)$ for the exponential distribution

n	HV_{mn}	HVE_{mn}	HC_{mn}	HA_{mn}	HS_n
			Absolute bias		
10	0.4384	0.1153	0.2427	0.0986	0.0988
20	0.2568	0.1119	0.1134	0.1042	0.1383
30	0.1887	0.1061	0.0685	0.1070	0.1464
50	0.1289	0.1014	0.0327	0.1060	0.1481
			Variance		
10	0.1287	0.1388	0.1315	0.1372	0.1413
20	0.0582	0.0622	0.0601	0.0629	0.0641
30	0.0377	0.0397	0.0390	0.0411	0.0414
50	0.0222	0.0228	0.0229	0.0244	0.0237
			RMSE		
10	0.5665	0.2164	0.4363	0.3833	0.3887
20	0.3524	0.2733	0.2701	0.2715	0.2885
30	0.2707	0.2258	0.2090	0.2292	0.2507
50	0.1970	0.1820	0.1548	0.1887	0.2137

Table 3. Absolute bias, variance and root of mean squared error of the estimators in estimating $H(f)$ for the uniform distribution (0,1).

n	HV_{mn}	HVE_{mn}	HC_{mn}	HA_{mn}	HS_n
			Absolute bias		
10	0.4215	0.0004	0.2413	0.0908	0.0693
20	0.2607	0.000007	0.1295	0.1071	0.0963
30	0.2018	0.000005	0.0928	0.1121	0.1012
50	0.1507	0.00034	0.0651	0.1142	0.1024
			Variance		
10	0.0274	0.0468	0.0278	0.0303	0.0305
20	0.0075	0.0145	0.0077	0.0106	0.0101
30	0.0035	0.0075	0.0037	0.0058	0.0056
50	0.0013	0.0034	0.0015	0.0027	0.0027
			RMSE		
10	0.4529	0.2164	0.2933	0.1962	0.1881
20	0.2747	0.1206	0.1566	0.1487	0.1392
30	0.2104	0.0864	0.1111	0.1354	0.1258
50	0.1552	0.0581	0.0758	0.1254	0.1148

According to the results, the proposed estimator shows strong performance in comparison with other estimators. Additionally, the new estimator shows superior behavior compared to Vasicek [6], Correa [10], Van Es [9] and Alizadeh [11] which are significant entropy estimators. In Table 4, we conduct a comparison of entropy estimators for both small and large sample sizes. The results show that, for small sample sizes, the new estimator performs better than Van Es [9] and Correa [10] estimators. Additionally, for large sample sizes, the new estimator exhibits superior performance when compared to Alizadeh [11] estimator.

3. Goodness-of-fit tests for Weibull distribution based on the entropy estimators

In this study, we will employ the entropy estimators to assess how well the sample conforms to the Weibull distribution. Since the exponential distribution shares a unique property with entropy, we transform the Weibull distribution into the exponential one.

Suppose we have a random variable X with a probability density function $f(x)$ and a cumulative distribution function $F(x)$. We are interested to test the following hypotheses.

$$H_0 : f(x) = f_0(x) \quad \text{against } H_1 : f(x) \neq f_0(x)$$

TABLE 4. Summary of comparisons of the estimators in terms of RMSE.

		Sample size		
		All sample sizes	Small	Large
HS_n	is better than:			
HV_{mn}		Almost for all the considered distributions		
HVE_{mn}		Normal	Exponential, Uniform	
HC_{mn}		Normal	Exponential, Uniform	
HA_{mn}		Uniform		Normal

where $f_0(x)$ is the density function of the Weibull distribution given by

$$f_0\left(x\right) = \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} \exp\left\{-\left(\frac{x}{\eta}\right)^{\beta}\right\}, \qquad x > 0, \quad \eta > 0, \beta > 0.$$

If X is an exponential random variable with mean λ, then the entropy of X can be calculated as $H(X) = log\lambda + 1$, and the entropy of the exponential distribution is maximum among all distributions with nonnegative support and mean λ. Therefore, this property of maximum entropy is utilized for the exponential distribution.

Substituting $V = X^{\beta}$ in Weibull density, we have

$$f\left(v\right) = \frac{1}{\eta^{\beta}} \exp\left(-\frac{v}{\eta^{\beta}}\right), \quad v > 0.$$

Clearly the random variable V has the exponential distribution with the scale parameter $\theta = \eta^{\beta}$ and its entropy is

$$\ln\theta + 1.$$

Therefore, for all nonnegative random variables X with mean θ, we have

$$H\left(X;\theta\right) \leq \ln\theta + 1,$$

and hence

$$\frac{\exp\left\{H\left(X;\theta\right)\right\}}{\theta} \leq e.$$

Consequently, Kang and Lee [13] proposed the test statistic provided below.

$$T = \frac{\exp\left\{\hat{H}\left(X;\hat{\theta}\right)\right\}}{\hat{\theta}},$$

where $\hat{H}$ is an estimator of entropy and $\hat{\theta}$ is the maximum likelihood estimators. Here, we examine the different entropy estimators and construct the following test statistics. If we substitute the aforementioned entropy estimators in place of $\hat{H}(X;\theta)$ in the proposed test statistic, the following test statistics can be obtained.

$$TV_{mn} = \frac{\exp\{HV_{mn}\}}{\hat{\theta}},$$
$$TC_{mn} = \frac{\exp\{HC_{mn}\}}{\hat{\theta}},$$
$$TVE_{mn} = \frac{\exp\{HVE_{mn}\}}{\hat{\theta}},$$
$$TA_{mn} = \frac{\exp\{HA_{mn}\}}{\hat{\theta}},$$
$$TS_{n} = \frac{\exp\{HS_{n}\}}{\hat{\theta}}.$$

Clearly, the test statistic T is invariant with respect to the scale transformation. The critical values T_α of T at the significance level α are defined by the equation

$$P[T \le T_\alpha] = \alpha.$$

In the next section, we compute the critical points and power of the above tests.

4. Simulation study

4.1. **Critical points.** The proposed test statistics are complicated to provide the possibility of extracting its exact distribution under the null hypothesis analytically. We used Monto Carlo methods with 50,000 replicates to obtain critical values and power of the proposed tests.
The critical values of the proposed test statistics are determined by the following steps:

(1) Generate a sample of size n from the Weibull distribution with parameters 1 and 1.
(2) Compute the proposed test statistics using the provided sample $X_1, ..., X_n$.
(3) Perform steps 1 and 2 numerous times, followed by calculating the αth percentile of the test statistic.

The critical value $C(\alpha)$ is calculated from α-quantile of the distribution of the test statistics under H_0. If the value of the test statistic is smaller than $C(\alpha)$, the null hypothesis H_0 at the significance level α is rejected.
We obtain the critical values of the test statistics TV_{mn}, TC_{mn}, TVE_{mn}, TA_{mn} and TS_n for different values of m and n at the significance level $\alpha = 0.05$. These values are presented in Tables 5-9. Since $\alpha = 0.05 = 2500/50000$, we evaluated the 2500^{th} order statistic to find the value of $C(\alpha)$. Moreover, Table 10 shows the type I error of the tests for different values of n and m and we can see that these values are acceptable. Therefore, we can use the proposed tests confidently in practice.

Table 5. Critical values of the TV_{mn} statistic for $\alpha = 0.05$.

	m									
n	1	2	3	4	5	6	7	8	9	10
5	0.8596	1.3593								
6	0.9462	1.3973	1.5073							
7	1.0199	1.4250	1.5866							
8	1.0952	1.4656	1.6311	1.6240						
9	1.1549	1.5059	1.6612	1.6962						
10	1.2146	1.5468	1.6770	1.7416	1.7009					
15	1.3987	1.7226	1.8156	1.8613	1.8842	1.8787	1.8428			
20	1.5053	1.8322	1.9289	1.9664	1.9752	1.9789	1.9738	1.9580	1.9313	1.9003
25	1.5780	1.9137	2.0147	2.0468	2.0571	2.0567	2.0539	2.0439	2.0304	2.0152
30	1.6333	1.9668	2.0688	2.1062	2.1187	2.1221	2.1148	2.1093	2.1014	2.0874
40	1.7094	2.0445	2.1513	2.1923	2.2068	2.2149	2.2098	2.2037	2.1982	2.1887
50	1.7551	2.0919	2.2003	2.2476	2.2698	2.2746	2.2773	2.2728	2.2689	2.2618

Table 6. Critical values of the TC_{mn} statistic for $\alpha = 0.05$.

	m									
n	1	2	3	4	5	6	7	8	9	10
5	1.0556	1.8030								
6	1.1376	1.8186	1.9263							
7	1.2120	1.8073	2.0284							
8	1.2872	1.8212	2.0663	2.0380						
9	1.3508	1.8380	2.0698	2.1172						
10	1.4120	1.8637	2.0607	2.1641	2.1148					
15	1.6005	2.0174	2.1287	2.1942	2.2411	2.2699	2.2516			
20	1.7115	2.9536	2.2171	2.2564	2.2773	2.2972	2.3176	2.3265	2.3255	2.3128
25	1.7855	2.1957	2.2906	2.3169	2.3290	2.3373	2.3485	2.3558	2.3621	2.3710
30	1.8417	2.2440	2.3347	2.3631	2.3710	2.3797	2.3800	2.3868	2.3929	2.3962
40	1.9206	2.3180	2.4066	2.4322	2.4373	2.4447	2.4426	2.4418	2.4429	2.4438
50	1.9666	2.3602	2.4504	2.4750	2.4875	2.4871	2.4868	2.4864	2.4870	2.4853

Table 7. Critical values of the TVE_{mn} statistic for $\alpha = 0.05$.

	m									
n	1	2	3	4	5	6	7	8	9	10
5	1.5708	2.0141								
6	1.5935	1.9931	2.1821							
7	1.6302	1.9833	2.0695							
8	1.6759	1.9821	2.0405	2.1700						
9	1.7040	2.0017	2.0476	2.1177						
10	1.7466	2.0125	2.0505	2.0889	2.1655					
15	1.8745	2.1003	2.1211	2.1103	2.1002	2.1165	2.1557			
20	1.9562	2.1522	2.1829	2.1732	2.1513	2.1375	2.1289	2.1372	2.1574	2.2015
25	2.0272	2.2022	2.2262	2.2113	2.1970	2.1751	2.1556	2.1540	2.1535	2.1492
30	2.0823	2.2457	2.26288	2.2543	2.2332	2.2160	2.1969	2.1765	2.1731	2.1686
40	2.1538	2.2978	2.3197	2.3127	2.2917	2.2769	2.2583	2.2379	2.2224	2.2157
50	2.2058	2.3414	2.3557	2.3512	2.3386	2.3196	2.3018	2.2865	2.2680	2.2539

TABLE 8. Critical values of the TA_{mn} statistic for $\alpha = 0.05$.

	m									
n	1	2	3	4	5	6	7	8	9	10
5	2.8985	3.0185								
6	2.9070	3.0381	3.2044							
7	2.9287	3.0111	3.1815							
8	2.9386	3.0105	3.1543	3.2999						
9	2.9451	2.9981	3.1259	3.2680						
10	2.9519	2.9912	3.0985	3.2338	3.3625					
15	2.9717	2.9522	2.9883	3.0603	3.1597	3.2754	3.3852			
20	2.9982	2.1227	2.9402	2.9660	3.0159	3.0875	3.1776	3.2773	3.3818	3.4851
25	3.0257	2.9700	2.9364	2.9289	2.9461	2.9844	3.0349	3.1028	3.1846	3.2718
30	3.0403	2.9887	2.9433	2.9220	2.9189	2.9288	2.9574	2.9969	3.0509	3.1136
40	3.0653	3.0199	2.9745	2.9432	2.9167	2.9009	2.8976	2.9057	2.9231	2.9491
50	3.0776	3.0395	3.0016	2.9651	2.9377	2.9148	2.8981	2.8881	2.8841	2.8900

TABLE 9. Critical values of the TS_n statistic for $\alpha = 0.05$.

n	5	6	7	8	9	10	15	20	25	30	40	50
TS_n	2.9163	2.9512	2.9868	3.0071	3.0223	3.0395	3.0613	3.0688	3.0784	3.0817	3.0868	3.0863

TABLE 10. The actual size of the tests at $\alpha = 0.05$.

$W(\beta, \eta)$	n	m	TV_{mn}	TC_{mn}	TVE_{mn}	TA_{mn}	TS_n
W(0.5,1)	10	3	0.0497	0.0489	0.0495	0.0457	0.0474
W(0.5,1)	20	4	0.0507	0.0507	0.0500	0.0493	0.0513
W(0.5,1)	30	5	0.0482	0.0480	0.0498	0.0509	0.0484
W(0.5,1)	40	6	0.0502	0.0505	0.0489	0.0509	0.0501
W(0.5,1)	50	7	0.0504	0.0508	0.0515	0.0494	0.0481
W(1,1)	10	3	0.0529	0.0505	0.0513	0.0511	0.0502
W(1,1)	20	4	0.0552	0.0520	0.0552	0.0502	0.0543
W(1,1)	30	5	0.0499	0.0505	0.0498	0.0514	0.0517
W(1,1)	40	6	0.0473	0.0528	0.0486	0.0457	0.0502
W(1,1)	50	7	0.0527	0.0509	0.0498	0.0475	0.0504
W(2,1)	10	3	0.0504	0.0501	0.0513	0.0488	0.0487
W(2,1)	20	4	0.0488	0.0497	0.0476	0.0466	0.0475
W(2,1)	30	5	0.0502	0.0510	0.0497	0.0509	0.0513
W(2,1)	40	6	0.0492	0.0496	0.0509	0.0479	0.0493
W(2,1)	50	7	0.0488	0.0497	0.0453	0.0486	0.0492
W(4,1)	10	3	0.0514	0.0501	0.0516	0.0497	0.0491
W(4,1)	20	4	0.0505	0.0513	0.0498	0.0494	0.0513
W(4,1)	30	5	0.0490	0.0485	0.0486	0.0501	0.0513
W(4,1)	40	6	0.0473	0.0478	0.0487	0.0482	0.0495
W(4,1)	50	7	0.0490	0.0493	0.0508	0.0514	0.0508

4.2. **Power study.** Practically, it is useful for the researchers to have a general recommendation for choosing the parameter m when the parameter n is fixed. Our simulations indicate that the ideal value of m (based on power) varies depending on the sample size and the alternative hypothesis. However, there is no single value of m that can be considered optimal in all scenarios. Hence, if one aims to protect against all potential alternative scenarios, a compromise needs to be reached.
The power values of the tests against different alternatives are computed using Monte Carlo simulations. For each alternative 50,000 samples are generated with size 10, 20, 30, 40 and 50 with $m = [\sqrt{n} + 0.5]$. We use $m = 3$ for $n = 10$, $m = 4$ for $n = 20$, $m = 5$ for $n = 30$, $m = 6$ for $n = 40$ and $m = 7$ for $n = 50$ to achieve good power against all alternatives. Typically, as n increases, the optimal value of m also increases, as the ratio m/n tends to zero.

To compare power, we examine the following alternatives, each of which possesses varying hazard rates: increasing hazard rate *(IHR)*, decreasing hazard rate *(DHR)*, bathtub-shaped hazard rate *(BT)*, and upside-down hazard rate *(UBT)* (see Krit et al. [14]). The common distributions considered are gamma *(G)*, lognormal *(LN)*, inverse-gamma *(IG)*, and inverse-gaussian *(IS)*. Several distributions with cumulative distribution function (cdf) denoted as $F(x)$ are outlined below.

- Exponentiated Weibull distribution $EW(\theta, \eta, \beta)$ (Mudholkar and Srivastava [15]).

$$F(x) = \left[1 - e^{-\left({}^{x}/{}_{\eta}\right)^{\beta}}\right]^{\theta}, \qquad \theta, \eta, \beta > 0.$$

- Generalized Gamma distribution $GG(\kappa, \eta, \beta)$ (Stacy [16]).

$$F(x) = \frac{1}{\Gamma(\kappa)} \gamma\left(\kappa, (x/\eta)^{\beta}\right), \ \ \kappa, \eta, \beta > 0,$$

where $\gamma(s, x) = \int\limits_{0}^{x} v^{s-1} e^{-v} dv$.

- Distribution I of Dhillon [17] $D1(\beta, b)$ with cdf:

$$F(x) = 1 - e^{-\left[e^{(\beta x)^{b}} - 1\right]}, \ \ b, \beta > 0.$$

- Distribution II of Dhillon [17] $D2(\lambda, b)$ with cdf:

$$F(x) = 1 - e^{-(\ln(\lambda x + 1))^{b+1}}, \ \ \lambda > 0, b \geq 0.$$

- Hjorth [18] distribution $H(\beta, \delta, \theta)$ with cdf:

$$F(x) = 1 - \frac{e^{-\frac{\delta x^2}{2}}}{(1 + \beta x)^{\theta/\beta}}, \ \ \beta, \delta, \theta > 0.$$

- Chen [19] distribution $C(\lambda, \beta)$ with cdf:

$$F(x) = \left[1 - e^{\lambda\left(1 - e^{x^{\beta}}\right)}\right], \quad \lambda, \beta > 0.$$

Under the above alternatives the power values of the tests are obtained by means of Monte Carlo simulations. Under each alternative 50,000 samples of size 10, 20, 30, 40 and 50 are generated and the test statistics (TV_{mn}, TC_{mn}, TVE_{mn}, TA_{mn} and TS_n) are calculated. Then power value of the corresponding test is computed by the frequency of the event "the statistic is in the critical region". The power values of the tests against various alternatives at the significance level 0.05 are presented in Tables 12-16. In these tables, the tests with the highest power are shown in bold type for each sample size and alternative.

TABLE 11. Alternative distributions

IHR	$g(2) \equiv g(2,1)$ $D2(2) \equiv D2(1,2)$	$g(3) \equiv g(3,1)$	$EW1 \equiv EW(6.5, 20, 6)$
UBT	$LN(0.8) \equiv LN(0, 0.8)$ $IS(0.25) \equiv IS(1, 0.25)$	$Ig(3) \equiv Ig(3,1)$ $IS(4) \equiv IS(1,4)$	$EW4 \equiv EW(4, 12, 0.6)$
DHR	$g(0.2) \equiv g(0.2, 1)$ $D2(0) \equiv D2(1,0)$	$EW2 \equiv EW(0.1, 0.01, 0.95)$	$H(0) \equiv H(0,1,1)$
BT	$EW3 \equiv EW(0.1, 100, 5)$ $C(0,4) \equiv C(2, 0.4)$	$gg1 \equiv gg(0.1, 1, 4)$ $D1(0.8) \equiv D1(1, 0.8)$	$gg2 \equiv gg(0.2, 1, 3)$

Tables 12-16 indicate that the TS_n statistic is powerful compared to other test statistics for two groups of hazard rates *IHR* and *BT*. These tables show that TS_n is powerful for the alternative hypotheses gamma(2), gamma(3) and exponentiated weibull distributions *(EW)* in the *IHR* group, and for all alternative hypotheses in the BT group with $n = 10, 20, 30, 40$ and 50.
Also, in the *IHR* group, the TVE_{mn} statistic is powerful for Dhillon type II distribution. Additionally, these tables show that in the UBT group, the TE_{mn} statistic is powerful for all values of n, with exception for the alternative hypothesis of the exponentiated Weibull distributions EW, where TS_n is powerful. The tests TS_n and TVE_{mn} are more powerful than the other tests against *IHR* and *UBT* alternatives. In the end, the TS_n and TVE_{mn} statistics in *DHR* group for $g(0.2)$, $H(0)$, $D2(0)$ alternatives, are powerful, and TC_{mn} and TA_{mn} are powerful in this group only for *EW2* alternative hypothesis for $n = 20$ and $n = 10, 30, 40, 50$, respectively.

5. Application to the real data

In this section, we illustrate the application of the tests for Weibull distribution in the real cases. We consider two real data sets as follows.
The first collection of data represents the duration of patient's survival times. The data show the duration of times, the patients from the time admitted

TABLE 12. Power comparisons of the tests based TV_{mn}, TC_{mn}, TVE_{mn}, TA_{mn} and TS_n statistics for sample size $n = 10$ under different alternatives at level 0.05.

Alternative	TV_{mn}	TC_{mn}	TVE_{mn}	TA_{mn}	TS_n
Increasing Hazard Rate					
$g(2)$	0.0555	0.0504	0.0712	0.0278	**0.1650**
$g(3)$	0.0622	0.0539	0.0853	0.0209	**0.7509**
$EW1$	0.0753	0.0617	0.1242	0.0108	**1.0000**
$D2(2)$	0.0542	0.0476	**0.0920**	0.0278	0.0257
Upside-down bathtub Hazard					
$LN(0.8)$	0.0955	0.0758	**0.1657**	0.0061	0.0093
$Ig(3)$	0.1928	0.1490	**0.3031**	0.0022	0.0018
$EW4$	0.0554	0.0602	0.0335	0.0831	**1.0000**
$IS(0.25)$	0.1867	0.1483	**0.2547**	0.0019	0.0018
$IS(4)$	0.1010	0.0803	**0.1634**	0.0053	0.0057
Decreasing Hazard Rate					
$g(0.2)$	0.0893	0.1066	0.0191	0.1732	**0.1761**
$EW2$	0.0516	0.0510	0.0482	**0.0530**	0.0523
$H(0)$	0.0500	0.0495	0.0493	0.0504	**0.0811**
$D2(0)$	0.1071	0.0832	**0.2172**	0.0100	0.0131
Bathtub Hazard Rate					
$EW3$	0.0746	0.0621	0.1247	0.0110	**0.9607**
$gg1$	0.1219	0.1489	0.0190	0.2235	**0.2573**
$gg2$	0.0884	0.1058	0.0208	0.1693	**0.2056**
$C(0.4)$	0.0570	0.0637	0.0304	0.0858	**0.0863**
$D1(0.8)$	0.0628	0.0730	0.0245	0.1127	**0.1140**

to the hospital until death due to COVID-19. There are 53 cases of patients in China during the initial two months of 2020 (see Liu et al. [20]). Among them, 37 patients (70%) were men and 16 women (30%). 40 patients (75%) were diagnosed with chronic disease, especially including high blood pressure, and diabetes, 47 patients (88%) had common clinical symptoms of the flu, 42 patients (81%) were coughing, 37 (69%) were short of breath, and 28 patients (53%) had fatigue. 50 (95%) patients had bilateral pneumonia showed by the chest computed tomographic scans. The observed data are (time is per 24 hours):

0.054, 0.064, 0.087, 0.087, 0.235, 0.352, 0.364, 0.421, 0.437, 0.458, 0.479, 0.548, 0.568, 0.704, 0.787, 0.796, 0.816, 0.865, 0.976, 0.976, 0.978, 1.756, 1.978, 2.089, 2.643, 2.869, 3.079, 3.348, 3.543, 3.646, 3.867, 3.890, 4.092, 4.093, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.058, 7.274, 8.273, 9.324, 10.827, 11.282,

TABLE 13. Power comparisons of the tests based TV_{mn}, TC_{mn}, TVE_{mn}, TA_{mn} and TS_n statistics for sample size $n = 20$ under different alternatives at level 0.05.

Alternative	TV_{mn}	TC_{mn}	TVE_{mn}	TA_{mn}	TS_n
Increasing Hazard Rate					
$g(2)$	0.0605	0.0566	0.0927	0.0389	**0.2059**
$g(3)$	0.0676	0.0624	0.1179	0.0363	**0.9296**
$EW1$	0.0973	0.0827	0.2075	0.0347	**1.0000**
$D2(2)$	0.0543	0.0473	**0.1454**	0.0487	0.0349
Upside-down bathtub Hazard					
$LN(0.8)$	0.1541	0.1239	**0.3116**	0.0329	0.0345
$Ig(3)$	0.3846	0.3116	**0.5841**	0.0427	0.0087
$EW4$	0.0557	0.0575	0.0259	0.0668	**1.0000**
$IS(0.25)$	0.3922	0.3315	**0.5108**	0.0350	0.0065
$IS(4)$	0.1730	0.1407	**0.3111**	0.0297	0.0190
Decreasing Hazard Rate					
$g(0.2)$	0.1267	0.1379	0.0110	0.1310	**0.2422**
$EW2$	0.0499	0.0513	0.0457	**0.0519**	0.0513
$H(0)$	0.0503	0.0509	0.0497	0.0495	**0.0892**
$D2(0)$	0.1684	0.1242	**0.4132**	0.0515	0.0766
Bathtub Hazard Rate					
$EW3$	0.0974	0.0823	0.2097	0.0342	**0.9992**
$gg1$	0.2314	0.2440	0.0154	0.1915	**0.3779**
$gg2$	0.1284	0.1372	0.0115	0.1305	**0.2706**
$C(0.4)$	0.0654	0.0701	0.0194	0.0630	**0.0910**
$D1(0.8)$	0.0738	0.0785	0.0151	0.0831	**0.1345**

13.324, 14.278, 15.287, 16.978, 17.209, 19.092, 20.083

The second dataset is a COVID-19 collection of data belonged to Italy and spans 172 days from the first of March to the twentieth of August, 2020, (see Alshanbari et al. [21]). These data constituted of mortality rate and the data are as follows.

0.0107, 0.0490, 0.0601, 0.0460, 0.0533, 0.0630, 0.0297, 0.0885, 0.0540, 0.1720, 0.0847, 0.0713, 0.0989, 0.0495, 0.1025, 0.1079, 0.0984, 0.1124, 0.0807, 0.1044, 0.1212, 0.1167, 0.1255, 0.1416, 0.1315, 0.1073, 0.1629, 0.1485, 0.1453, 0.2000, 0.2070, 0.1520, 0.1628, 0.1666, 0.1417, 0.1221, 0.1767, 0.1987, 0.1408, 0.1456, 0.1443, 0.1319, 0.1053, 0.1789, 0.2032, 0.2167, 0.1387, 0.1646, 0.1375, 0.1421, 0.2012, 0.1957, 0.1297, 0.1754, 0.1390,0.1761, 0.1119, 0.1915, 0.1827, 0.1548, 0.1522, 0.1369,0.2495, 0.1253, 0.1597, 0.2195, 0.2555, 0.1956, 0.1831,0.1791, 0.2057, 0.2406, 0.1227, 0.2196, 0.2641, 0.3067, 0.1749, 0.2148, 0.2195, 0.1993,

TABLE 14. Power comparisons of the tests based TV_{mn}, TC_{mn}, TVE_{mn}, TA_{mn} and TS_n statistics for sample size $n = 30$ under different alternatives at level 0.05.

Alternative	TV_{mn}	TC_{mn}	TVE_{mn}	TA_{mn}	TS_n
Increasing Hazard Rate					
$g(2)$	0.0620	0.0593	0.1083	0.0547	**0.2909**
$g(3)$	0.0746	0.0667	0.1468	0.0583	**0.9869**
$EW1$	0.1213	0.1009	0.2890	0.0768	**1.0000**
$D2(2)$	0.0565	0.0466	**0.1899**	0.0829	0.0517
Upside-down bathtub Hazard					
$LN(0.8)$	0.2155	0.1706	**0.4438**	0.0915	0.0854
$Ig(3)$	0.5586	0.4664	**0.7669**	0.1531	0.0587
$EW4$	0.0576	0.0607	0.0198	0.0472	**1.0000**
$IS(0.25)$	0.5798	0.5011	**0.7090**	0.1173	0.0236
$IS(4)$	0.2404	0.1931	**0.4419**	0.0865	0.0491
Decreasing Hazard Rate					
$g(0.2)$	0.1812	0.1860	0.0087	0.0778	**0.2700**
$EW2$	0.0498	**0.0499**	0.0435	0.0491	0.0460
$H(0)$	0.0488	0.0491	0.0502	0.0490	**0.1017**
$D2(0)$	0.2281	0.1660	**0.5612**	0.1507	0.2009
Bathtub Hazard Rate					
$EW3$	0.1221	0.1019	0.2899	0.0758	**1.0000**
$gg1$	0.3586	0.3658	0.0145	0.1314	**0.4653**
$gg2$	0.1823	0.1879	0.0093	0.0789	**0.3087**
$C(0.4)$	0.0712	0.0750	0.0144	0.0407	**0.0825**
$D1(0.8)$	0.0871	0.0901	0.0105	0.0499	**0.1343**

0.2421, 0.2430, 0.1994, 0.1779, 0.0942, 0.3067, 0.1965, 0.2003, 0.1180, 0.1686, 0.2668, 0.2113, 0.3371, 0.1730, 0.2212, 0.4972, 0.1641, 0.2667, 0.2690, 0.2321, 0.2792, 0.3515, 0.1398, 0.3436,0.2254, 0.1302, 0.0864, 0.1619, 0.1311, 0.1994, 0.3176, 0.1856, 0.1071, 0.1041, 0.1593, 0.0537, 0.1149, 0.1176, 0.0457, 0.1264, 0.0476, 0.1620, 0.1154, 0.1493, 0.0673, 0.0894, 0.0365, 0.0385, 0.2190, 0.0777, 0.0561, 0.0435, 0.0372, 0.0385, 0.0769, 0.1491, 0.0802, 0.0870, 0.0476, 0.0562, 0.0138, 0.0684, 0.1172, 0.0321, 0.0327, 0.0198,0.0182, 0.0197, 0.0298, 0.0545, 0.0208, 0.0079, 0.0237,0.0169, 0.0336, 0.0755, 0.0263, 0.0260, 0.0150, 0.0054, 0.0375, 0.0043, 0.0154, 0.0146, 0.0210, 0.0115, 0.0052,0.2512, 0.0084, 0.0125, 0.0125, 0.0109, 0.0071.

Now, using the proposed tests, we test whether the data come from a Weibull distribution. The value of the proposed test statistic and the critical points are computed and presented in Table 17. According to the results presented in Table 17, the value of each test statistics is greater than the corresponding

TABLE 15. Power comparisons of the tests based TV_{mn}, TC_{mn}, TVE_{mn}, TA_{mn} and TS_n statistics for sample size $n = 40$ under different alternatives at level 0.05.

Alternative	TV_{mn}	TC_{mn}	TVE_{mn}	TA_{mn}	TS_n
Increasing Hazard Rate					
$g(2)$	0.0628	0.0585	0.1252	0.0668	**0.3946**
$g(3)$	0.0782	0.0692	0.1820	0.0792	**0.9983**
$EW1$	0.1397	0.1137	0.3672	0.1152	**1.0000**
$D2(2)$	0.0552	0.0451	**0.2337**	0.1148	0.0609
Upside-down bathtub Hazard					
$LN(0.8)$	0.2679	0.2111	**0.5631**	0.1558	0.1453
$Ig(3)$	0.6846	0.5854	**0.8847**	0.2792	0.1056
$EW4$	0.0624	0.0637	0.0162	0.0369	**1.0000**
$IS(0.25)$	0.7114	0.6296	**0.8414**	0.1960	0.0422
$IS(4)$	0.3037	0.2472	**0.5651**	0.1435	0.0810
Decreasing Hazard Rate					
$g(0.2)$	0.2384	0.2362	0.0075	0.0530	**0.3013**
$EW2$	0.0502	**0.0510**	0.0428	0.0477	0.0400
$H(0)$	0.0484	0.0497	0.0469	0.0499	**0.1244**
$D2(0)$	0.2848	0.2106	**0.6851**	0.2694	0.3390
Bathtub Hazard Rate					
$EW3$	0.1416	0.1161	0.3644	0.1154	**1.0000**
$gg1$	0.4898	0.4886	0.0163	0.1077	**0.5517**
$gg2$	0.2410	0.2402	0.0071	0.0531	**0.3555**
$C(0.4)$	0.0819	**0.0846**	0.0109	0.0289	0.0772
$D1(0.8)$	0.1023	0.1042	0.0074	0.0334	**0.1343**

critical value. As a result, the Weibull hypothesis cannot be rejected at the significance level of 0.05. Consequently, we can conclude that the probability density of these data sets follows a Weibull distribution.

6. Conclusions

In this paper, we have presented a new estimator for the entropy of a continuous random variable and then computed its RMSE to conduct a comparison with the other estimators. We found that the new entropy estimator outperforms the others in situations where the sample size is small for exponential and uniform distributions, as well as in cases where the sample size is large for normal distribution.

Moreover, we have proposed some GOF tests for Weibull distribution based on the entropy estimators. The proposed test statistics were easy to compute. Percentage points and power of the proposed tests against various alternatives for different sample sizes were reported. Power of the tests showed that this

Table 16. Power comparisons of the tests based TV_{mn}, TC_{mn}, TVE_{mn}, TA_{mn} and TS_n statistics for sample size $n = 50$ under different alternatives at level 0.05.

Alternative	TV_{mn}	TC_{mn}	TVE_{mn}	TA_{mn}	TS_n
Increasing Hazard Rate					
$g(2)$	0.0652	0.0600	0.1486	0.0774	**0.5082**
$g(3)$	0.0847	0.0764	0.2212	0.0932	**0.9999**
$EW1$	0.1604	0.1309	0.4500	0.1551	**1.0000**
$D2(2)$	0.0586	0.0478	**0.2876**	0.1500	0.0744
Upside-down bathtub Hazard					
$LN(0.8)$	0.3155	0.2512	**0.6723**	0.2156	0.2225
$Ig(3)$	0.7753	0.6791	**0.9453**	0.3956	0.1613
$EW4$	0.0648	0.0669	0.0142	0.0319	**1.0000**
$IS(0.25)$	0.8052	0.7303	**0.9261**	0.2794	0.0687
$IS(4)$	0.3620	0.2952	**0.6800**	0.1994	0.1247
Decreasing Hazard Rate					
$g(0.2)$	0.3103	0.3054	0.0073	0.0398	**0.3346**
$EW2$	0.0491	**0.0492**	0.0447	0.0475	0.0383
$H(0)$	0.0495	0.0509	0.0498	0.0491	**0.1473**
$D2(0)$	0.3326	0.2449	**0.7814**	0.3752	0.4809
Bathtub Hazard Rate					
$EW3$	0.1633	0.1327	0.4549	0.1556	**1.0000**
$gg1$	0.6156	0.6065	0.0197	0.0967	**0.6273**
$gg2$	0.3079	0.3022	0.0069	0.0389	**0.4000**
$C(0.4)$	0.0921	**0.0944**	0.0101	0.0230	0.0694
$D1(0.8)$	0.1276	0.1295	0.0058	0.0236	**0.1317**

Table 17. Results for Examples 1 and 2.

	Example 1			Example 2		
	Value of the test statistic	Critical value	Decision	Value of the test statistic	Critical value	Decision
TV_{mn}	2.5500	2.2929	Not rejected H_0	2.5409	2.5175	Not rejected H_0
TC_{mn}	2.9277	2.4997	Not rejected H_0	2.6717	2.6414	Not rejected H_0
TVE_{mn}	2.2859	2.3084	Not rejected H_0	2.4944	2.4507	Not rejected H_0
TA_{mn}	3.1446	2.8967	Not rejected H_0	3.0073	2.9531	Not rejected H_0
TS_n	3.1034	3.0864	Not rejected H_0	3.1211	3.0532	Not rejected H_0

GOF tests are viable for testing the hypothesis of Weibull. Generally, we observed that the tests TE_{mn} and TS_n have the most power for the Weibull distribution.

7. Aknowledgement

The authors are very grateful to anonymous referees and the associate editor for providing some useful comments on an earlier version of this manuscript.

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