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GENERALIZED TOTAL TIME ON TEST TRANSFORM FOR WEIGHTED VARIABLES, PROPERTIES AND APPLICATIONS

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Special issue Dedicated to memory of professor Mahbanoo Tata

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Abstract. In this article, the generalized total time on test transform and some related transforms for weighted variables are stated. Their characteristics and relationship with each other have been considered and also these transforms have been investigated in the weighted mode from the point of view of stochastic orders. Also, by presenting graphs of generalized total time on test transform for some common weight functions, its behavior with respect to the weighted function is studied. Then the relationship of this transform with its initial state is expressed. In the following, the topic under discussion is explained with some practical examples. Then providing a comprehensive exploration of the applications of the studied transforms within the domains of insurance and reliability. By delving into these practical contexts, we gain valuable insights into how these mathematical tools can be effectively utilized to address complex challenges in risk assessment, decision-making, and resource allocation. Additionally, the examination of the NBU class of distributions offers a deeper understanding of their behavior, shedding light on their relevance and applicability in various statistical analyses. Finally, the article concludes with a detailed discussion of a specific real dataset, offering a concrete demonstration of how the topic under study can be applied in practice.

Keywords: Total Time on Test Transform, Weighted Variable, Location Independent Riskier Transform, Excess Wealth Transform, Distortion Pricing Principle.
2020 MSC: Primary 62N05 - 62E15 - 62G30.

1. Introduction

The total time on test (TTT) transform and its plot is the most widely used and useful tools in reliability. [2–5,15,18] described their use in reliability. Recently, [10] have described in detail the applications of the TTT plot.

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[16] introduced a family of stochastic orders using a real function $h(\cdot)$. They investigated its characteristics and obtained interesting results regarding the relationship of stochastic orders with each other as well as applications of this family in actuarial science and reliability theory. The introduced family with appropriate choices of $h(\cdot)$ includes some well-known stochastic orders, such as the usual stochastic order, the location-independent riskier order, and the TTT transform order. [26–28] obtained interesting results in this field.

Weighted distributions were widely developed by statisticians and data analysts to provide efficient statistical models for the arising data from various domains, including medicine, industry, ecology, reliability, and many other fields. Studies on weighted variables have been started by [9]. Then [23] developed a unifying approach that can be applied to a variety of sampling situations and visualized using weighted distributions. Recently, [19, 21, 24] have conducted studies in this field and expressed the features and applications of weighted distributions. Also, [6] and [1] have presented results using weight functions.

This article is organized as follows: In Section 2, some essential preliminaries are reviewed. In Section 3, we describe the generalized total time on test transform for weighted variables, and its relationship with excess wealth and location independent riskier transforms be described. We also present the features of these transforms in weighted mode and we have studied the conditions for establishing some stochastic orders by considering weighted variables. In Section 4, applications of the generalized total time on test transform in the weighted mode in insurance, the reliability of expression and the behavior of NBU class by considering the weighted variables are also considered. Finally, in Section 5, we examine and study the breaking stress of the carbon fibers dataset.

2. Preliminaries

In this section, we briefly state the concepts needed in the article. Assume throughout the article that X and Y are two random variables with the corresponding distribution functions $F(\cdot)$ and $G(\cdot)$ and probability density functions $f(\cdot)$ and $g(\cdot)$, respectively. In addition, let $\bar{F}(\cdot) = 1 - F(\cdot)$ $\left(\bar{G}(\cdot) = 1 - G(\cdot)\right)$ be the survival function, and also let $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in (0, 1)$, be the quantile function.

Definition 2.1. For the nonnegative continuous random variable X, the TTT, excess wealth (EW), and location independent riskier (LIR) transforms are, respectively, defined as

$$T_X(p) = \int_0^{F^{-1}(p)} \bar{F}(x)dx, \quad p \in (0, 1),$$

$$EW_X(p) = \int_{F^{-1}(p)}^{\infty} \bar{F}(x)dx, \quad p \in (0, 1),$$

$$LIR_X(p) = \int_0^{F^{-1}(p)} F(x)dx, \quad p \in (0,1).$$

Note that $T_X(1) = E(X)$, where the expectation $E(X)$ can be finite or infinite. It can also be easily concluded that $T_X(p) + EW_X(p) = E(X)$ and $T_X(p) + LIR_X(p) = F^{-1}(p)$.

Definition 2.2. Let X and Y be two random variables with distribution functions $F(\cdot)$ and $G(\cdot)$ that have finite means. Then X is said to be smaller than Y in the

- TTT transform order, denoted by $X \leq_{ttt} Y$, if $T_X(p) \leq T_Y(p), \quad p \in (0,1)$;
- EW transform order, denoted by $X \leq_{ew} Y$, if $EW_X(p) \leq EW_Y(p), \quad p \in (0,1)$;
- LIR transform order, denoted by $X \leq_{lir} Y$, if $LIR_X(p) \leq LIR_Y(p), \quad p \in (0,1)$.

[15] used the TTT and EW transforms orders to compare two nonnegative random variables and also presented various applications for them. The LIR order and its features were also described by [11]. [16] introduced a general family of orders as follows.

Definition 2.3. Let ξ denote the set of all functions $h(\cdot)$ such that $h(u) > 0$ for $u \in (0,1)$, and $h(u) = 0$ for $u \notin [0,1]$. For $h \in \xi$, if

$$T_X^{(h)}(p) \leq T_Y^{(h)}(p), \qquad p \in (0,1),$$

then we say that X is smaller than Y in the $GTTT$ transform order with respect to $h(\cdot)$ and denote this by $X \leq_{ttt}^{(h)} Y$, wherein

$$T_X^{(h)}(p) = \int_{-\infty}^{F^{-1}(p)} h(F(x))dx, \quad p \in (0,1), \tag{1}$$

and is called $GTTT$ transform. Note that for $h(u) = u$, the $GTTT$ transform is the same as the LIR transform, and for $h(u) = 1 - u$, it gives the TTT transform. Similarly, the generalized EW transform is defined by

$$EW_X^{(h)}(p) = \int_{F^{-1}(p)}^{\infty} h(F(x))dx, \qquad p \in (0,1). \tag{2}$$

It is clear that equation (1) can also be considered for the generalized LIR transform.

A brief description of the weight function and its characteristics as well as some of the required definitions, are given in what follows. Let $w(\cdot) : R \longrightarrow R^+$ be a function for which $0 < E[w(X)] < \infty$. Then

$$F_w(x) = \frac{1}{E[w(X)]} \int_{-\infty}^{x} w(u)dF(u) = \frac{1}{E[w(X)]} \int_0^{F(x)} wF^{-1}(z)dz,$$

is a distribution function, called the weighted distribution associated with the baseline distribution $F(\cdot)$. If the density $f(\cdot)$ of $F(\cdot)$ exists, then $f_w(x) = \frac{w(x)f(x)}{E[w(x)]}$ is the density of $F_w(x)$. If $F(0) = 0$ and $w(x) = x^k$, where k is a positive integer, then we call $F_w(x)$ the length-biased (or size-biased) distribution of order $k(\cdot)$ and denote it by $F_{w_k}(x)$ and simply by $F_{w_1}(x)$ if $k = 1$. If $E(X) < \infty$, then $F_{w_1}(x) = \frac{1}{E(X)} \int_0^x udF(u)$ and $f_{w_1}(x) = \frac{xf(x)}{E(X)}, x > 0$. It is obvious that the function $\frac{F^{-1}(u)}{E(X)}$ is a probability density function on $(0, 1)$.

Throughout this study, we have used notations similar to [25]. On the other hand, stochastic orders are relationships between probability distributions.

Definition 2.4. Let X and Y be two random variables with distribution functions $F(\cdot)$ and $G(\cdot)$.

- X is stochastically smaller than Y ($X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x values.
- X is smaller than Y in the convex transform order ($X \leq_c Y$) if $G^{-1}(F(x))$ be convex.
- X is smaller than Y in the dispersive order ($X \leq_{disp} Y$) if $F^{-1}(b) - F^{-1}(a) \leq G^{-1}(b) - G^{-1}(a)$ whenever $0 < a \leq b < 1$.

[25] is a complete reference for studying the conditions for establishing the opposite of the above relations.

Definition 2.5. Let $F(\cdot)$ be a lifetime distribution.

- $F(\cdot)$ is said to be $IFR(DFR)$ [increasing(decrasing) failure rate] if $r(x)$ is increasing (decreasing) on S_F(support of $F(.)$) being an interval, where $r(x) = \frac{f(x)}{\bar{F}(x)}$ is the hazard rate function of $F(\cdot)$.
- $F(\cdot)$ is said to be $NBU(NWU)$ [new better(worse) than used] if for $x, y \geq 0$,
$$\bar{F}(x + y) \leq (\geq)\bar{F}(x)\bar{F}(y).$$

3. Main Results

In this section, the $GTTT$ transform for weighted variables is first defined, and then its properties are examined. Let $Y = w(X)$ be a random variable, where $w(\cdot)$ is a derivative weight function with probability density function $f_w(x)$. [8] developed the TTT transform for the weighted random variables and investigated the behavior of the failure rate function of such variables based on the TTT transform. In addition, the conditions for establishing the TTT transform ordering for weight variables and its relationship with some stochastic orders are investigated, and the conditions for establishing the TTT transform order as well as the presentation of the new better than used in total time on test transform ($NBUT$) class of the weighted variables were also studied. Finally, expressed interesting applications of it in different fields and by analyzing the real data sets, applications of the transform introduced in the fit of a model is presented, and it is shown that weighted models have a significant advantage over the base models.

Proposition 3.1. *If $w(\cdot)$ is an increasing function, then for the nonnegative random variable Y, we have $F_Y^{-1}(p) = w(F^{-1}(p))$ and $\bar{F}_Y(x) = \bar{F}(w^{-1}(x))$. Hence, by using equations (1) and (2), the GTTT and generalized EW transforms for weighted variable Y are*

(3)
$$T_Y(p) = \int_0^{w(F^{-1}(p))} h(F(w^{-1}(x))dx = \int_0^{F^{-1}(p)} h(F(z))w^{'}(z)dz, \quad p \in (0,1),$$

(4)
$$EW_Y(p) = \int_{w(F^{-1}(p))}^{\infty} h(F(w^{-1}(x))dx = \int_{F^{-1}(p)}^{\infty} h(F(z))w^{'}(z)dz, \quad p \in (0,1),$$

where the last equalities are obtained from changing the variable $z = w^{-1}(x)$. If $w(x) = K(F(x))$, then $w^{'}(x) = f(x)K'(F(x))$, and therefore, by changing the variable $z = F(x)$, we achieve

$$T_Y(p) = \int_0^{F^{-1}(p)} f(x)h(F(x))K^{'}(F(x))dx = \int_0^{p} h(z)K^{'}(z)dz, \tag{5}$$

$$EW_Y(p) = \int_{F^{-1}(p)}^{\infty} f(x)h(F(x))K^{'}(F(x))dx = \int_p^{1} h(z)K^{'}(z)dz. \tag{6}$$

When the weight function is a function of the distribution function, equation (5) is more suitable for obtaining the TTT transform of weighted variables.

Note that if $w(\cdot)$ be a decreasing function, then $F_Y^{-1}(p) = w(F^{-1}(1-p))$ and

$$T_Y(p) = \int_0^{w(F^{-1}(1-p))} h(\bar{F}(w^{-1}(x))dx.$$

Remark 3.2. In the special case for the weight function $w(x) = K(F(x)) + K(\bar{F}(x))$, the above transforms are,

$$\begin{aligned} T_Y(p) &= \int_0^{F^{-1}(p)} f(x)h(F(x))[K^{'}(F(x)) - K^{'}(\bar{F}(x))]dx \\ &= \int_0^{p} h(z)[K^{'}(z) - K^{'}(1-z)]dz, \\ EW_Y(p) &= \int_{F^{-1}(p)}^{\infty} f(x)h(F(x))[K^{'}(F(x)) - K^{'}(\bar{F}(x))]dx \\ &= \int_p^{1} h(z)[K^{'}(z) - K^{'}(1-z)]dz. \end{aligned}$$

TABLE 1. $GTTT$ transforms for $h(u) = (1-u)^2$ and regarding some well-known weight functions

	Weight function	$GTTT$ Transform
$w_1(x)$	x	$T_X^{(h)}(p)$
$w_2(x)$	$F(x) + \bar{F}(x) \ln \bar{F}(x)$	$\frac{1}{3}(1-p)^3 \ln(1-p) - \frac{1}{9}(1-p)^3 + \frac{1}{9}$
$w_3(x)$	$1 - (1-\alpha)\bar{F}(x)$	$\frac{(1-\alpha)}{3}(1-(1-p)^3)$
$w_4(x)$	$1 - \bar{F}^a(x)$	$\frac{a}{a+2}(1-(1-p)^{a+2})$
$w_5(x)$	$F(x)(1 + a\bar{F}(x))$	$\frac{1}{3} - \frac{a}{4} - \frac{(1-p)^3}{3} + \frac{a(1-p)^4}{4} + \frac{ap^2}{2} - \frac{a}{3}p^3 + \frac{a}{4}p^4$

Remark 3.3. With some slight calculations, for each $h(\cdot) \in \xi$ and $a, b \in \mathbb{R}$ it can be simply derived that for $Y = w(X) = aX + b \in \mathbb{R}^+$, the relation $T_Y(p) = |a| T_X^{(h)}(p)$ is hold that result in scale invariant property.

As can be seen, transform (3) is the general state of transform (1) presented in [8], when $h(u) = 1 - u$.

If X is a nonnegative random variable with distribution function $F(\cdot)$ and $Y = w(X)$ is a weighted random variable, then we can simply show that $F_Y(.) \in IFR(DFR)$ if

$$T_Y^{'}(p) = \frac{h(p) w^{'}(F^{-1}(p))}{f(F^{-1}(p))} \quad \searrow (\nearrow) in \quad p, \qquad \text{for all } p \in (0,1).$$

Also, by replacement $p = F(x)$, in certain cases, it can be concluded that

- $T_Y(p)$ is convex (concave) for $h(u) = 1 - u$ if and only if $\frac{w^{'}(x)}{r(x)}$ is increasing (decreasing), where $r(x) = \frac{f(x)}{\bar{F}(x)}$ is the hazard rate function of F.
- $T_Y(p)$ is convex (concave) for $h(u) = u$ if and only if $\frac{w^{'}(x)}{\check{r}(x)}$ is increasing (decreasing), where $\check{r}(x) = \frac{f(x)}{F(x)}$ is the reversed hazard rate function of F.

For $h(u) = (1-u)^2$ and some weight functions, the weighted $GTTT$ transform has an explicit form, which is presented in Table 1.

Recently, [7] showed that if X is a random variable with a distribution function $F(\cdot)$ and $w(x) = ae^{-a\bar{F}(x)} + e^{-a}, a > 0$, is an increasing weight function, then X_w is a weighted variable with distribution function of the form

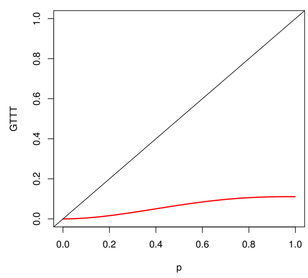
$$F_w(x) = e^{-a\bar{F}(x)} - e^{-a}\bar{F}(x).$$

For $h(u) = (1-u)^i, i \geq 2$, we have

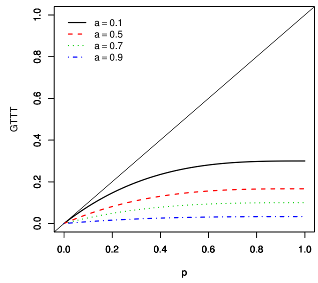
$$T_w^{(h)}(u) = \frac{\Gamma(i+1)}{a^{i-1}} [F_{Gamma}(1, i+1, a) - F_{Gamma}(1-u, i+1, a)],$$

wherein $F_{Gamma}(x, a, b)$ is the Gamma distribution function in x with parameters a and b.

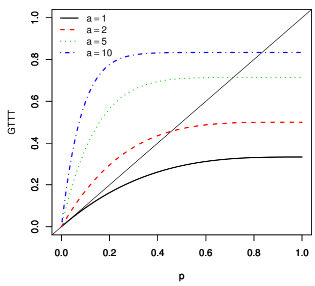
Figure 1 shows the TTT plots for some of the weight functions of Table 1 and $h(u) = (1-u)^2$. It is easy to examine the behavior of the failure rate function of the weighted distribution over the $GTTT$ plot.



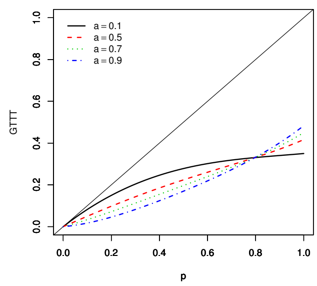
(A) w_2.



(B) w_3.



(C) w_4.



(D) w_5.

FIGURE 1. $GTTT$ plot for some weight functions with different values of parameter a.

3.1. **Stochastic orders for weighted variables.** In this section, we study the conditions of establishing some stochastic orders by weighted variables and provide useful results.

Proposition 3.4. *Let X and Y be two nonnegative random variables with distribution functions $F(\cdot)$ and $G(\cdot)$, respectively.*

a)Let $w(x)$ be a weighted random variable, then $X <_{ttt}^{(h)} X_w$ if $w^{'}(x) \geq 1$.

b) For two increasing weight functions $w_1(\cdot)$ and $w_2(\cdot)$, such that $w_1^{'}(x) \leq w_2^{'}(x)$ and decreasing function $h(\cdot) \in \xi$, if $X \leq_{st} Y$ then $X_{w_1} \leq_{ttt}^{(h)} Y_{w_2}$.

Proof. To prove them, it is enough to use the definition of stochastic orders. Here, we prove the problem for the TTT transform order.

a) For the weight function $w(\cdot)$, we have

$$w^{'}(s) \geq 1 \Longrightarrow h(F(s)) \leq h(F(s))w^{'}(s) \Longrightarrow \int_0^{F^{-1}(p)} h(F(s))ds \leq \int_0^{F^{-1}(p)} h(F(s))w^{'}(s)ds$$

$$\Longleftrightarrow T_X^{(h)}(p) \leq T_{X_w}^{(h)}(p) \Longleftrightarrow X \leq_{ttt}^{(h)} X_w.$$

b) To prove it, from the decreasing of h and $X \leq_{st} Y$, it can be concluded that

$$h(F(s))w_1^{'}(s) \leq h(G(s))w_2^{'}(s) \Longrightarrow \int_0^{F^{-1}(p)} h(F(s))w_1^{'}(s)ds \leq \int_0^{G^{-1}(p)} h(G(s))w_2^{'}(s)ds.$$

$$\Longleftrightarrow T_{X_{w_1}}^{(h)}(p) \leq T_{Y_{w_2}}^{(h)}(p) \Longleftrightarrow X_{w_1} <_{ttt}^{(h)} Y_{w_2}.$$

□

Corollary 3.5. *Let $w_1(\cdot)$ and $w_2(\cdot)$ be two derivative weight functions. In this case, it holds that*

$$w_1^{'}(x) \leq w_2^{'}(x) \Longleftrightarrow X_{w_1} \leq_{ttt}^{(h)} X_{w_2}.$$

Now, according to the results presented by [16], the following results can be rewritten for the $GTTT$ transform order.

Proposition 3.6. *Let X and Y be two nonnegative random variables with distribution functions $F(\cdot)$ and $G(\cdot)$, respectively, and let $w_1(\cdot)$ and $w_2(\cdot)$ be weighted random variables. Let $h \in \xi$.*

a) If $h(\cdot)$ is decreasing on $(0,1)$, then $X_{w_1} \leq_{st} Y_{w_2} \Longrightarrow X_{w_1} \leq_{ttt}^{(h)} Y_{w_2}$.

b) If $h(\cdot)$ is increasing on $(0,1)$, then $X_{w_1} \leq_{st} Y_{w_2} \Longrightarrow X_{w_1} \geq_{ttt}^{(h)} Y_{w_2}$.

Proof. Let $X_{w_1} \leq_{st} Y_{w_2}$, then $\bar{F}_{w_1}(x) \leq \bar{G}_{w_2}(x)$, $F_{w_1}^{-1}(p) \leq G_{w_2}^{-1}(p)$, and thus for the increasing function h, we have $h(\bar{F}_{w_1}(x)) \leq h(\bar{G}_{w_2}(x))$. So it can be easily concluded that

$$\int_0^{F_{w_1}^{-1}(p)} h(\bar{F}_{w_1}(s))w_1^{'}(s)ds \leq \int_0^{G_{w_2}^{-1}(p)} h(\bar{G}_{w_2}(x))w_2^{'}(s)ds,$$

which means $X_{w_1} \leq_{ttt}^{(h)} Y_{w_2}$ and the proof of (b) is complete. Proof (a) is similar to (b).

□

The following results were proved for nonnegative random variables X and Y by [16], which can be easily generalized for the weighted variables.

Corollary 3.7. *Let X and Y be two nonnegative random variables and let $w(\cdot)$ be a weighted function. Also, let $h_1(\cdot), h_2(\cdot) \in \xi$ and let $K_1(\cdot)$ and $K_2(\cdot)$ be two absolutely continuous distribution functions with supports $[0, a)$ and $[0, b)$ for some finite or infinite constants a and b, and density functions $k_1(\cdot)$ and $k_2(\cdot)$, respectively.*

a) If $\frac{h_2(u)}{h_1(u)}$ is decreasing on $(0, 1)$, then $X_w \leq_{ttt}^{(h_1)} Y_w \Longrightarrow X_w \leq_{ttt}^{(h_2)} Y$.

b) If $X_w \leq_{ttt}^{(k_1 K_1^{-1})} Y_w$ and $K_1 \leq_c K_2$, then $X_w \leq_{ttt}^{(k_2 K_2^{-1})} Y_w$.

c) If $K_1 \in IFR$ and $K_2 \in DFR$, then
- $X_w \leq_{ttt}^{(k_1 K_1^{-1})} Y_w \Longrightarrow X_w \leq_{ttt} Y_w$,
- $X_w \leq_{ttt} Y_w \Longrightarrow X_w \leq_{ttt}^{(k_2 K_2^{-1})} Y_w$,
- $X_w \leq_{ttt}^{(k_1 K_1^{-1})} Y_w \Longrightarrow X_w \leq_{ttt}^{(k_2 K_2^{-1})} Y_w$.

d) If $h(\cdot)$ is increasing on $(0, 1)$, then $X_w \leq_{st} Y_w \Longrightarrow X_w \leq_{ttt}^{(h)} Y_w$.

e) If $h(\cdot)$ is decreasing on $(0, 1)$, then $X_w \leq_{ttt}^{(h)} Y_w \Longrightarrow X_w \leq_{st} Y_w$.

Proposition 3.8. *Let X and Y be two nonnegative random variables with distribution functions $F(\cdot)$ and $G(\cdot)$, respectively, and let $w(\cdot)$ be a weighted random variable. Then*

$$X_w \leq_{lir} Y_w \Longrightarrow X_w \leq_{st} Y_w \Longrightarrow X_w \leq_{ttt} Y_w.$$

Proof. If we put $h_1(u) = u$ and $h_2(u) = c$, then $\frac{h_2(u)}{h_1(u)}$ is decreasing on $(0, 1)$. Therefore, it follows from part (a) of Corollary 3.7 that $X_w \leq_{ttt}^{(h_1)} Y_w \Longrightarrow X_w \leq_{ttt}^{(h_2)} Y$, and this also means $X_w \leq_{lir} Y_w \Longrightarrow X_w \leq_{st} Y_w$. So we have $\bar{F}_{X_w}(x) \leq \bar{G}_{Y_w}(x)$ and $F_{X_w}^{-1}(p) \leq G_{Y_w}^{-1}(p)$. Hence

$$\int_0^{F_{X_w}^{-1}(p)} \bar{F}_{X_w}(s) ds \leq \int_o^{G_{Y_w}^{-1}(p)} \bar{G}_{Y_w}(s) ds \Longleftrightarrow X_w \leq_{ttt} Y_w.$$

□

It is known that for nonnegative random variables X and Y, if $X \leq_{disp} Y$, then $X \leq_{st} Y$. So for the weight function $w(\cdot)$, we can conclude that

$$X_w \leq_{disp} Y_w \Longrightarrow X_w \leq_{st} Y_w \Longrightarrow X_w \leq_{ttt} Y_w.$$

Also, based on the results presented by [26], it can be concluded that if X and Y are two random variables with continuous distribution functions and interval supports and also $w_1(\cdot)$ and $w_2(\cdot)$ are two weight functions, then the following results hold:

- $X_{w_1} \leq_{disp} Y_{w_2} \iff X_{w_1} \leq_{ttt}^{(h)} Y_{w_2}$ for all $h \in \xi$.
- $X_{w_1} \leq_{lir} Y_{w_2} \iff X_{w_1} \leq_{ttt}^{(h)} Y_{w_2}$ for all concave $h \in \xi$.
- $X_{w_1} \leq_{ew} Y_{w_2} \iff -X_{w_1} \leq_{lir} -Y_{w_2}$.

Closure under increasing concave transformations can also be investigated and expressed in terms of weight variables for the three studied stochastic orders. For this purpose, let X and Y be two continuous random variables with interval supports (l_X, u_X) and (l_Y, u_Y). Also, let $w_1(\cdot)$ and $w_2(\cdot)$ be two weight functions and let $h(\cdot) \in \xi$ be differentiable and strictly increasing on $[0, 1]$. For any increasing concave function φ, we have

- if $u_Y \leq u_X < \infty$, then $X_{w_1} \leq_{ttt}^{(h)} Y_{w_2} \implies \varphi(X_{w_1}) \leq_{ttt}^{(h)} \varphi(Y_{w_2})$,
- if $u_Y \leq u_X < \infty$, then $X_{w_1} \leq_{lir} Y_{w_2} \implies \varphi(X_{w_1}) \leq_{lir} \varphi(Y_{w_2})$,

and for any increasing convex function φ, if $-\infty < l_X \leq l_Y$ and $X_{w_1} \leq_{ew} Y_{w_2}$, then $\varphi(X_{w_1}) \leq_{ew} \varphi(Y_{w_2})$.

4. **Applications**

The applications of the presented concepts in insurance and reliability theory described in follow.

4.1. **Insurance.** [16] presented some applications of the $GTTT$ transform in actuarial science and insurance and reliability. In this section, we describe the application of insurance based on weighted variables.

Let X be the loss of an insurance contract and let $w(\cdot)$ be a weight function. Then the expected loss of X_w is

$$E[X_w] = \int_0^\infty \bar{F}_w(x)dx.$$

The expected loss can be utilized as the premium paid for the insurance contract. To give more loss weight to higher risks, a distortion pricing principle is sometimes utilized practically. Let φ be a distortion function; that is, $\varphi : [0, 1] \longrightarrow [0, 1]$ is an increasing function such that $\varphi(0) = 0$ and $\varphi(1) = 1$. The distortion pricing principle that is based on φ states that

$$\rho_\varphi(X_w) = \int_0^\infty \varphi(\bar{F}_w(x))dx \tag{7}$$

is the premium paid for the insurance contract. If $\varphi \in \xi$, then

$$X_w \leq_{ttt}^{g_\varphi} Y_w \Longrightarrow \rho_\varphi(X_w) \leq \rho_\varphi(Y_w),$$

wherein $g_\varphi(u) = \varphi(1 - u)$.

The value of the distortion pricing principle presented in (7) can be obtained for each random variable and different weighted functions. In the following examples, we obtain the distortion pricing principle for some distribution functions that have the desired conditions.

Example 4.1. *Let X be a random variable with distribution function $F(x) = 1 - \frac{1}{(1+\alpha x)^\theta}, x, \alpha, \theta > 0$. If $\psi(u) = u^{\frac{\theta+1}{\theta}}$, then $\rho_\varphi(X) = \frac{1}{\alpha\theta}$. Now let $X_i \sim F_{\alpha_i,\theta_i}(x_i), i = 1, 2$. It is clear that for fix value α if $\theta_1 > \theta_2$, then $\rho_\varphi(X_1) < \rho_\varphi(X_2)$. Also for fix value θ if $\alpha_1 > \alpha_2$, then $\rho_\varphi(X_1) < \rho_\varphi(X_2)$. Investigating the behavior of distortion pricing principle in terms of simultaneous changes α and θ can also be presented by performing simple mathematical calculations.*

Example 4.2. *Let X be a random variable with distribution function $F(x) = \frac{x}{1+x}, x > 0$. If $\psi(u) = u^2$, then $\rho_\varphi(X) = \frac{\pi}{2}$.*

4.2. **Reliability theory.** Let X be a nonnegative random variable with life distribution $F(\cdot)$. In reliability theory, $F(\cdot)$ belongs to the mean time to failure or replaced $(MTFR)$ class if the function $M_X(t)$ is decreasing wherein

$$M_X(t) = \frac{\int_0^t \bar{F}_X(x)dx}{F_X(t)}, \quad t \in \{x : F_X(x) > 0\},$$

which was considered by [2,13,14,17]. It is clear that, for $t = F^{-1}(p), p \in (0, 1)$,

$$M_X(F^{-1}(p)) = \frac{T_X(p)}{p}, \quad p \in (0, 1).$$

Corollary 4.3. *Nonnegative random variable X belongs to $IMTFR$ class if $\frac{\partial}{\partial p} \ln(T_X(p)) \geq \frac{\partial}{\partial p} \ln p$ for $p \in (0, 1)$.*

Proof. We have

$$\frac{\partial}{\partial p} M_X(F^{-1}(p)) = \frac{p\frac{\partial}{\partial p}T_X(p) - T_X(p)}{p^2}.$$

For the above relationship to be positive, it is sufficient that $p\frac{\partial}{\partial p}T_X(p) - T_X(p) \geq 0$. Therefore $\frac{\frac{\partial}{\partial p}T_X(p)}{T_X(p)} \geq \frac{1}{p}$. Also, this means $\frac{\partial}{\partial p} \ln(T_X(p)) \geq \frac{\partial}{\partial p} \ln p$. □

The residual life of nonnegative random variable X at time $t > 0$ is characterized as $X_t = [X - t | X > t]$, and the mean residual life of X is

$$m_X(t) = E(X_t) = \frac{\int_t^\infty \bar{F}_X(x)dx}{\bar{F}_X(t)}, \quad t \geq 0.$$

If $m_X(t)$ is decreasing (increasing), then X (or F_X) is in the decreasing (increasing) mean residual [DMRL(IMRL)] life distribution class and denoted by X(or F_X) $\in DMRL(IMRL)$.

Similarly, for $t = F^{-1}(p)$ we have $m_X(F^{-1}(p)) = \frac{EW_X(p)}{1-p}, p \in (0, 1)$. By performing simple mathematical calculations, we can conclude that $X \in DMRL(IMRL)$ if $\frac{\partial}{\partial p} \ln(EW_X(p)) \leq (\geq) \frac{\partial}{\partial p} \ln(1 - p)$, for $p \in (0, 1)$.

4.3. **NBU class for weighted variables.** Let X be a nonnegative random variable with distribution function F describing the lifetime of an item and let X_t be the residual life of the item at time t. For the weight function w, the weighted residual life variable is defined by $X_t^w = [X_w - t|X_w > t]$ for $t > 0$. Based on the results presented by [3, 22], the characterization of the NBU and IFR can also be determined based on weighted variables and simply written as follows:

- X_w is $NBU \iff X_t^w \leq_{st} X_w$ for all $t > 0$;
- X_w is $IFR \iff X_t^w \leq_{disp} X_w$ for all $t > 0$.

Remark 4.4. According to the relationship of total time on test transform order and its generalized with stochastic and dispersive orders, it can be easily concluded that

- X_w is $NBU \implies X_t^w \leq_{ttt} X_w$ for all $t > 0$;
- X_w is $IFR \implies X_t^w \leq_{ttt}^{(h)} X_w$ for all $t > 0$ and for all $h \in \xi$.

Also, $NBU_{(h)}$ (new better than used with respect to $h(\cdot)$) class introduced by [12] can be rewritten in terms of weight variables. In such a way that X_w is $NBU_{(h)}$ if $X_t^w \leq_{ttt}^{(h)} X_w$ for all $t > 0$.

Cosidering the above results, it can be easily concluded that for all $h(\cdot) \in \xi$

$$X_w \text{ is } IFR \iff X_w \text{ is } NBU_{(h)},$$

and also

$$X_w \text{ is } IFR \implies X_w \text{ is } NBU_{(h)} \implies X_w \text{ is } NBU.$$

5. **Real Data Analysis**

An analysis of the breaking stress of carbon fibers dataset is illustrated. These data are obtained from a process is producing carbon fibers to be used in constructing fibrous composite materials. Carbon fibers of 50 mm in length were sampled from the process, tested, and their tensile strength observed. As discussed earlier, the Weibull distribution is a reasonable model for the tensile strength of such material. [20] shown that the Weibull distribution with parameters shape 4.8 and the scale 3.2 is an appropriate distribution for this data. The TTT plot of these data is shown in Figure 2. According to this graph, the breaking stress of carbon fibers dataset have a distribution with a decreasing failure rate function.

Considering the function $h(u) = (1-u)^2$, the $GTTT$ plot for this data under the weight functions presented in Table 1 is shown in Figure 3.

By examining the graphs in Figure 3, it can be concluded that the weight function $w_2(\cdot)$ causes the Bathtub-shaped failure rate in the studied data. The weighted data with the weight function $w_3(\cdot)$ have a increasing failure rate and their failure rate decreases as the a value increases. Also the weighted data with the weight function $w_4(\cdot)$ have a increasing failure rate and their failure rate increases as the a value increases. Finally the weighted data with the weight function $w_5(\cdot)$ have a increasing failure rate and their failure rate decreases

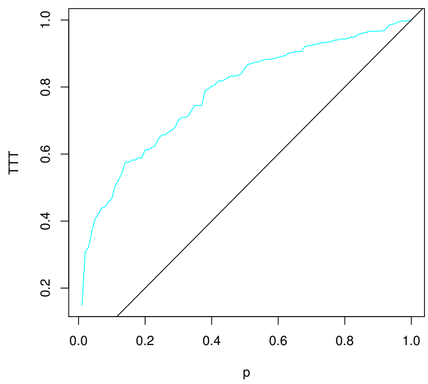


FIGURE 2. TTT plot of the breaking stress of carbon fibers dataset.

as the a value increases. Therefore, it can be concluded that weighting the data in general does not always create a positive result and in some cases it increases the failure rate. As a result, finding a suitable weight function that can positively change and reduce the failure rate of the original data is of great importance. The results obtained by [8] also confirm this. Because the weighted transform provided there is a special mode of transform 3 and the use of some special weight functions improve the failure rate of the data.

Discussion and conclusions

In this paper, the weighted form of generalized total time on test transform was presented. The characteristics of this transform, as well as the relationship between the generalized total time on test order and other stochastic orders in the weighted state, were expressed. Afterward, the application of this transform in insurance is presented and explained with an example. Also, its application in reliability has been investigated and its relationship with concepts NBU, $NBU_{(h)}$ and IFR has been investigated and the conditions for establishing total time on test order and its generalized have been stated based on those concepts. In the end, by analyzing a real dataset, we have studied and investigated the features of the newly introduced transform.

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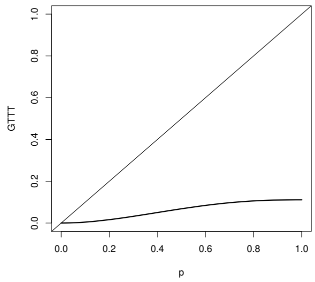
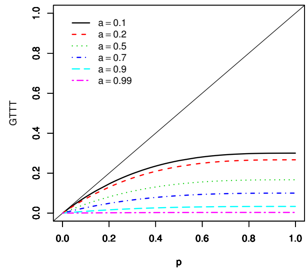
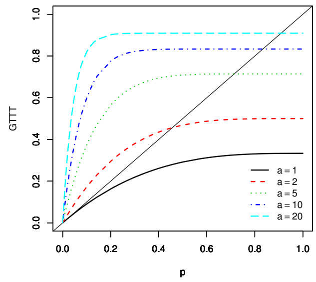


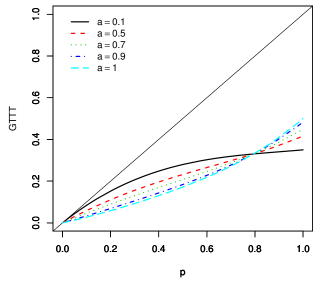
(A) w_2.



(B) w_3.



(C) w_4.



(D) w_5.

FIGURE 3. $GTTT$ plot of the breaking stress of carbon fibers dataset for some weight functions with different values of parameter a.

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