Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

EQUALIZER IN THE KLEISLI CATEGORY OF THE n-FUZZY POWERSET MONAD

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Special issue dedicated to Professor Esfandiar Eslami
Article type: Research Article
(Received: 14 February 2024, Received in revised form 31 May 2024)
(Accepted: 16 June 2024, Published Online: 18 June 2024)

Abstract. In this article, we first consider the L-fuzzy powerset monad on a completely distributive lattice L. Then for $L = [n]$, we investigate the fuzzy powerset monad on $[n]$ and we introduce simple, subsimple and quasisimple L-fuzzy sets. Finally, we provide necessary and sufficient conditions for the existence of an equalizer of a given pair of morphisms in the Kleisli category associated to this monad. Several illustrative examples are also provided.

Keywords: (completely distributive) lattice, (simple, subsimple, quasisimple) fuzzy set, fuzzy powerset functor.
2020 MSC: Primary 06B23, 18A30, 18C15, 18C20.

1. Introduction and Preliminaries

Kleisli categories appear in different areas of mathematics such as the semantics of linear logic, [2], computing, [7], Maltsev varieties, [5], extension of functors, [12], factorization-related monads, [3], and information systems, [10], to mention a few. Such categories are not complete in general. The limits are tougher than the colimits and the equalizers are more sophiticated to deal with than products. When dealing with the Kleisli categories, monads come to play and different monads make the situation vary considerably as far as limits are concerned. So investigating the existence of limits for various monads gains significance.

In [11], the completeness/cocompleteness of Kleisli categories are investigated, however as the author mentions, the results are powerless in concrete instances. In [6], the authors attempt the problem of the existence of equalizers in Kleisli categories by giving some equivalent conditions for the existence of equalizers of a given pair of maps in a general Kleisli category; then they present more elegant criterions for the existence of equalizers in a number of cases of interesting monads. In this article we investigate when the Kleisli category corresponding to the n-fuzzy powerset monad has equalizers. In particular, necessary and sufficient conditions for the existence of equalizer of a

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https://doi.org/10.22103/jmmr.2024.22946.1584
Publisher: Shahid Bahonar University of Kerman
How to cite: S.N. Hosseini, Y. Qasemi Nezhad, *Equalizer in the Kleisli category of the n-fuzzy powerset monad*, J. Mahani Math. Res. 2024; 13(4): 83 - 99.

parallel pair of morphisms are given. For more information on fuzzy sets and monads we refer the reader to [8] and [4]. To this end, we recall:

Definition 1.1. [1]. A monad in a category $\mathcal{E}$ is a triple $\mathbb{T} = (T, \eta, \mu)$, where $T : \mathcal{E} \longrightarrow \mathcal{E}$ is a functor, $\eta : Id \longrightarrow T$ (with Id the identity functor) and $\mu : T^2 \longrightarrow T$ are natural transformations rendering commutative the following square and triangles.

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] \arrow[dr, phantom, "///"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] \arrow[dl, phantom, "///"] \arrow[dr, phantom, "///"] & T \arrow[l, "\eta T"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd}$$

Definition 1.2. Let R be a relation on a set L. For every set X we define a relation R_X on $L^X = \{ f : X \longrightarrow L : f$ is a function$\}$ by $f \ R_X \ g$ if for every $x \in X$, $f(x) \ R \ g(x)$.

Lemma 1.3. *Let R be a relation on a set L*
(a) If R is a partial order on L, then for every X, R_X is a partial order on L^X.
(b) If L is a meet semi-lattice, then for every X, L^X is a meet semi-lattice.
(c) If L is a complete lattice, then for every X, L^X is a complete lattice.
(d) If L is a distributive lattice, then for every X, L^X is a distributive lattice.

Proof. They can be verified directly. □

Definition 1.4. Let L be a completely distributive lattice with the smallest element 0 and the largest element 1. Define:

(a) the map $\mathcal{L} : Set \longrightarrow Set$ on objects by $\mathcal{L}(X) = L^X$ and on morphisms by,

$$\mathcal{L}(X \xrightarrow{f} Y) = L^X \xrightarrow{\mathcal{L}(f)} L^Y$$

where for every $A \in L^X$ and $y \in Y$, $\mathcal{L}(f)(A)(y) = \bigvee_{x \in f^{-1}(y)} A(x)$.

(b) the natural transformation $\eta : Id \longrightarrow \mathcal{L}$ as follows:

for every $X \in Set$ and $x \in X$, $\eta_X(x) : X \longrightarrow L$ is the function that takes $y \in X$ to $\eta_X(x)(y) = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}$.

(c) the natural transformation $\mu : \mathcal{L}^2 \longrightarrow \mathcal{L}$ as follows:

for every $X \in Set$ and $A \in \mathcal{L}^2(X)$, $\mu_X(A) : X \longrightarrow L$ is the function that takes $x \in X$ to $\mu_X(A)(x) = \bigvee_{c \in \mathcal{L}(X)} (A(c) \wedge c(x))$.

Lemma 1.5. $\mathbb{L} = (\mathcal{L}, \eta, \mu)$ *is a monad.*

Proof. See [9], Example 5. □

The functor $\mathcal{L}$ is called the L-fuzzy powerset functor and the monad $\mathbb{L} = (\mathcal{L}, \eta, \mu)$ is called the L-fuzzy powerset monad.

2. n-Fuzzy Powerset Monad, Simple, Subsimple and Quasisimple

In this section, we introduce the n-fuzzy power set monad and prove some results that are needed in the subsequent sections. In the rest of the paper we consider the lattice $L = [n] = \{0, 1, 2, ..., n-1\}$ for a fixed $n \in \mathbb{N}$, which is a complete lattice under the usual order. The associated functor, respectively monad, is called the n-fuzzy powerset functor, respectively the n-fuzzy powerset monad. The unit η, for $x, y \in X$, is now given by:

$$\eta_X(x)(y) = \begin{cases} n-1 & y = x \\ 0 & y \neq x, \end{cases}$$

and the multiplication μ, for $A \in \mathcal{L}^2(X)$ and $x \in X$, is given by:

$$\mu_X(A)(x) = \max\{\min\{A(c), c(x)\} : c \in \mathcal{L}(X)\}$$

Lemma 2.1. *Let* $f : X \longrightarrow \mathcal{L}(Y)$ *be a function. For* $A \in \mathcal{L}(X)$ *and* $y \in Y$*, we have*

$$\mu_Y \mathcal{L}(f)(A)(y) = \max\{\min\{A(x), f(x)(y)\} : x \in X\}$$

Proof. We have:

$$\begin{aligned}
\mu_Y \mathcal{L}(f)(A)(y) &= \mu_Y(\mathcal{L}(f)(A))(y) \\
&= \bigvee_{c \in \mathcal{L}(Y)} (\mathcal{L}(f)(A)(c) \wedge c(y)) \\
&= \bigvee_{c \in \mathcal{L}(Y)} ((\bigvee_{x \in f^{-1}(c)} A(x)) \wedge c(y)) \\
&= \max\{\bigvee_{x \in f^{-1}(c)} A(x) \wedge c(y) : c \in \mathcal{L}(Y)\} \\
&= \max\{\bigvee_{x \in f^{-1}(c)} (A(x) \wedge c(y)) : c \in \mathcal{L}(Y)\} \\
&= \max\{\max\{A(x) \wedge c(y) : x \in f^{-1}(c)\} : c \in \mathcal{L}(Y)\} \\
&= \max\{\min\{A(x), c(y)\} : x \in f^{-1}(c), c \in \mathcal{L}(Y)\} \\
&= \max\{\min\{A(x), f(x)(y)\} : x \in X\}
\end{aligned}$$

□

Definition 2.2. Let $k \in [n]$. For every set X, we define $\eta_X^k : X \longrightarrow \mathcal{L}(X)$ by $\eta_X^k(x) = \bar{k} \wedge \eta_X(x)$, where $\bar{k} : X \to L$ is the constant function with value k.

Remark: Note that $\eta_X^{n-1} = \eta_X$, $\eta_X^0 = 0$ and $\eta_X^k(x)(y) = \begin{cases} k & y = x \\ 0 & y \neq x \end{cases}$.

Lemma 2.3. *For $k \in [n]$, $\eta^k : Id \longrightarrow \mathcal{L}$ is a natural transformation.*

Proof. We need to show for any function $f : X \longrightarrow Y$, the diagram:

$$\begin{tikzcd} X \arrow[r, "\eta_X^k"] \arrow[d, "f"'] & \mathcal{L}(X) \arrow[d, "\mathcal{L}(f)"] \\ Y \arrow[r, "\eta_Y^k"'] & \mathcal{L}(Y) \end{tikzcd}$$

commutes. For every $x \in X$ and $y \in Y$, we have:

$$\mathcal{L}(f)(\eta_X^k(x))(y) = \bigvee_{c \in f^{-1}(y)} \eta_X^k(x)(c) =$$

$$\bigvee_{c \in f^{-1}(y)} (\begin{cases} k & c = x \\ 0 & c \neq x \end{cases}) = \begin{cases} k & x \in f^{-1}(y) \\ 0 & x \notin f^{-1}(y) \end{cases} = \begin{cases} k & y = f(x) \\ 0 & y \neq f(x) \end{cases} = \eta_Y^k(f(x))(y)$$

$\square$

Lemma 2.4. *For $j, k \in [n]$ and $s, t \in X$, $\eta_X^j(s)$ and $\eta_X^k(t)$ are comparable if and only if $j = 0$ or $k = 0$ or $s = t$.*

Proof. Suppose $\eta_X^j(s)$ and $\eta_X^k(t)$ are comparable. If $\eta_X^j(s) \leq \eta_X^k(t)$, then $\eta_X^j(s)(s) \leq \eta_X^k(t)(s)$ and so $j \leq \eta_X^k(t)(s)$. It follows that $j = 0$ or $s = t$. Similarly if $\eta_X^k(s) \leq \eta_X^j(t)$, then $k = 0$ or $s = t$. The converse is obvious. $\square$

Lemma 2.5. *Suppose $B \in \mathcal{L}(X)$ and $t \in X$. If $0 \leq B \leq \eta_X(t)$, then $B = \eta_X^k(t)$, for some $k \in [n]$.*

Proof. For each $s \in X$, we have $0 \leq B(s) \leq \eta_X(t)(s)$. So for $s = t$, $0 \leq B(s) \leq n - 1$ and for $s \neq t$, $0 \leq B(s) \leq 0$. The result then follows. $\square$

In the sequel, for $A \in \mathcal{L}(X)$, we denote *supp*$(A) = A^{-1}([n] - \{0\})$ by A^*.

Lemma 2.6. *If $B \in \mathcal{L}(X)$, then $B = \bigvee_{t \in B^*} \eta_X^{B(t)}(t)$.*

Proof. For every $s \in X$, we have:

$$\bigvee_{t \in B^*} \eta_X^{B(t)}(t)(s) = \max\{\eta_X^{B(t)}(t)(s) : t \in B^*\} \begin{cases} B(s) & B(s) \neq 0 \\ 0 & B(s) = 0 \end{cases} = B(s)$$

$\square$

Definition 2.7. For a set X, an inclusion $I \overset{i}{\hookrightarrow} \mathcal{L}(X)$ and a function $e : E \longrightarrow \mathcal{L}(X)$, we say (e, i) is μ_X-compatible if $\mu_X \mathcal{L}(e)$ factors through i, i.e., if there is a (necessarily unique) function φ rendering commutative the following triangle.

$$\begin{tikzcd} I \arrow[r, hook, "i"] & \mathcal{L}(X) \\ \mathcal{L}(E) \arrow[u, "\varphi"] \arrow[ur, bend right, "\mu_X \mathcal{L}(e)"'] \arrow[ur, phantom, "///"] & \end{tikzcd}$$

In this case, the map φ is called the restricted multiplication.

Example 2.8. *Consider any pair of functions* $X \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{L}(Y)$ *and let* i *be an equalizer of the pair* $\mathcal{L}(X) \underset{\mu_Y \mathcal{L}(g)}{\overset{\mu_Y \mathcal{L}(f)}{\rightrightarrows}} \mathcal{L}(Y)$. *Given any function* $e : E \to \mathcal{L}(X)$, *using the naturality of* η, μ *and the monad equations, one can show that* (e, i) *is* μ_X*-compatible.*

Lemma 2.9. *If* (e, i) *is* μ_X*-compatible, then the restricted multiplication* φ *preserves order.*

Proof. Suppose that $A, B \in \mathcal{L}(E)$ and $A \leq B$. Then by 2.1,

$$\varphi(A)(x) = \mu_X \mathcal{L}(e)(A)(x) = \max\{\min\{A(t), f(t)(x)\} : t \in E\} \text{ and}$$
$$\varphi(B)(x) = \mu_X \mathcal{L}(e)(B)(x) = \max\{\min\{B(t), f(t)(x)\} : t \in E\}.$$

Since for all $t \in E$ $A(t) \leq B(t)$, the result follows. □

Lemma 2.10. *If* (e, i) *is* μ_X*-compatible, then the restricted multiplication* φ *preserves arbitrary joins.*

Proof. We show that $\varphi(\bigvee_{j \in J} A_j) = \bigvee_{j \in J} \varphi(A_j)$. Because φ preserves order and for all $j \in J$, $A_j \leq \bigvee_{j \in J} A_j$, so $\varphi(A_j) \leq \varphi(\bigvee_{j \in J} A_j)$ therefore $\bigvee_{j \in J} \varphi(A_j) \leq \varphi(\bigvee_{j \in J} A_j)$. To show $\varphi(\bigvee_{j \in J} A_j) \leq \bigvee_{j \in J} \varphi(A_j)$, let $x \in X$ and $m = \varphi(\bigvee_{j \in J} A_j)(x)$. We have:

$$m = \varphi(\bigvee_{j \in J} A_j)(x) = \max\{\min\{a, b\} : \exists t \in (\bigvee_{j \in J} A_j)^{-1}(a) \text{ such that } x \in e(t)^{-1}(b)\}$$

So there are a, b such that $m = \min\{a, b\}$ and there exists $t \in (\bigvee_{j \in J} A_j)^{-1}(a)$ and $x \in e(t)^{-1}(b)$. Therefore $a = \bigvee_{j \in J} A_j(t) = \max\{A_j(t) : j \in J\}$. So there exists $j_0 \in J$ such that $t \in A_{j_0}^{-1}(a)$. So $m \leq \varphi(A_{j_0})(x) \leq \bigvee_{j\in J} \varphi(A_j)(x)$. Hence $\varphi(\bigvee_{j\in J} A_j) \leq \bigvee_{j\in J} \varphi(A_j)$.

□

Lemma 2.11. *If (e, i) is μ_X-compatible and the restricted multiplication φ is an isomorphism, then φ^{-1} preserves order.*

Proof. Suppose that $Q, R \in I$, $Q \leq R$. Let $A = \varphi^{-1}(Q)$ and $B = \varphi^{-1}(R)$. By 2.10, we have $\varphi(A \vee B) = \varphi(A) \vee \varphi(B) = Q \vee R = R = \varphi(B)$. So $A \vee B = B$ and therefore $\varphi^{-1}(Q) = A \leq B = \varphi^{-1}(R)$. □

Definition 2.12. For a set X, let $I \subseteq \mathcal{L}(X)$ and $A \in \mathcal{L}(X)$.

(i) An I-chain for A, of length m, is a chain $0 < A^1 < A^2 < ... < A^m < A$, where $A^j \in I$ for all j.

(ii) A canonical I-chain for A, of length m, is an I-chain as in (i), such that $B \in I$ and $0 < B < A$, implies $B = A^k$, for some $1 \leq k \leq m$.

(iii) A is said to be simple if $A \in I$ and it has a canonical I-chain of length $n - 2$.

(iv) Let $1 \leq k \leq n - 1$. $B \in I$ is called k-subsimple if there is a simple A with the canonical I-chain $0 < A^1 < A^2 < ... < A^{n-2} < A^{n-1} = A$ such that $B = A^k$.

(v) $B \in I$ is called subsimple if B is k-subsimple for some $1 \leq k \leq n - 1$.

(vi) Calling a subset of a preordered set mutually incomparable, if any two distinct members of it are incomparable, A is called quasisimple if it can be written as a unique join of incomparable subsimples, i.e., $A = \bigvee_{k\in K} A_k$, where each A_k is subsimple and $\{A_k : k \in K\}$ is mutually incomparable; and if $A = \bigvee_{j\in J} B_j$, where each B_j is subsimple and $\{B_j : j \in J\}$ is mutually incomparable, then $\{A_k : k \in K\} = \{B_j : j \in J\}$.

Lemma 2.13. *Let A be simple and $0 < A^1 < A^2 < ... < A^{n-2} < A$ be a canonical I-chain for A.*

(a) If $0 < B^1 < B^2 < ... < B^m < A$ is an I-chain for A, then $\mathcal{B} \subseteq \mathcal{A}$, where $\mathcal{B} = \{B^1, B^2, \cdots, B^m\}$ and $\mathcal{A} = \{A^1, A^2, \cdots, A^{n-2}\}$.

(b) Any I-chain of length $n - 2$ for A is a canonical I-chain for A.

(c) The canonical I-chain for A is unique.

(d) If S is j-subsimple and k-subsimple, then $j = k$.

Proof. (a) Suppose $0 < B^1 < \cdots < B^m < A$ is an I-chain for A. Since for each j, $0 < B^j < A$, we get $B^j = A^k$, for some k, i.e., $B^j \in \mathcal{A}$. So $\mathcal{B} \subseteq \mathcal{A}$ and thus $m \leq n - 2$.

(b) Suppose the I-chain of part (a) has length $n - 2$. Then obviously $\mathcal{B} = \mathcal{A}$. Since the two chains are strictly increasing, we get $B^j = A^j$, for all j.

(c) Suppose $0 < B^1 < \cdots < B^m < A$ is another canonical I-chain for A. By (a), $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \mathcal{B}$. So $\mathcal{B} = \mathcal{A}$ and therefore $m = n-2$. Now by (b), for all j, $B^j = A^j$.

(d) Since S is j-subsimple, there is a simple A such that $S = A^j$. Also, there is a simple B such that $S = B^k$. So $B^k = S = A^j$. Now the I-chain $0 < B^1 < \cdots < B^k = A^j < A^{j+1} < \cdots < A^{n-2} < A$ for A has length $k + n - 2 - j$. By part (a) $k + n - 2 - j \leq n - 2$, implying $k \leq j$. Similarly $j \leq k$. Hence $j = k$. □

Lemma 2.14. *If (e, i) is μ_X-compatible and the restricted multiplication φ is an isomorphism, then $e(E)$ is the set of all simples.*

Proof. First notice that since $e = \mu_X \mathcal{L}(e)\eta_E = i\varphi\eta_E$, e is mono and for each $t \in E$, $e(t) = \varphi(\eta_E(t)) \in I$.

Now let $t \in E$. It is clear that $0 < \eta_E^1(t) < \eta_E^2(t) < \ldots < \eta_E^{n-2}(t) < \eta_E^{n-1}(t) = \eta_E(t)$. By 2.9, φ preserves order, so $0 < \varphi(\eta_E^1(t)) < \varphi(\eta_E^2(t)) < \ldots < \varphi(\eta_E^{n-2}(t)) < \varphi(\eta_E(t)) = e(t)$. If there exists $B \in I$ such that $0 < B < e(t) = \varphi(\eta_E(t))$, then by 2.11, $0 < \varphi^{-1}(B) < \eta_E(t)$. So by 2.5, there exists $1 \leq k \leq n-2$ such that $\varphi^{-1}(B) = \eta_E^k(t)$. i.e, $B = \varphi(\eta_E^k(t))$. Hence $e(t)$ is simple.

Now suppose A is simple. So A has a canonical I-chain $0 < A^1 < \ldots < A^{n-2} < A^{n-1} = A$. If there are s and s' such that $m = \varphi^{-1}(A)(s) \neq 0$ and $m' = \varphi^{-1}(A)(s') \neq 0$, then since $0 < \eta_E^m(s) \leq \varphi^{-1}(A)$ and $0 < \eta_E^{m'}(s') \leq \varphi^{-1}(A)$, we get $0 < \varphi(\eta_E^m(s)) \leq A$ and $0 < \varphi(\eta_E^{m'}(s')) \leq A$. A being simple, yields $\varphi(\eta_E^m(s)) = A^k$ and $\varphi(\eta_E^{m'}(s')) = A^{k'}$, for some k, k'. Now since A^k and $A^{k'}$ are comparable, we get $\varphi(\eta_E^m(s))$ and $\varphi(\eta_E^{m'}(s'))$ and thus $\eta_E^m(s)$ and $\eta_E^{m'}(s')$ are comparable. So by 2.4, $s = s'$. This proves $\varphi^{-1}(A)$ can be nonzero at only one point, s. Therefore $\varphi^{-1}(A) = \eta_E^j(s)$, for some $0 < j \leq n-1$. So $A = \varphi(\eta_E^j(s))$, implying $\varphi(\eta_E^j(s))$ is simple. Now by the first part of the proof, since $s \in E$, $e(s)$ is simple and we have the chain $0 < \varphi(\eta_E^1(s)) < \varphi(\eta_E^2(s)) < \ldots < \varphi(\eta_E^{n-2}(s)) < \varphi(\eta_E(s)) = e(s)$. This shows $j = n-1$ and so $A = \varphi(\eta_E(s)) = e(s) \in e(E)$. This completes the proof. □

Lemma 2.15. *Suppose (e, i) is μ_X-compatible.*

(a) If $t \in E$ with $e(t)$ simple, then $\varphi(\eta_E^k(t))$ is k-subsimple.

(b) If every simple is in $e(E)$, then every k-subsimple is of the form $\varphi(\eta_E^k(t))$, for some $t \in E$ with $e(t)$ simple.

Proof. (a) Follows from the canonical I-chain $0 < \varphi(\eta_E^1(t)) < \varphi(\eta_E^2(t)) < \ldots < \varphi(\eta_E^{n-2}(t)) < \varphi(\eta_E(t)) = e(t)$ for $e(t)$.

(b) Suppose A is k-subsimple. Then there is a simple B, such that $0 < A < B$. By hypothesis, $B = e(t)$ with $t \in E$, so we have the I-chain, $0 < \varphi(\eta_E^1(t)) < \varphi(\eta_E^2(t)) < \ldots < \varphi(\eta_E^{n-2}(t)) < \varphi(\eta_E(t)) = e(t)$, of length $n-2$ for $e(t)$; which by 2.13, is the canonical I-chain for $e(t)$. So $A = \varphi(\eta_E^k(t))$.

□

Calling a function $h : M \longrightarrow N$ one to one on $K \subseteq M$, whenever its restriction $h_{|K} : K \longrightarrow N$ is one to one, we have:

Lemma 2.16. *If* (e, i) *is* μ_X*-compatible, then* $\mu_X \mathcal{L}(e)$ *is one to one on* $\Sigma_j = \{\eta_E^j(t) : t \in E\}$*, for each* $j \in [n]$ *if and only if it is one to one on* $\Sigma = \{\eta_E^j(t) : j \in [n],\ t \in E\}$.

Proof. Suppose $\mu_X \mathcal{L}(e)$ is one to one on Σ_j, for each $j \in [n]$. If $\mu_X \mathcal{L}(e)(\eta_E^j(s)) = \mu_X \mathcal{L}(e)(\eta_E^k(t))$, for $j, k \in [n]$ and $s, t \in E$, then $\varphi(\eta_E^j(s)) = \varphi(\eta_E^k(t))$. By 2.15, $\varphi(\eta_E^j(s))$ is j-subsimple and $\varphi(\eta_E^k(t))$ is k-subsimple. Now by 2.13, $j = k$. Since $\mu_X \mathcal{L}(e)$ is one to one on Σ_j, we get $\eta_E^j(s) = \eta_E^k(s)$. The converse is obvious. $\square$

3. Equalizer in the kleisli category

In this section, we give necessary and sufficient conditions for the existence of an equalizer of a pair of morphisms in the Kleisli category $Set_{\mathbb{L}}$.

Proposition 3.1. *Let* $X \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{L}(Y)$, $e : E \longrightarrow \mathcal{L}(X)$ *be functions and* $I \overset{i}{\hookrightarrow} \mathcal{L}(X)$ *be the equalizer of the pair* $\mathcal{L}(X) \underset{\mu_Y \mathcal{L}(g)}{\overset{\mu_Y \mathcal{L}(f)}{\rightrightarrows}} \mathcal{L}(Y)$. *The diagram:*

$$\mathcal{L}(E) \xrightarrow{\mu_X \mathcal{L}(e)} \mathcal{L}(X) \underset{\mu_Y \mathcal{L}(g)}{\overset{\mu_Y \mathcal{L}(f)}{\rightrightarrows}} \mathcal{L}(Y)$$

is an equalizer in Set if and only if $e : E \longrightarrow \mathcal{L}(X)$ *is mono with* $e(E)$ *the set of all simples,* $\mu_X \mathcal{L}(e)$ *is one to one on* $\Sigma_j = \{\eta_E^j(t) : t \in E\}$*, for each* $j \in [n]$ *and every nonzero member of* I *is quasisimple.*

Proof. Suppose that

$$\mathcal{L}(E) \xrightarrow{\mu_X \mathcal{L}(e)} \mathcal{L}(X) \underset{\mu_Y \mathcal{L}(g)}{\overset{\mu_Y \mathcal{L}(f)}{\rightrightarrows}} \mathcal{L}(Y)$$

is an equalizer. Then $e = \mu_X \mathcal{L}(e)\eta_E$ is mono and the restricted multiplication φ is an isomorphism. So by 2.14, $e(E)$ is the set of all simples. Since $\mu_X \mathcal{L}(e)$ is one to one, it is obviously one to one on Σ_j for each j. To prove the last assertion, let $0 \neq A \in I$. Then there exists $B \in \mathcal{L}(E)$ such that $A = \varphi(B)$. By 2.6, $B = \bigvee_{t \in B^*} \eta_E^{B(t)}(t)$. Therefore by 2.10, $A = \varphi(B) = \varphi(\bigvee_{t \in B^*} \eta_E^{B(t)}(t)) = \bigvee_{t \in B^*} \varphi(\eta_E^{B(t)}(t))$. For all $t \in E$ with $B(t) \neq 0$, by 2.15, $\varphi(\eta_E^{B(t)}(t))$ is subsimple, and by 2.4, $\{\varphi(\eta_E^{B(t)}(t))\}$ is mutually incomparable. So A equals a join of incomparable (f, g)-subsimples.

To show the uniqueness of the join representation of A, suppose $A = \bigvee_{j\in J} A_j$, where each A_j is subsimple and $\{A_j : j \in J\}$ is mutually incomparable. By 2.15, there exists $t_j \in E$ and $0 \leq \alpha_j \leq n-1$ such that $\varphi^{-1}(A_j) = \eta_E^{\alpha_j}(t_j)$. Because $\{A_j : j \in J\}$ is mutually incomparable, by 2.4, each α_j is nonzero and every two t_j are unequal. So we have:

$$B = \varphi^{-1}(A) = \varphi^{-1}(\bigvee_{j\in J} A_j) = \bigvee_{j\in J} \varphi^{-1}(A_j) = \bigvee_{j\in J} \eta_E^{\alpha_j}(t_j)$$

On the other hand, we know:

$$B = \bigvee_{t\in B^*} \eta_E^{B(t)}(t)$$

It is enough to show that:

$$\{\eta_E^{\alpha_j}(t_j) : j \in J\} = \{\eta_E^{B(t)}(t) : t \in B^*\}$$

Suppose $D = \eta_E^{\alpha_{j_0}}(t_{j_0})$, for some $j_0 \in J$. Since every two t_j are unequal, we have:

$$B(t_{j_0}) = \bigvee_{j\in J} \eta_E^{\alpha_j}(t_j)(t_{j_0}) = \alpha_{j_0} \neq 0$$

Therefore $D = \eta_E^{\alpha_{j_0}}(t_{j_0}) = \eta_E^{B(t_{j_0})}(t_{j_0})$.

Now suppose $D = \eta_E^{B(t_0)}(t_0)$, for some $t_0 \in E$, with $B(t_0) \neq 0$. Since $B(t_0) \neq 0$ and every two t_j are unequal, there is one and only one $j_0 \in J$ such that $t_{j_0} = t_0$. The equation $B(t_0) = \bigvee_{j\in J} \eta_E^{\alpha_j}(t_j)(t_0)$ now gives $B(t_0) = \alpha_{j_0}$. It follows that $D = \eta_E^{B(t_0)}(t_0) = \eta_E^{\alpha_{j_0}}(t_{j_0})$. This establishes uniqueness.

Conversely, suppose $e : E \longrightarrow \mathcal{L}(X)$ is mono with $e(E)$ the set of all simples, that every nonzero element of I is quasisimple and $\mu_X\mathcal{L}(e)$ is one to one on Σ_j, for each j. We claim that $\mu_X\mathcal{L}(e)$ is an equalizer of $\mu_Y\mathcal{L}(f)$ and $\mu_Y\mathcal{L}(g)$. We have:

$$\begin{gathered}\mu_Y\mathcal{L}(f)\mu_X\mathcal{L}(e) = \mu_Y\mu_{\mathcal{L}(Y)}\mathcal{L}^2(f)\mathcal{L}(e) = \mu_Y\mathcal{L}(\mu_Y)\mathcal{L}^2(f)\mathcal{L}(e) = \\ \mu_Y\mathcal{L}(\mu_Y\mathcal{L}(f)e) = \\ \mu_Y\mathcal{L}(\mu_Y\mathcal{L}(g)e) = \mu_Y\mathcal{L}(\mu_Y)\mathcal{L}^2(g)\mathcal{L}(e) = \mu_Y\mu_{\mathcal{L}(Y)}\mathcal{L}^2(g)\mathcal{L}(e) = \mu_Y\mathcal{L}(g)\mu_X\mathcal{L}(e)\end{gathered}$$

So there exists $\varphi : \mathcal{L}(E) \longrightarrow I$ such that $i\varphi = \mu_X\mathcal{L}(e)$. It suffices to show that φ is an isomorphism.

To show φ is epi, let $A \in I$. If $A = 0$, then $A = \varphi(0)$, otherwise $A = \bigvee_{j\in J} A_j$ where each A_j is subsimple and $\{A_j : j \in J\}$ is mutually incomparable. We define $B \in \mathcal{L}(E)$ as follows:

$$B(t) = \begin{cases} k & \text{if } \varphi(\eta_E^k(t)) = A_j \text{ for some } j \in J \\ 0 & \text{otherwise} \end{cases}$$

Since $\{A_j : j \in J\}$ is mutually incomparable and by 2.4, $\varphi(\eta_E^k(t))$ and $\varphi(\eta_E^{k'}(t))$ are comparable, if $\varphi(\eta_E^k(t)), \varphi(\eta_E^{k'}(t)) \in \{A_j : j \in J\}$, then $\varphi(\eta_E^k(t)) = \varphi(\eta_E^{k'}(t))$. So $\mu_X \mathcal{L}(e)(\eta_E^k(t)) = \mu_X \mathcal{L}(e)(\eta_E^{k'}(t))$. Since by 2.16, $\mu_X \mathcal{L}(e)$ is one to one on Σ, we get $\eta_E^k(t) = \eta_E^{k'}(t)$. So $k = k'$ and B is well defined. We show $\varphi(B) = A$. Since each A_j is subsimple, by 2.15, for every $j \in J$, there exists $t_j \in E$ and $k_j \in [n]$, such that $A_j = \varphi(\eta_E^{k_j}(t_j))$. So $A = \bigvee_{j\in J} A_j = \bigvee_{j\in J} \varphi(\eta_E^{k_j}(t_j)) = \varphi(\bigvee_{j\in J} \eta_E^{k_j}(t_j))$. Since $\{A_j : j \in J\}$ is mutually incomparable, by 2.4, every two t_j are unequal and each k_j is nonzero. So for every $t \in E$,

$$\bigvee_{j\in J} \eta_E^{k_j}(t_j)(t) = \begin{cases} k_j & t = t_j \\ 0 & t \neq t_j \end{cases}$$

On the other hand, since $A_j = \varphi(\eta_E^{k_j}(t_j))$, $B(t) = \begin{cases} k_j & t = t_j \\ 0 & t \neq t_j \end{cases}$. So $\varphi(B) = A$.

To show φ is mono, let $B_1, B_2 \in \mathcal{L}(E)$ and $\varphi(B_1) = \varphi(B_2)$. By 2.6, we have $B_i = \bigvee_{t\in B_i^*} \eta_E^{B_i(t)}(t)$ for $i = 1, 2$. Therefore

$$\bigvee_{t\in B_1^*} \varphi(\eta_E^{B_1(t)}(t)) = \varphi(\bigvee_{t\in B_1^*} \eta_E^{B_1(t)}(t)) = \varphi(B_1) = \varphi(B_2) = \\ \varphi(\bigvee_{t\in B_2^*} \eta_E^{B_2(t)}(t)) = \bigvee_{t\in B_2^*} \varphi(\eta_E^{B_2(t)}(t))$$

If $B_2 = 0$, then $\bigvee_{t\in B_1^*} \varphi(\eta_E^{B_1(t)}(t)) = \bigvee_{t\in B_2^*} \varphi(\eta_E^{B_2(t)}(t)) = 0$, implying for each $t \in B_1^*$, $\varphi(\eta_E^{B_1(t)}(t)) = 0$. Now if $B_1^* \neq \emptyset$, then let $t \in B_1^*$. We have $\mu_X \mathcal{L}(e)(\eta_E^{B_1(t)}(t)) = \varphi(\eta_E^{B_1(t)}(t)) = 0$. Since by 2.16, $\mu_X \mathcal{L}(e)$ is one to one on Σ, we get $B_1(t) = 0$, that is a contradiction to $t \in B_1^*$. It follows that $B_1^* = \emptyset$, thus $B_1 = 0$. Similarly if $B_1 = 0$, then so is B_2. Finally if $B_1 \neq 0$ and $B_2 \neq 0$, then $0 \neq \varphi(B_1) \in I$ and is therefore quasisimple. By 2.15, for all $t \in E$, $\varphi(\eta_E^{B_1(t)}(t))$ is subsimple. On the other hand $\{\varphi(\eta_E^{B_1(t)}(t))\}$ is mutually incomparable, because if say $\varphi(\eta_E^{B_1(s)}(s)) \leq \varphi(\eta_E^{B_1(t)}(t))$, then $0 < \varphi(\eta_E^{B_1(s)}(s)) \leq t$. It follows that $\varphi(\eta_E^{B_1(s)}(s)) = \varphi(\eta_E^j(t))$ for some $j = 1, 2, \cdots, n-1$. Since $\mu_X \mathcal{L}(e)$ is one to one on Σ, $\eta_E^{B_1(s)}(s) = \eta_E^j(t)$, yielding $s = t$. So $\varphi(B_1)$ is a join of noncomparable subsimples $\eta_E^{B_1(t)}(t)$, for $t \in E$. Similarly $\varphi(B_2)$ is a join of noncomparable subsimples $\eta_E^{B_2(t)}(t)$, for $t \in E$. Since $\varphi(B_1) = \varphi(B_2)$ is quasisimple, $\{\varphi(\eta_E^{B_1(t)}(t)) : t \in B_1^*\} = \{\varphi(\eta_E^{B_2(t)}(t)) : t \in B_2^*\}$. So for every $t \in B_1^*$, there exists $s \in B_2^*$ such that $\varphi(\eta_E^{B_1(t)}(t)) = \varphi(\eta_E^{B_2(s)}(s))$. Again since $\mu_X \mathcal{L}(e)$ is one to one on Σ, we get $\eta_E^{B_1(t)}(t) = \eta_E^{B_2(s)}(s)$, implying $s = t$ and $B_1(t) = B_2(t)$. Hence $B_1 = B_2$.

$\square$

With $Set_{\mathbb{L}}$ the Kleisli category of the n-fuzzy powerset monad $\mathbb{L}$, we have:

Lemma 3.2. *The functor* $U : Set_{\mathbb{L}} \longrightarrow Set$ *that takes* $\hat{f} : X \longrightarrow Y$ *to* $\mu_Y \mathcal{L}(f) : \mathcal{L}(X) \longrightarrow \mathcal{L}(Y)$ *preserves and reflects equalizers.*

Proof. See Proposition 3.2 of [6], with $\mathcal{E} = Set$ and $\mathbb{T} = \mathbb{L}$, where $L = [n]$. $\square$

Theorem 3.3. *Let* $X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$ *be morphisms in* $Set_{\mathbb{L}}$ *with* $X \overset{f}{\underset{g}{\rightrightarrows}} \mathcal{L}(Y)$ *the corresponding functions and* $I \overset{i}{\hookrightarrow} \mathcal{L}(X)$ *be the equalizer of the pair* $\mathcal{L}(X) \overset{\mu_Y \mathcal{L}(f)}{\underset{\mu_Y \mathcal{L}(g)}{\rightrightarrows}} \mathcal{L}(Y)$*. The diagram:*

$$E \xrightarrow{\hat{e}} X \overset{\hat{f}}{\underset{\hat{g}}{\rightrightarrows}} Y$$

is an equalizer in $Set_{\mathbb{L}}$ *if and only if the corresponding map* $e : E \longrightarrow \mathcal{L}(X)$ *is mono with* $e(E)$ *the set of all simples,* $\mu_X \mathcal{L}(e)$ *is one to one on* $\Sigma_j = \{\eta_E^j(t) : t \in E\}$*, for each* $j \in [n]$ *and every nonzero member of* I *is quasisimple.*

Proof. The proof follows from 3.2 and 3.1. $\square$

Corollary 3.4. *If an equalizer of* $\hat{f}, \hat{g} : X \longrightarrow Y$ *exists in* $Set_{\mathbb{L}}$*, then* I *is a join-semilattice that is freely generated by a set of subsimples.*

Proof. To show I is closed under join, let $A, B \in I$. So for each $y \in Y$, we have:

$$\mu_Y \mathcal{L}(f)(A)(y) = \mu_Y \mathcal{L}(g)(A)(y)$$

and

$$\mu_Y \mathcal{L}(f)(B)(y) = \mu_Y \mathcal{L}(g)(B)(y)$$

Therefore,

$$\max\{\min\{A(x), f(x)(y)\} : x \in X\} = \max\{\min\{A(x), g(x)(y)\} : x \in X\}$$

and

$$\max\{\min\{B(x), f(x)(y)\} : x \in X\} = \max\{\min\{B(x), g(x)(y)\} : x \in X\}$$

Using general facts that for $a, b, c \in L$, $\min\{a \vee b, c\} = \min\{a, c\} \vee \min\{b, c\}$ and for functions $C, D : X \to L$, $\max\{C(x) \vee D(x) : x \in X\} = \max\{C(x) : x \in X\} \vee \max\{D(x) : x \in X\}$, by 2.1 for each $y \in Y$ we have:

$$\begin{aligned}
&\mu_Y \mathcal{L}(f)(A \vee B)(y) \\
&= \max\{\min\{A(x) \vee B(x), f(x)(y)\} : x \in X\} \\
&= \max\{\min\{A(x), f(x)(y)\} \vee \min\{B(x), f(x)(y)\} : x \in X\} \\
&= \max\{\min\{A(x), f(x)(y)\} : x \in X\} \vee \max\{\min\{B(x), f(x)(y)\} : x \in X\} \\
&= \max\{\min\{A(x), g(x)(y)\} : x \in X\} \vee \max\{\min\{B(x), g(x)(y)\} : x \in X\} \\
&= \max\{\min\{A(x), g(x)(y)\} \vee \min\{B(x), g(x)(y)\} : x \in X\} \\
&= \max\{\min\{A(x) \vee B(x), g(x)(y)\} : x \in X\} \\
&= \mu_Y \mathcal{L}(g)(A \vee B)(y)
\end{aligned}$$

Hence $A \vee B \in I$. Now by 3.3, we know every non-zero member of I is quasisimple and so it is a join of subsimples. □

4. **Examples**

In this section to illustrate some of the results obtained in the previous sections, we present several examples.

Example 4.1. *In this example, we discuss the results obtained for the case $n = 2$. In this case we have $L = \{0, 1\}$. Given a function $f : X \to \mathcal{L}(Y)$, for each $x \in X$, $f(x)$ is the characteristic function χ_{F_x}, where $F_x = f(x)^{-1}(1) \subseteq Y$. By 2.1,*

$$\begin{aligned}
\mu_Y \mathcal{L}(f)(A)(y) &= \max\{\min\{A(x), f(x)(y)\} : x \in X\} \\
&= \max\{\min\{A(x), \chi_{F_x}(y)\} : x \in X\} \\
&= \begin{cases} 0 & \text{if } \forall x,\ F_x \not\ni y \\ \max\{A(x) : x \in X, F_x \ni y\} & \text{otherwise} \end{cases} \\
&= \max\{A(x) : x \in C_{f,y}\}
\end{aligned}$$

where $C_{f,y} = \{x \in X : F_x \ni y\}$. So $\mu_Y \mathcal{L}(f)(A)(y) = 1$ if and only if $C_{f,y} \cap A^{-1}(1) \neq \emptyset$ if and only if there is $x \in C_{f,y} \cap A^{-1}(1)$ if and only if there is $x \in A^{-1}(1)$ such that $y \in F_x$ if and only if $y \in \bigcup_{x \in A^{-1}(1)} F_x$. Setting $A_f = \bigcup_{x \in A^{-1}(1)} F_x$, we have $\mu_Y \mathcal{L}(f)(A) = \chi_{A_f}$.

Now suppose functions $X \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{L}(Y)$ *are given and* $I \overset{i}{\hookrightarrow} \mathcal{L}(X)$ *is the equalizer of the pair* $\mathcal{L}(X) \underset{\mu_Y\mathcal{L}(g)}{\overset{\mu_Y\mathcal{L}(f)}{\rightrightarrows}} \mathcal{L}(Y)$. *We have,*

$$\begin{aligned} I &= \{A \in \mathcal{L}(X) : \mu_Y\mathcal{L}(f)(A) = \mu_Y\mathcal{L}(g)(A)\} \\ &= \{A \in \mathcal{L}(X) : \chi_{A_f} = \chi_{A_g}\} \\ &= \{A \in \mathcal{L}(X) : A_f = A_g\} \end{aligned}$$

Since $n = 2$, $A \in I$ *is simple if and only if there is no* $B \in I$ *such that* $0 < B < A$. *Recalling that, an element* a *in a lattice with* 0 *is an atom if it is non-zero and there is no element strictly between* 0 *and* a, *we have* $A \in I$ *is simple if and only if it is an atom in* I, *or an* I*-atom. Note an element* $A \in I$ *that is an atom in* $\mathcal{L}(X)$ *is an* I*-atom, but not vice versa. So* $A \in \mathcal{L}(X)$ *is simple provided that* $A_f = A_g$ *and if* $0 < B \leq A$ *such that* $B_f = B_g$, *then* $B = A$. *It follows that the subsimple elements are only the simple ones.*

Let E *be the set of all simples and* $E \overset{e}{\hookrightarrow} \mathcal{L}(X)$ *be the inclusion. The sets* Σ_j, $j \in L$ *given in 3.3 are* $\Sigma_0 = \{\bar{0} : E \to L\}$ *and* $\Sigma_1 = \{\eta_E(t) : E \to L \mid t \in E\}$. *Obviously* $\mu_X\mathcal{L}(e)$ *is one to one on* Σ_0. *To see it is one to one on* Σ_1 *we have,* $\mu_X\mathcal{L}(e)(\eta_E(s)) = \mu_X\mathcal{L}(e)(\eta_E(t))$ *if and only if* $\chi_{A_e} = \chi_{B_e}$, *where* $A = \eta_E(s)$ *and* $B = \eta_E(t)$ *if and only if* $A_e = B_e$ *if and only if* $e(s)^{-1}(1) = e(t)^{-1}(1)$ *if and only if* $e(s) = e(t)$ *if and only if* $s = t$ *if and only if* $\eta_E(s) = \eta_E(t)$. *So by 3.3,*

$$E \xrightarrow{\hat{e}} X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} Y$$

is an equalizer in $Set_{\mathbb{L}}$ *if every non-zero element of* I *is quasisimple.*

We also have $\eta_X(x)$ *is in* I *if and only if* $A_f = A_g$, *where* $A = \eta_X(x)$ *if and only if* $f(x)^{-1}(1) = g(x)^{-1}(1)$ *if and only if* $f(x) = g(x)$, *as* $A^{-1}(1) = \{x\}$ *if and only if* $x \in Eq(f,g)$, *where* $Eq(f,g)$ *is the equalizer of* f *and* g. *Thus* $\eta_X(x)$ *is simple only when* $x \in Eq(f,g)$.

Example 4.2. *In this example we let* $n = 2$. *So the results obtained in Example 4.1 applies.*

a) We show the existence of simples other than $\eta_X(x)$ *for all* x.

Let $X = \{1,2\}$, Y *be any set,* $f : X \to \mathcal{L}(Y)$ *be any function with* $f(1) \neq f(2)$ *and let* $g : X \to \mathcal{L}(Y)$ *be defined by* $g(1) = f(2)$ *and* $g(2) = f(1)$. *Since* $Eq(f,g) = \emptyset$, $\eta_X(x) \notin I$ *for all* $x \in X$. *On the other hand the constant function* $\bar{1} : X \to L$ *with value 1 is in* I, *because* $\bar{1}_f = f(1)^{-1}(1) \cup f(2)^{-1}(1) = g(2)^{-1}(1) \cup g(1)^{-1}(1) = \bar{1}_g$. *Since* $\eta_X(1)$ *and* $\eta_X(2)$ *do not lie in* I, $\bar{1}$ *is simple.*

b) We give a pair of morphisms in $Set_{\mathbb{L}}$ *whose equalizer exists.*

Let $X = Y = \{1,2,3\}$ *and denote by* $(a,b,c) : X \to L$ *the function that takes* 1 *to* a, 2 *to* b *and* 3 *to* c, *where* a, b *and* $c \in L = \{0,1\}$. *Let* $f = \eta_X : X \to \mathcal{L}(X)$ *and define* $g : X \to \mathcal{L}(X)$ *by* $g(1) = (0,1,0)$, $g(2) = (1,0,0)$, $g(3) = (0,0,0)$.

With $F_x = f(x)^{-1}(1)$ and $G_x = g(x)^{-1}(1)$, we summerize the corrsponding data in the following table.

x	$f(x)$	$g(x)$	F_x	G_x
1	$(1,0,0)$	$(0,1,0)$	$\{1\}$	$\{2\}$
2	$(0,1,0)$	$(1,0,0)$	$\{2\}$	$\{1\}$
3	$(0,0,1)$	$(0,0,0)$	$\{3\}$	$\emptyset$

Since besides the empty union, only $F_1 \cup F_2 = G_1 \cup G_2$, we get $I = \{(0,0,0),(1,1,0)\}$. So $E = \{(1,1,0)\}$ and thus the only subsimple is $(1,1,0)$. Obviously every non-zero member of I is quasisimple. Hence by Example 4.1 the equalizer of $X \underset{\hat{g}}{\overset{1_X}{\rightrightarrows}} X$ exists in $Set_{\mathbb{L}}$.

c) Finally we give a pair of morphisms in $Set_{\mathbb{L}}$ whose equalizer does not exist.

With X and Y as in part (b), $f = \eta_X$ and g defined by $g(1) = (1,0,0)$, $g(2) = (1,1,0)$, $g(3) = (1,1,1)$. We have,

x	$f(x)$	$g(x)$	F_x	G_x
1	$(1,0,0)$	$(1,0,0)$	$\{1\}$	$\{1\}$
2	$(0,1,0)$	$(1,1,0)$	$\{2\}$	$\{1,2\}$
3	$(0,0,1)$	$(1,1,1)$	$\{3\}$	$\{1,2,3\}$

It follows that $I = \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\}$, and so $E = \{(1,0,0)\}$. Thus the only subsimple is $(1,0,0)$. Since $(1,1,0) \in I$ is not quasisimple, by Example 4.1 the equalizer of $X \underset{\hat{g}}{\overset{1_X}{\rightrightarrows}} X$ does not exist.

Example 4.3. *In this example we set $n = 3$ so that $L = \{0,1,2\}$.*

a) We give a pair of morphisms whose equalizer exists in $Set_{\mathbb{L}}$.

Let $X = Y = \{1,2\}$ and $(a,b) : X \to L$ denote the function that takes 1 to a and 2 to b. Consider functions $X \underset{\eta_X}{\overset{f}{\rightrightarrows}} \mathcal{L}(X)$, where $f(1) = (2,0)$ and $f(2) = (1,1)$. Let I be the equalizer of $\mathcal{L}(X) \underset{\mu_X \mathcal{L}(\eta_X)=1_{\mathcal{L}(X)}}{\overset{\mu_X \mathcal{L}(f)}{\rightrightarrows}} \mathcal{L}(X)$. For $A \in \mathcal{L}(X)$,

$$\begin{aligned} \mu_X \mathcal{L}(f)(A)(1) &= \max\{\min\{A(1),2\}, \min\{A(2),1\}\} \\ &= \max\{A(1), \min\{A(2),1\}\} \end{aligned}$$

and

$$\begin{aligned}\mu_X\mathcal{L}(f)(A)(2) &= \max\{\min\{A(1),0\},\min\{A(2),1\}\}\\ &= \min\{A(2),1\}\end{aligned}$$

So

$$\begin{aligned}A \in I \text{ if and only if } &\max\{A(1),\min\{A(2),1\}\} = A(1) \text{ and } \min\{A(2),1\} = A(2)\\ &\text{if and only if } \max\{A(1),A(2)\} = A(1) \text{ and } \min\{A(2),1\} = A(2)\\ &\text{if and only if } A(2) \le A(1) \text{ and } A(2) \le 1\\ &\text{if and only if } A = (0,0), A = (1,0), A = (2,0), A = (1,1) \text{ or } A = (2,1).\end{aligned}$$

Therefore $I = \{(0,0),(1,0),(2,0),(1,1),(2,1)\}$. It then follows that the set of simples is $E = \{(2,0),(1,1)\}$ and that the set of subsimples is $\{(1,0),(2,0),(1,1)\}$. One can easily verify that the non-zero elements of I are quasisimple.

With $E \xhookrightarrow{e} \mathcal{L}(X)$ the inclusion, we show that $\mu_X\mathcal{L}(e)$ is one to one on Σ_j, for each $j \in [3]$. Since $\Sigma_0 = \{0\}$, the assertion holds for $j = 0$. For $1 \le j \le 2$, suppose $\mu_X\mathcal{L}(e)(\eta_E^j(t_1)) = \mu_X\mathcal{L}(e)(\eta_E^j(t_2))$, where $t_1, t_2 \in E$. if $t_1 \neq t_2$, then say $t_1 = (2,0)$ and $t_2 = (1,1)$. The equality $\mu_X\mathcal{L}(e)(\eta_E^j(t_1)) = \mu_X\mathcal{L}(e)(\eta_E^j(t_1))$ can be shown to be equivalent to $\min\{j,t_1(1)\} = \min\{j,t_2(1)\}$ and $\min\{j,t_1(2)\} = \min\{j,t_2(2)\}$. It follows that $\min\{j,2\} = \min\{j,1\}$ and $\min\{j,0\} = \min\{j,1\}$ or equivalently $j = \min\{j,1\}$ and $0 = \min\{j,1\}$. So $j = 0$ which is a contradiction. It follows that $t_1 = t_2$, and thus $\eta_E^j(t_1) = \eta_E^j(t_2)$ as desired. Hence by 3.3 the equalizer of $X \underset{1_X}{\overset{\hat{f}}{\rightrightarrows}} X$ exists in $Set_{\mathbb{L}}$.

b) Now we give a pair of morphisms whose equalizer does not exist in $Set_{\mathbb{L}}$. Set $X = \{1,2\}$, $Y = \{1,2,3\}$ and define f and g as in the following table.

x	$f(x)$	$g(x)$
1	$(1,1,1)$	$(2,2,2)$
2	$(2,2,2)$	$(1,1,1)$

Given any $A : X \to L$ one can easily see that for each $y \in Y$,

$$\mu\mathcal{L}(f)(A)(y) = \max\{\min\{A(1),1\},A(2)\} = \begin{cases}\max\{A(1),A(2)\} & \text{if } A(1) \le 1\\ \max\{1,A(2)\} & \text{if } A(1) > 1\end{cases}$$

$$\mu\mathcal{L}(g)(A)(y) = \max\{A(1),\min\{A(2),1\}\} = \begin{cases}\max\{A(1),A(2)\} & \text{if } A(2) \le 1\\ \max\{A(1),1\} & \text{if } A(2) > 1\end{cases}$$

It follows that $A \in I$ if and only if ($A(1) \le 1$ and $A(2) \le 1$) or $A(1) = A(2) = 2$. Thus $I = \{(0,0),(0,1),(1,0),(1,1),(2,2)\}$. Since $n = 3$, a simple is an element of I that has only one non-zero member of I smaller than it. Thus we have no simples here and therefore no subsimples and no quasisimples. So the condition *that every non-zero member of I is quasisimple does not hold and thus by 3.3 the equalizer of* $X \underset{\hat{g}}{\overset{\hat{f}}{\rightrightarrows}} X$ *does not exist in* $Set_{\mathbb{L}}$.

5. Aknowledgement

The authors want to express their gratitude to the referees whose insightful comments greatly improved the quality of the manuscript.

6. Conflict of interest

The authors declare no conflict of interest.

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