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NONPARAMETRIC ESTIMATORS FOR VAREXTROPY UNDER α-MIXING CONDITION WITH APPLICTION IN EXPONENTIAL AR(1) MODEL

R. Zamini, F. Goodarzi ✉, and M. Salimi

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Abstract. The goal of this paper is to study the problem of estimation of varextropy function under α-mixing dependence condition. We propose nonparametric estimators for varextropy, residual varextropy and past varextropy. Asymptotic properties of the proposed estimators are investigated under regularity conditions. Moreover, the comparison of the proposed estimators for varextropy in terms of the bias and mean squared error has been done by Monte Carlo method. Furthermore, a real data example is presented.

Keywords: Asymptotic properties, Strong mixing, Varextropy function, Kernel estimator, Simulation.
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1. Introduction

Let X be an absolutely continuous nonnegative random variable with density and distribution function f and F, respectively. Shannon entropy is defined as:

$$H(X) = -\int_0^{+\infty} f(x)\log f(x)dx, \tag{1}$$

where $\log(\cdot)$ is the natural logarithm with standard convention $0\log 0 = 0$. This measure is defined by Shannon [29] and is used in many fields such as data compression, thermodynamics, error-detecting codes, image and signal processing, cryptology, economics, linguistics, physiology, physics, wavelet analysis and computer sciences. It should be noted that the variance entropy (varentropy) of a random variable X is defined as

$$\begin{aligned} V(X) &:= Var[\log f(X)] = E[(-\log f(X))^2] - [H(X)]^2 \\ &= \int_0^{+\infty} f(x)[\log f(x)]^2 dx - \left[\int_0^{+\infty} f(x)\log f(x)dx\right]^2. \end{aligned} \tag{2}$$

✉ f-goodarzi@kashanu.ac.ir, ORCID: 0000-0002-3783-6632
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Song [30] defined the concept of varentropy for measuring the variability in the information content of a random variable. This measure can be used to compare the heavy-tailed distributions instead of kurtosis measure. Goodarzi et al. [7] obtained an upper bound for varentropy in terms of some concepts of reliability theory. Also Maadani et al. [15] studied the properties of varentropy of order statistics. Alizadeh and Shafaei [1] proposed some estimators for varentropy and applied them to test the uniformity.

Several generalizations of Shannon entropy have been considered by many authors over the years. One of the important generalizations of the Shannon entropy is due to Lad et al. [12], which is known as extropy function and is given by:

$$J(X) = -\frac{1}{2}\int_0^{+\infty} f^2(x)dx. \tag{3}$$

This measure is the complementary dual of Shannon entropy. This function compares the uncertainties of two random variables. The extropy of two random variables may be the same. For example, the extropy function is the same for standard uniform distribution and exponential distribution with mean $1/2$. The need for another measure in such situations is quite evident. Some applications of extropy were investigated by Gneiting and Raftery [6]. Becerra et al. [4] showed the application of extropy in automatic speech recognition. The extropy of order statistics was studied by Qui and Jia [24]. Qui and Jia [23] proposed some estimators for extropy with applications in testing uniformity. Under length-biased sampling, Rajesh et al. [25] obtained kernel estimators of extropy function and investigated the asymptotic properties of the proposed estimators such as consistency and asymptotic normality.
The extropy of the residual lifetime $X_t = [X - t|X \geq t]$ with the density function of

$$f_t(x) = \frac{f(x+t)}{\bar{F}(t)}, \;\; x \geq 0,\, t \geq 0,$$

is given by

$$\begin{aligned} J(X_t) &= -\frac{1}{2}\int_0^{+\infty} f_t^2(x)dx \\ &= \frac{-1}{2(\bar{F}(t))^2}\int_t^{+\infty} f^2(x)dx, \;\; t \geq 0. \end{aligned} \tag{4}$$

It is easy to see that $J(X_0) = J(X)$.
$J(X_t)$ measures residual uncertainty of the random variable X. Some properties of this measure is investigated by Qui and Jia (2018). Krishnan et al. [11]

defined the extropy of past lifetime $X_{[t]} = [X|X \leq t]$ as:

$$\begin{aligned} J(X_{[t]}) &= -\frac{1}{2}\int_0^{+\infty} f_{[t]}^2(x)dx \\ &= \frac{-1}{2(F(t))^2}\int_0^t f^2(x)dx, \end{aligned} \tag{5}$$

where $f_{[t]}(x) = \dfrac{f(x)}{F(t)}$, $0 < x < t$, is the density function of $X_{[t]}$. When $t \to \infty$, the past extropy tends to $J(X)$. This measure has applications in fields such as information theory, survival analysis, reliability and insurance. Kamari and Buono [10] elucidated certain characterization results of the past extropy using order statistics. Recently, Vaselabadi et al. [31] defined statistically the term of varextropy as the another alternative measure to Shannon entropy. For a random variable X with density function f, its varextropy is defined as

$$\begin{aligned} VJ(X) &= Var\left(-\frac{1}{2}f(X)\right) \\ &= \frac{1}{4}E(f^2(X)) - J^2(X). \end{aligned} \tag{6}$$

The varextropy measure indicates how the information content is scattered around the extropy. If the value of the extropy function is the same for two distributions, the lower value of varextropy determines which extropy is more appropriate for measuring uncertainty because of the least information volatility. Vaselabadi et al. [31] showed that varextropy function is free of the model parameters in some cases and therefore is more flexible than the varentropy function. One of the applications of $VJ(X)$ is to measure the information volatility contained in the associated residual and past lifetimes. The residual varextropy of X at time t is defined as:

$$\begin{aligned} VJ\left(X_t\right) &= \frac{1}{4}E\left(f_t^2\left(X_t\right)\right) - J^2\left(X_t\right) \\ &= \frac{1}{4}\int_0^{+\infty} f_t^3\left(x\right)dx - J^2\left(X_t\right) \\ &= \frac{1}{4}\frac{1}{(\bar{F}(t))^3}\int_t^{+\infty} f^3(x)dx - \frac{1}{4(\bar{F}(t))^4}\left(\int_t^{+\infty} f^2(x)dx\right)^2. \end{aligned} \tag{7}$$

Also, the past varextropy of X at time t can be defined as:

$$\begin{aligned} VJ\left(X_{[t]}\right) &= \frac{1}{4}E\left(f_{[t]}^2\left(X_{[t]})\right)\right) - J^2\left(X_{[t]}\right) \\ &= \frac{1}{4}\int_0^{+\infty} f_{[t]}^3(x)dx - J^2\left(X_{[t]}\right) \\ &= \frac{1}{4}\frac{1}{(F(t))^3}\int_0^t f^3(x)dx - \frac{1}{4(F(t))^4}\left(\int_0^t f^2(x)dx\right)^2. \end{aligned} \tag{8}$$

The properties of these measures can be found in Vaselabadi et al. [31]. Al-Labadi et al. [2] derived Bayesian nonparametric estimators for varentropy and varextropy. Recently, Zamini et al. [33] provided kernel estimations for varextropy function under length-biased sampling. For discrete random variables, Goodarzi [8] investigated the discrete residual varentropy and obtained an upper bound for it.

In this paper, our main goal is to develop nonparametric estimator for varextropy function via recursive kernel type estimation based on dependence data. Practically, replacing independence with some type of dependence seems more realistic. One of the dependence structures is α-mixing and since it is the better mixing and has applications in various fields including time series modeling, we focus on it in our research. α-mixing was introduced to look at time series of random variables coming from dynamical systems. This is, the assumption of independence is not appropriate for such dynamical systems, but rather a form of asymptotic independence is what a practitioner would be interested in. Moreover, when the complexity of a dynamical system precludes the proposal of a particular parametric family, there is the need to introduce nonparametric estimation, e.g. kernel-based estimators, for random variables describing the evolution of such systems.

In this paper, the analysis devoted requires asymptotic independence; that is, random samples at times which are very distant from each other should behave as if they are independent. The α-mixing (strong mixing) condition was introduced by Rosenblatt [26] as follows:

Definition 1.1. Let $\{X_n, n \geq 1\}$ be a sequence of real-valued random variables on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_i^k = \sigma(X_j, i \leq j \leq k)$ be the σ-algebra generated by the indicated random variables. The sequence $\{X_n, n \geq 1\}$ is said to be α-mixing (strong mixing) if

$$\alpha_n = \sup_{k\geqslant 1} \sup_{A\in\mathcal{F}_1^k} \sup_{B\in\mathcal{F}_{k+n}^\infty} |P(A\cap B) - P(A)P(B)| \to 0 \ as\ n \ \to \infty.$$

There is a large literature on basic properties of α-mixing conditions. Loynes [14] investigated some limit theorems for maxima of stationary processes under α-mixing condition. Wolverton and Wanger [32] studied the problem of kernel density estimation under α-mixing dependence conditions. Maya et al. [21], Irshad and Maya [9] and Maya and Irshad [18] obtained the non-parametric kernel type estimations for the extropy, past extropy and residual extropy under α-mixing dependence condition, respectively. Maya et al. [20] have studied the recursive and non recursive kernel estimation of negative cumulative extropy under α-mixing dependence condition. The kernel estimations of the Mathai-Haubold entropy and the residual Mathai-Haubold entropy function under α-mixing dependence conditions were proposed by Maya and Irshad [19]. Assuming that $X_1, \ldots, X_n$ is an α-mixing sequence, Wolverton and Wanger [32] proposed the following non-parametric recursive density estimator:

$$f_n(x) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{\psi_j}K\left(\frac{x-X_j}{\psi_j}\right), \tag{9}$$

where K is a kernel function of order s and satisfies the following conditions:
A_1: K is bounded, non-negative and symmetric function.
A_2: $\int_{-\infty}^{+\infty} K(x)dx = 1$, and $\int_{-\infty}^{+\infty} xK(x)dx = 0$.
Also, $\{\psi_n\}$ is a positive bandwidth sequence, where it satisfies the following conditions:
A_3: $\lim_{n\to\infty}\psi_n = 0$, and $\lim_{n\to\infty} n\psi_n = +\infty$.
A_4: $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\psi_j}{\psi_n}\right)^l = \beta_l < +\infty, \quad l = 1, 2, \ldots, s+1.$
A_5 : $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\psi_n}{\psi_j}\right)^l = \theta_l < +\infty, \quad 1 \le l < 2.$
Under Assumptions of $\mathbf{A_1} - \mathbf{A_5}$, Masry [16] obtained the following expressions for the bias and variance of $f_n(x)$:

$$\text{Bias}\,(f_n(x)) \simeq \frac{\psi_n^s Z_s}{s!} f^{(s)}(x)\beta_s, \tag{10}$$

and

$$\text{Var}\,(f_n(x)) \simeq \frac{\theta_1 f(x)}{n\psi_n} Z_k, \tag{11}$$

where $Z_s = \int_{-\infty}^{+\infty} u^s K(u)du$, $Z_k = \int_{-\infty}^{+\infty} K^2(u)du$, and $f^{(s)}(x)$ is the s^{th} derivative of the density function of f.

In this paper, we propose kernel estimators for $VJ(X), VJ(X_t)$ and $VJ(X_{[t]})$ based on a stationary α-mixing sequence and investigate some asymptotic properties of them in Sections 2 and 3, respectively. A Monte Carlo simulation and a real data study of the proposed estimators is carried out in Section 4.

2. Kernel-based estimation of varextropy

In this section, we develop kernel-based estimators for the varextropy using the α-mixing sequence of $X_1, \ldots, X_n$ and study asymptotic properties of them. A natural estimator for $VJ(X)$ can be obtained by substituting f by $f_n(x)$, and is given by:

$$\begin{aligned}\hat{V}J_1(X) &= \frac{1}{4}\left[\int f_n^2(x)dF_n(x) - \left(\int f_n(x)dF_n(x)\right)^2\right]\\ &= \frac{1}{4}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\tilde{f}_n\left(X_i\right)\right)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\tilde{f}_n(X_i)\right)^2\right],\end{aligned} \tag{12}$$

where

$$\tilde{f}_n(X_i) = \frac{1}{(n-1)} \sum_{j \neq i = 1}^{n} \frac{1}{\psi_j} K\left(\frac{X_i - X_j}{\psi_j}\right), \tag{13}$$

and

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq x). \tag{14}$$

$F_n(x)$ is the empirical distribution function under dependence condition. Some properties of $F_n(x)$ have been investigated by Roussas [27]. Another estimator for $VJ(X)$ is directly obtained by plug in $f_n(x)$ in the expression for $VJ(X)$ defined in (6), and is given by

$$\hat{V}J_2(X) = \frac{1}{4}\left[\int_0^{+\infty} f_n^3(x)dx - \left(\int_0^{+\infty} f_n^2(x)dx\right)^2\right]. \tag{15}$$

In the following theorems, we establish some asymptotic properties of $\hat{V}J_1(X)$ and $\hat{V}J_2(X)$. Before stating theorems, let for any distribution function F, τ denotes the right endpoint of its support with $\tau := \sup\{x, F(x) < 1\} \leq \infty$.

Theorem 2.1. *Suppose that: (i) as $n \to \infty$, the mixing rate satisfies $\alpha(n) = O(n^{-6})$,*

(ii) $\tau < \infty$ and $f(x)$ is bounded, then we can write

$$\lim_{n \to \infty} \hat{V}J_1(X) = VJ(X), \quad a.s.$$

Proof. From (6) and (12), we have

$$\hat{V}J_1(X) - VJ(X) = I_n + II_n, \tag{16}$$

where

$$I_n = \frac{1}{4n} \sum_{i=1}^{n} \left(\tilde{f}_n(X_i)\right)^2 - \frac{1}{4}\int_0^{\tau} f^2(x)dF(x),$$

and

$$II_n = J^2(X) - \frac{1}{4}\left(\frac{1}{n}\sum_{i=1}^{n} \tilde{f}_n(X_i)\right)^2.$$

On the other hand

$$\begin{aligned} I_n &= \frac{1}{4}\int_0^{\tau} f_n^2(x)dF_n(x) - \frac{1}{4}\int_0^{\tau} f^2(x)dF(x) \\ &:= I_{n_1} + I_{n_2}, \end{aligned} \tag{17}$$

where

$$I_{n_1} = \frac{1}{4}\int_0^{\tau} f_n^2(x)d(F_n(x) - F(x)), \tag{18}$$

and

$$I_{n_2} = \frac{1}{4}\int_0^{\tau} (f_n^2(x) - f^2(x))dF(x). \tag{19}$$

A direct algebra shows that

$$f_n^2(x) - f^2(x) = \sum_{i=1}^{2} (f(x))^{i-1} (f_n(x))^{2-i} (f_n(x) - f(x)). \tag{20}$$

The almost sure convergence of $f_n(x)$ given in Masry and Gyorfi [17] and (20) ensure that

$$\lim_{n\to\infty} \sup_{-\infty<x<\infty} \left|f_n^2(x) - f^2(x)\right| = 0, \quad a.s. \tag{21}$$

Using (21) we can see that

$$\lim_{n\to\infty} |I_{n_2}| \leq \lim_{n\to\infty} \frac{1}{4} \sup_{-\infty<x<\infty} \left|f_n^2(x) - f^2(x)\right| = 0, \quad a.s. \tag{22}$$

and hence

$$\lim_{n\to\infty} |I_{n_2}| = 0, \quad a.s. \tag{23}$$

Now, let $\beta_n(t) = \sqrt{n}(F_n(t) - F(t))$ and $g_i(t) = I\{X_i \leq t\} - F(t), i \geq 1$. Put

$$\Gamma(s,t) = E[g_1(s)g_1(t)] + \sum_{j=2}^{\infty} E[g_1(s)g_j(t) + g_j(s)g_1(t)]. \tag{24}$$

Under condtion (i) of the theorem, Philipp and Pinzur [22] proved the following approximation for some $\lambda > 0$,

$$\sup_{n} \sup_{-\infty<s<\infty} |\sqrt{n}\beta_n(s) - K(s,n)| = O(n^{\frac{1}{2}}(\log n)^{-\lambda}), \quad a.s., \tag{25}$$

where $K(s,n)$ is a Kiefer process with covariance

$$E(K(s,n)K(s',m)) = (m \wedge n)\Gamma(s,s'), \quad m \wedge n = min(m,n). \tag{26}$$

Therefore

$$
\begin{aligned}
I_{n_1} &= \frac{1}{4}\int_0^\tau f_n^2(x)d(F_n(x)-F(x))\\
&= \frac{1}{4}n^{-\frac{1}{2}}\int_0^\tau f_n^2(x)d(\beta_n(x))\\
&= -\frac{1}{4}n^{-1}\int_0^\tau \sqrt{n}\beta_n(s)df_n^2(s)\\
&= \frac{1}{4}n^{-1}\int_0^\tau (K(s,n)-\sqrt{n}\beta_n(s))df_n^2(s) - \frac{1}{4}n^{-1}\int_0^\tau K(s,n)df_n^2(s)\\
&\le \frac{1}{4}n^{-1}\sup_{-\infty<s<\infty}|\sqrt{n}\beta_n(s)-K(s,n)|\int_0^\tau df_n^2(s) + \frac{1}{4}n^{-1}\sup_{-\infty<s<\infty}|K(s,n)|\int_0^\tau df_n^2(s)\\
&:= I_{n_{11}} + I_{n_{12}}. \qquad (27)
\end{aligned}
$$

From the law of the iterated logarithm (LIL) for Kiefer process (see, Corollary 1.15.1 of [5]), we can observe that

$$
\limsup_n (\log\log n)^{-\frac{1}{2}} n^{-\frac{1}{2}} \sup_{-\infty<s<\infty}|K(s,n)| = 2^{-\frac{1}{2}}, \quad a.s. \qquad (28)
$$

On the other hand, we can write

$$
\begin{aligned}
\lim_{n\to\infty}\int_0^\tau df_n^2(s) &= \lim_{n\to\infty}\left[\int_0^\tau d(f_n^2(s)-f^2(s)) + \int_0^\tau df^2(s)\right]\\
&= \lim_{n\to\infty}(f_n^2(\tau)-f^2(\tau)-f_n^2(0)-f^2(0)+f^2(\tau)-f^2(0))\\
&\le \lim_{n\to\infty} 2\sup_{0<x<\infty}|f_n^2(x)-f^2(x)| + 2\sup_{0<x<\infty}|f^2(x)| < M < \infty, \quad a.s., \qquad (29)
\end{aligned}
$$

where the last equality follows from (21) and boundedness of $f(x)$.

(28) and (29) imply that

$$
\limsup_n I_{n_{12}} = 0, \quad a.s. \qquad (30)
$$

Also, from (25) and (29) we can see that

$$
\lim_{n\to\infty} I_{n_{11}} = 0, \quad a.s. \qquad (31)
$$

From (27), (30) and (31), we can conclude that

$$
\lim_{n\to\infty} I_{n_1} = 0, \quad a.s. \qquad (32)
$$

(17), (23) and (32) imply that

$$
\lim_{n\to\infty} I_n = 0, \quad a.s. \qquad (33)
$$

An argument similar to the convergence of I_n shows that

$$
\lim_{n\to\infty}\frac{1}{2}\int_0^\tau f_n dF_n(x) = \frac{1}{2}\int_0^\tau f dF(x), \quad a.s. \qquad (34)
$$

And hence

$$\lim_{n\to\infty}\left(\frac{1}{2}\int_0^{\tau} f_n dF_n(x)\right)^2 = J^2(X), \quad a.s.$$

Or in other word,

(35) $$\lim_{n\to\infty} II_n = 0, \quad a.s.$$

(16), (33) and (35) complete the proof. □

Theorem 2.2. *Under Assumptions* $\mathbf{A_1} - \mathbf{A_5}$ *we can write*

$$\hat{V}J_2(X) \xrightarrow{p} VJ(X), \text{ as } n\to\infty.$$

Proof.

(36) $$\hat{V}J_2(X) - VJ(X) = I_n + II_n,$$

where

(37) $$I_n = \frac{1}{4}\left(\int_0^{+\infty} f_n^3(x)dx - \int_0^{+\infty} f^3(x)dx\right),$$

and

(38) $$II_n = J^2(X) - \left(-\frac{1}{2}\int_0^{+\infty} f_n^2(x)dx\right)^2.$$

Theorem 2 of Maya et al. [21] implies that

(39) $$II_n \xrightarrow{p} 0.$$

By using Taylor's series expansion, we can see that

(40) $$\int_0^{+\infty} f_n^{\alpha}(x)dx - \int_0^{+\infty} f^{\alpha}(x)dx \simeq \alpha\int_0^{+\infty} f^{\alpha-1}(x)\left(f_n(x) - f(x)\right)dx, \quad \alpha = 1,2,3.$$

Now, using (10), (11) and (40), the expression for the bias and variance of $\int_0^{+\infty} f_n^3(x)dx$ is given by:

(41) $$\text{Bias}\left(\int_0^{+\infty} f_n^3(x)dx\right) \simeq \frac{3\psi_n^s Z_s}{s!}\beta_s\int_0^{+\infty} f^2(x)f^{(s)}(x)dx,$$

and

(42) $$\text{Var}\left(\int_0^{+\infty} f_n^3(x)dx\right) \simeq \frac{9\theta_1 Z_k}{n\psi_n}\int_0^{+\infty} f^5(x)dx.$$

The corresponding MSE is obtained by

(43) $$\text{MSE}\left(\int_0^{+\infty} f_n^3(x)dx\right) \simeq \left(\frac{3\psi_n^s Z_s}{s!}\beta_s\int_0^{+\infty} f^2(x)f^{(s)}(x)dx\right)^2 + \frac{9\theta_1 Z_k}{n\psi_n}\int_0^{+\infty} f^5(x)dx.$$

Now, from (43) as $n \to +\infty$

$$\text{MSE}\left(\int_0^{+\infty} f_n^3(x)dx\right) \longrightarrow 0. \tag{44}$$

Therefore,

$$I_n \xrightarrow{p} 0, \text{ as } n \to +\infty. \tag{45}$$

From (36), (39) and (45), we can conclude that

$$\hat{V}J_2(X) \xrightarrow{p} VJ(X), \text{ as } n \to +\infty.$$

□

3. Kernel-based estimators for residual and past varextropy

In this section, we propose some estimators for $VJ(X_t)$ and $VJ(X_{[t]})$. An estimator for $VJ(X_{[t]})$ can be written as:

$$\begin{aligned} \hat{V}J\left(X_{[t]}\right) = & \frac{1}{4(\tilde{F}_n(t))^3}\int_0^t f_n^3(x)dx \\ & - \left(\frac{1}{2\left(\tilde{F}_n(t)\right)^2}\int_0^t f_n^2(x)dx\right)^2, \end{aligned} \tag{46}$$

where

$$\tilde{F}_n(t) = \int_0^t f_n(x)dx, \tag{47}$$

is the kernel estimator of the distribution function $F(t)$.

In a similar way, we can propose an estimator for $VJ(X_t)$ as follows:

$$\begin{aligned} \hat{V}J\left(X_t\right) = & \frac{1}{4\left(1-\tilde{F}_n(t)\right)^3}\int_t^{+\infty} f_n^3(x)dx \\ & - \left(\frac{1}{2\left(1-\tilde{F}_n(t)\right)^2}\int_t^{+\infty} f_n^2(x)dx\right)^2, \end{aligned} \tag{48}$$

In the following theorems we prove some asymptotic properties for $VJ\left(X_{[t]}\right)$ and $VJ\left(X_t\right)$.

Theorem 3.1. *Under Assumptions* $\mathbf{A_1} - \mathbf{A_5}$ *, we can observe that* $\hat{V}J(X_{[t]})$ *is a consistent estimator of* $VJ(X_{[t]})$ *for each* $t < \tau < \infty$*. That is, as* $n \to +\infty$

$$\hat{V}J\left(X_{[t]}\right) \xrightarrow{p} VJ(X_{[t]}).$$

Proof.

$$\hat{V}J\left(X_{[t]}\right) - VJ\left(X_{[t]}\right) = II_{n_1} + II_{n_2}, \tag{49}$$

where

$$\begin{aligned} II_{n_1} &= \frac{1}{4(\tilde{F}_n(t))^3}\int_0^t f_n^3(x)dx - \frac{1}{4(F(t))^3}\int_0^t f^3(x)dx, \\ II_{n_2} &= \left(\frac{1}{2(F(t))^2}\int_0^t f^2(x)dx\right)^2 - \left(\frac{1}{2(\tilde{F}_n(t))^2}\int_0^t f_n^2(x)dx\right)^2. \end{aligned} \tag{50}$$

Theorem 3.2 of Irshad and Maya [9] ensures that for each $t < \tau$,

$$II_{n_2} \xrightarrow{p} 0, \text{ as } n \to \infty. \tag{51}$$

By using (40), (10) and (11) the expression for the bias and variance of $\int_0^t f_n^3(x)dx$ are obtained as follows:

$$\text{Bias}\left(\int_0^t f_n^3(x)dx\right) \simeq \frac{3\psi_n^s Z_s}{s!}\beta_s \int_0^t f^2(x)f^{(s)}(x)dx, \tag{52}$$

and

$$\text{Var}\left(\int_0^t f_n^3(x)dx\right) \simeq \frac{9\theta_1 Z_k}{n\psi_n}\int_0^t f^5(x)dx. \tag{53}$$

The corresponding MSE is given by

$$\begin{aligned} \text{MSE}\left(\int_0^t f_n^3(x)dx\right) \simeq& \left(\frac{3\psi_n^s Z_s}{s!}\beta_s \int_0^t f^2(x)f^{(s)}(x)dx\right)^2 \\ &+ \frac{9\theta_1 Z_k}{n\psi_n}\int_0^t f^5(x)dx. \end{aligned} \tag{54}$$

Under conditions $\psi_n \to 0$ and $n\psi_n \to +\infty$, as $n \to \infty$ we can observe that

$$\text{MSE}\left(\int_0^t f_n^3(x)dx\right) \to 0. \tag{55}$$

(55) ensures that

$$\int_0^t f_n^3(x)dx \xrightarrow{p} \int_0^t f^3(x)dx, \ \ as\ n \to \infty. \tag{56}$$

Also, by using Taylor's series expansion, we can see that

$$(\tilde{F}_n(t))^3 \simeq F^3(t) + 3F^2(t)(\tilde{F}_n(t) - F(t)), \quad a.s. \tag{57}$$

Using (57), (10) and (11), the expression for the bias and variance of $(\tilde{F}_n(t))^3$ are obtained as

$$\text{Bias}\left((\tilde{F}_n(t))^3\right) \simeq 3\frac{\psi_n^s Z_s}{s!}\beta_s F^2(t)\int_0^t f^{(s)}(x)dx, \tag{58}$$

and

$$\operatorname{Var}\left((\tilde{F}_n(t))^3\right) \simeq \frac{9\theta_1 Z_k}{n\psi_n} F^5(t).$$

Also

$$\text{(59)} \quad \begin{aligned} \operatorname{MSE}\left((\tilde{F}_n(t))^3\right) \simeq & \left(3\frac{\psi_n^s Z_s}{s!}\beta_s F^2(t)\int_0^t f^{(s)}(x)dx\right)^2 \\ & + \frac{9\theta_1 Z_k}{n\psi_n} F^5(t). \end{aligned}$$

(59) ensures that as $n \to \infty$, MSE $\left((\tilde{F}_n(t))^3\right) \to 0$. Therefore we can write:

$$\text{(60)} \qquad \tilde{F}_n^3(t) \xrightarrow{p} F^3(t), \;\; as \;\; n \to \infty.$$

From (56) and (60), we can conclude that

$$\text{(61)} \qquad II_{n_1} \xrightarrow{p} 0, \;\; as \;\; n \to \infty.$$

(49), (51) and (61) complete the proof. □

Theorem 3.2. *Under conditions of Theorem 3.1 for each $t < \tau$, we can write*

$$\text{(62)} \qquad \hat{V}J\left(X_t\right) \xrightarrow{p} VJ(X_t), \;\; as \;\; n \to \infty.$$

Proof. The proof is similar to the proof of Theorem 3.1 and is omitted. □

4. Simulation study and real data analysis

In this section, we present some results on Monte-Carlo simulation study to assess the performance of the proposed estimators under different sample sizes. We assume $X_1, \ldots, X_n$ is generated from the exponential AR(1) process with correlation coefficient $\rho = 0.2$ and parameter $\theta = 1.5$, so we have:

$$X_i = 0.2X_{i-1} + \epsilon_i, \quad i = 1, \ldots, n,$$

where ϵ_i's are i.i.d. Exp(θ) random variables with density:

$$f(x) = \frac{2}{3}e^{-2x/3}, \quad x > 0, \quad \theta > 0,$$

and X_0 is fixed. To see more details of this model, see for example Saadatmand and Nematollahi [28] and Balakrishna [3]. 500 samples of size $n = 10, 20, 50, 100, 150, 200, 250, 300, 350$ and 400 are taken into consideration. The Gaussian kernel is used as the kernel function and $\psi_j = 1/(j^{0.5})$ is considered for $j = 1, \ldots, n$, as given in Maya et al. [21]. The bias and MSEs of the varextropy estimaors are computed for different sample sizes. The values are shown in Tables 1 and 2.

TABLE 1. Bias and MSE of $\hat{V}J_1(X)$

n	Bias	MSE
10	-0.03053	0.00097
20	-0.02512	0.00068
50	-0.01757	0.00036
100	-0.01307	0.00021
150	-0.01047	0.00015
200	-0.00938	0.00012
250	-0.00862	0.00010
300	-0.00745	0.00008
350	-0.00701	0.00007
400	-0.00655	0.00006

TABLE 2. Bias and MSE of $\hat{V}J_2(X)$

n	Bias	MSE
10	-0.03392	0.00117
20	-0.031828	0.00104
50	-0.02828	0.00083
100	-0.02657	0.00070
150	-0.02599	0.00069
200	-0.02524	0.00065
250	-0.02477	0.00063
300	-0.02454	0.00061
350	-0.02423	0.00060
400	-0.02414	0.00059

From Tables 1 and 2, we can see that the proposed estimators perform well in the sense of minimizing the MSE and the bias as the sample size increases. Meanwhile, $\hat{V}J_1(X)$ has a better performance as compared with $\hat{V}J_2(X)$. Figure 1 shows the numerically simulated values of $\hat{V}J_1(X)$ and $\hat{V}J_2(X)$ corresponding to the exponential AR(1) model. As it can be seen in the figure, the estimations are close to $VJ(X)$, especially when n is large. According to Figure 1, the values of the $\hat{V}J_1(X)$ (blue points) are closer to the real value (the red dashed line) than the values of $\hat{V}J_2(X)$ (black points).

In the following, we use the time series data of computer failures to illustrate the performance of our estimators in the real case. 257 observations are gathered in Lewis [13]. The quantities relate to successive times-to-failures. These data have been analyzed by Maya et al. [21] in the context of kernel estimation of the extropy function under α-mixing dependent data. They fitted an $AR(1)$ model to the data with the correlation coefficient $\phi = 0.5854$.

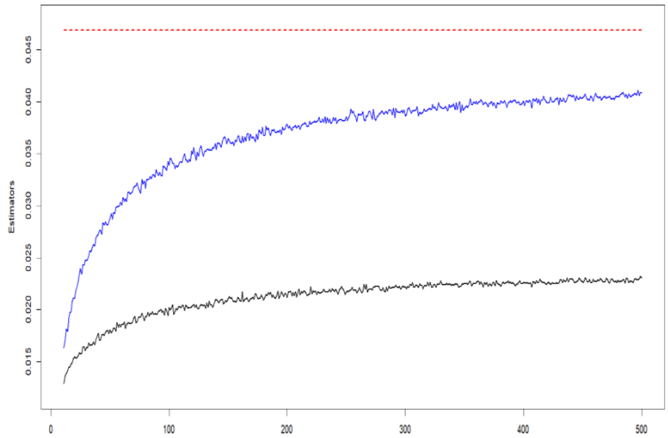

FIGURE 1. Simulated values of $\hat{V}J_1(X)$ (blue points) and $\hat{V}J_2(X)$ (black points). The red dashed line indicates the varextropy value in the exponential AR(1) model.

Also, they fitted an exponential distribution with the rate parameter θ to the data and obtained the observed value of the Kolmogorov-Smirnov statistic (KS=0.07501 with p-value =0.1121). The maximum likelihood estimate of the parameter is obtained as $\hat{\theta} = 0.00327$. Considering exponential distribution and its maximum likelihood estimate, we have obtained the varextropy of the whole data as $VJ(X) = 1.476113 \times 10^{-8}$. The estimated varextropy values are $\hat{V}J_1(X) = 2.980256 \times 10^{-8}$ and $\hat{V}J_2(X) = 3.022876 \times 10^{-8}$. The estimated results are close to $VJ(X) = 1.476113 \times 10^{-8}$ and this shows that our estimators are good in the real scenario.

5. Conclusion

In this paper, we study the problem of estimation of varextropy function under α-mixing dependence condition. Asymptotic properties of the proposed estimators are investigated under regularity conditions. We used Monte-Carlo simulation to compute MSE and bias of the estimators under exponential AR(1) process. Also we give a real data analysis.

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Raheleh Zamini
Orcid number: 0009-0001-4502-7763
Department of Mathematics
Faculty of Mathematical Sciences and Computer
Kharazmi University
Tehran, Iran
Email address: zamini@khu.ac.ir

Faranak Goodarzi
Orcid number: 0000-0002-3783-6632
Department of Statistics
Faculty of Mathematical Sciences
University of Kashan
Kashan, Iran
Email address: f-goodarzi@kashanu.ac.ir

Mohammad salimi
Orcid number: 0009-0007-8908-144X
Department of Mathematics
Faculty of Mathematical Sciences and Computer
Kharazmi University
Tehran, Iran
Email address: salimimohammad239@gmail.com