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ON THE KERNELS OF FROBENIUS GROUPS

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Abstract. A Frobenius group is a transitive permutation group on a finite set, such that no non-trivial element fixes more than one point and some non-trivial element fixes a point. Using character theory, it is proved that the Frobenius kernel is a normal subgroup of its Frobenius group. In this paper, we present some group-theoretical proofs that the Frobenius kernel is a subgroup of its Frobenius group under certain conditions.

Keywords: Frobenius group, Frobenius complement, Frobenius kernel.
2020 MSC: 20H20, 20F50.

1. Introduction

Suppose G is a Frobenius group consisting of permutations of a set Ω. A subgroup H of G fixing a point of Ω is called a Frobenius complement. The identity element together with all elements not in any conjugate of H form a normal subgroup called the Frobenius kernel K. The Frobenius group G is the semidirect product of K and H. Both Frobenius kernel and Frobenius complement have very restricted structures.

J. G. Thompson in 1960 proved that the Frobenius kernel K is a nilpotent group [16]. If H has an even order, then K is abelian. The Frobenius complement H has the property that every subgroup whose order is the product of 2 primes is cyclic; this implies that its Sylow subgroups are cyclic or generalized quaternion groups. A finite group is a Frobenius complement if and only if it has a faithful, finite-dimensional representation over a finite field in which non-identity group elements correspond to linear transformations without nonzero fixed points. The Frobenius kernel K is uniquely determined by G as it is the Fitting subgroup, and the Frobenius complement is uniquely determined up to conjugacy. In particular, a finite group G is a Frobenius group in at most one way.

Definition 1.1. Let G be a finite group acting on a set Ω with $|\Omega| > 1$. Then G is called a Frobenius group if

(a) G acts transitively on Ω,
(b) $G_\alpha \neq 1$ for any $\alpha \in \Omega$,
(c) $G_\alpha \cap G_\beta = 1$ for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$.

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Let $H = G_\alpha$ for some $\alpha \in \Omega$. Then for any $\beta \in \Omega$, the group G_β is conjugate to G_α, in other words, $G_\beta = G_\alpha^g = H^g$ for some $g \in G$. Therefore, let $F = G \setminus \bigcup_{g \in G} H^g$ be the set of elements of G that don't fix any element of Ω. We define

$$K = F \cup \{1\} = (G \setminus \bigcup_{g \in G} H^g) \cup \{1\}.$$

The subgroup H is called a Frobenius complement, and the set K is called the Frobenius kernel of G.

It is easy to prove that $N_G(H) = H$, and thus,

$$|K| = |G| + 1 - (|H|)[G : H] - 1 = [G : H] = n.$$

Therefore, $|G| = |K||H|$, and $H \cap K = 1$. An equivalent definition of a Frobenius group is as follows.

Definition 1.2. G is called a Frobenius group with complement H if

$$1 \neq H \lneqq G$$

and $H \cap H^g = 1$ for all $g \in G \setminus H$.

It was proved by G. Frobenius in 1901 that the Frobenius kernel K is a normal subgroup of G [5]. The proof by Frobenius uses the character theory of finite groups. But since 1901, many attempts have been made to prove the normality of K without using character theory. Of course, K contains the unit element 1 and is a normal subset of G , but the difficulty lies in proving that K is closed under multiplication. A Fourier-analytic proof is given in [15].

2. Character Theory

Furthere proofs of the normality of K in G can be found in references such as [3], [4], [6], [8], [9], and [10], where character theory is utilized.

Theorem 2.1. *If G is a Frobenius group with complement H and kernel K, then K is a normal subgroup of G.*

Proof. This proof is a modification of the proof in [10]. Define the function $\psi : G \longrightarrow \mathbb{C}$ by

$$\psi(g) = \begin{cases} |H|, & \text{if } g \in K; \\ 0, & \text{otherwise.} \end{cases}$$

Since K is a normal subset of G, ψ is a class function on G. We will prove ψ is a character of G with $ker\psi = K$, thus proving $K \trianglelefteq G$.

Let $\chi \in Irr(G)$. We will show that C_χ, with the following definition, is a non-negative integer,

$$C_\chi = \langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{x \in G} \chi(x)\psi(x) = \frac{1}{|G|} \sum_{x \in K} \chi(x)|H| = \frac{1}{n} \sum_{x \in K} \chi(x).$$

If $\chi = 1_G$ the trivial character of G, then $\langle \chi, \psi \rangle = 1$. So assume $\chi \neq 1_G$. Since $G - K = \bigcup_{g \in G} (H-1)^g$ is a disjoint union of n conjugates of $H - 1$, $\langle \chi, 1_G \rangle$ is zero, and we have

$$\begin{aligned} 0 = \langle \chi, 1_G \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \\ &= \frac{1}{|H|} \sum_{h \in H-1} \chi(h) + \frac{1}{n|H|} \sum_{x \in K} \chi(x) \\ &= \langle \chi_H, 1_H \rangle - \frac{\chi(1)}{|H|} + \frac{C_\chi}{|H|}. \end{aligned}$$

We conclude $C_\chi = \chi(1) - |H| \langle \chi_H, 1_H \rangle$, so $|H|$ divides $C_\chi - \chi(1)$. Therefore, ψ equals a linear combination of irreducible characters of G with integer coefficients.

Now we have

$$1 = \langle \chi, \chi \rangle = \frac{1}{|H|} \sum_{h \in H-1} |\chi(h)|^2 + \frac{1}{|G|} \sum_{x \in K} |\chi(x)|^2,$$

where

$$\frac{1}{|H|} \sum_{h \in H-1} |\chi(h)|^2 = \langle \chi_H, 1_H \rangle - \frac{\chi(1)^2}{|H|}$$

is a non-negative rational number. By Cauchy-Schwartz inequality,

$$\sum_{x \in K} |\chi(x)|^2 \geq \frac{1}{n} (\sum_{x \in K} |\chi(x)|)^2$$

with equality if and only if $|\chi(x)| = \chi(1)$ for all x. Moreover,

$$\frac{1}{n} \sum_{x \in K} |\chi(x)|^2 \geq C_\chi^2$$

with equality if and only if $\chi(x)$ is a real number with the same sign for all x. Therefore,

$$1 \geq (\langle \chi_H, \chi_H \rangle - \frac{\chi(1)^2}{|H|}) + \frac{C_\chi^2}{|H|} \qquad (*)$$

But

$$C_\chi^2 - \chi(1)^2 = (C_\chi - \chi(1))(C_\chi + \chi(1))$$

is divisible by $|H|$. Therefore, the right-hand side of $(*)$ is a positive integer, and consequently, we must have equality. Thus $C_\chi = \frac{1}{n} \sum_{x \in K} |\chi(x)| = \chi(1)$ is the degree of χ, proving that $ker\psi = K \trianglelefteq G$. □

In [2], we obtained some character-theoretic properties of the finite Frobenius groups as follows. Recall that G is a Frobenius group with complement H and kernel K, $G = KH$, $K \cap H = 1$, where K is a normal subgroup of G, $n = [G : H]$. G acts on the set of cosets of H as a transitive permutation group of degree n, and the number of orbits of H on this set is called the rank of G and is denoted by s.

Proposition 2.2. *Let χ be the permutation character of H acting on K by conjugation. Then $\chi = s\rho_H + 1_H$, where ρ_H and 1_H are the regular and the identity characters of H, respectively.*

Proposition 2.3. *Let G be a Frobenius group with kernel K as a subset. If all elements of K commute, then K is a normal subgroup of G.*

Proof. K is a normal subset of G with identity, and G acts on it by conjugation. Let η be the permutation character associated with this action. For $g \in G$, $\eta(g)$ is the number of $k \in K$ such that $k^g = g^{-1}kg = k$. We have $\eta(1) = |K|$, and if $g \neq 1$ and $k^g = k$, then $k \in C_G(g) \cap K = C_K(g)$. Since $G = K \underset{g\in G}{\cup} H^g$, we distinguish the following cases.

Case 1. If $g \neq 1$ and $g \in K$, then $k^g = k$ implies $k \in C_K(g)$, so we conclude $\eta(g) = |C_K(g)|$.

Case 2. If $g \neq 1$ and $g \in \bigcup_{g\in G} H^g$, then g belongs to some conjugate of H.

So, without loss of generality, we can take $g \in H$. Then $k^g = k$ implies $g = g^k \in H \cap H^k$, which implies $k \in H \cap K = 1$. Then $\eta(g) = 1$. Therefore,

$$\eta(g) = \begin{cases} |K|, & \text{if } g = 1, \\ 1, & \text{if } 1 \neq g \in \{H^x \mid x \in G\}, \\ |C_K(g)|, & \text{if } 1 \neq g \in K. \end{cases}$$

By assumption, all elements of K commute, hence

$$\eta(g) = \begin{cases} |K|, & \text{if } g \in K, \\ 1, & \text{if } 1 \neq g \in \{H^x \mid x \in G\}. \end{cases}$$

Now we see that $ker\eta = K \trianglelefteq G$. □

3. Group Theory

As mentioned earlier, there is no group-theoretic proof establishing the Frobenius kernel as a subgroup in general. However, in some special cases, there exists a proof, which we will present here. If G is a Frobenius group with complement H and kernel K, then $N_G(H) = H$ and $|K| = [G : H]$.

Lemma 3.1. *If N is a normal subgroup of G such that $G = NH$ and $N \cap H = 1$, then $N \leq K$.*

Lemma 3.2 (Burnside)**.** *If G is a finite group and P is a Sylow p-subgroup of G such that $N_G(P) = C_G(P)$, then P has a normal complement in G, in other words, there exists $N \trianglelefteq G$ such that $G = NP$ and $N \cap P = 1$.*

Proof. See [13] page 137, section 6.2, proposition 6.2.9. □

Theorem 3.3. *Let G be a Frobenius group with complement H and kernel K. Assume that H is an abelian p-group. Then K is a normal subgroup of G.*

Proof. From the fact that $N_G(H) = H$ and the fact that H is abelian, we obtain

$$H = N_G(H) \geq C_G(H) \geq H.$$

Therefore, $N_G(H) = C_G(H)$. But $(|H| : [G : H]) = 1$, from which it follows that H is a Sylow p-subgroup of G. Now by Burnside's theorem (Lemma 3.2), H has a normal complement N in G, in other words, $G = NH$, $N \cap H = 1$, and $N \trianglelefteq G$. It follows that $N = K$. □

Corollary 3.4. *Let G be a finite Frobenius group with complement H and kernel K. Suppose H is centralized by a Sylow p-subgroup of G. Then $K \trianglelefteq G$.*

Proof. By assumption, $H \leq C_G(P)$, where P is a Sylow p-subgroup of G. If $1 \neq x \in H$, then $C_G(x) \leq H$. Therefore $C_G(P) \leq H$, then

$$C_G(P) = H = N_G(P).$$

Now, by Burnside's theorem, there exists a normal subgroup $N \trianglelefteq G$ such that $G = NP$. Therefore, $|N| = [G : P]$.

By Lemma 3.1, if $N \leq K$, then $|N| = [G : P] \leq |K| = [G : H]$. Therefore, $|P| \geq |H|$. This implies $P = H$. By Theorem 3.3, $K \leq G$. □

Another group-theoretical proof under different conditions exists, which we mention below. If $2 \mid |H|$, there is an elementary proof that $K \leq G$ due to Bender [12]. The fact is also proved in [1] page 172.

Theorem 3.5. *Let G be a Frobenius group with a complement H and kernek K. If H has even order, then K is a normal subgroup of G.*

Proof. Let t be an element of order 2 in H and $g \in G \backslash H$. Then either

$$a = t \cdot g^{-1} t g = t t^g = [t, g]$$

is in K, or there exists $x \in G$ such that $1 \neq a \in H^x$. If $a \in H^x$, then $a \in H^x \cap H^{xt} \cap H^{xt^g}$. Since $a^t = a^{-1} = a^{tg}$, we have $H^x = H^{xt} = H^{xt^g}$, where $t, t^g \in H^x$. If $H^x = H$ contradicts that $t \in H$ and $t^g \notin H$. Therefore, $tt^g \in K$ if $g \in G \backslash H$. Let $\{g_1, \cdots, g_n\}$ be a transversal of H in G, where $n = [G : H]$. We have

$$tt^{g_i} = tt^{g_j} \iff t^{g_i} = t^{g_j} \iff t^{g_i g_j^{-1}} = t \iff g_i g_j^{-1} \in H.$$

The elements $tt^{g_1}, \cdots, tt^{g_n}$ are pairwise distinct, so $K = \{t^{g_1}t, \cdots, t^{g_n}t\}$. Now we show that $K \leq G$. For $t^{g_i t}$, there exists g_s such that $t^{g_i t} = tt^{g_s}$. Therefore,

$$(tt^{g_i})(tt^{g_j}) = t(t^{g_i}t)t^{g_j} = t(tt^{g_s})t^{g_j} = t^{g_s}t^{g_j} = (tt^{g_i g_s^{-1}})g_s \in K^{g_s} = K.$$

$tt^g \in K$ for $g \in G \backslash H$. □

If a Frobenius complement H is solvable, then K is a subgroup of G.

Theorem 3.6. *[14] If G is a Frobenius group with a solvable complement, then the Frobenius kernel is a normal subgroup of G.*

In [7] related results are obtained.

H acts on $K - \{1\}$ by conjugation without a fixed point, creating orbits of size $|H|$. Therefore, $|H|$ divides $|K| - 1$, implying $(|H|, |K|) = 1$. If $K \leq G$, then K is a Hall-subgroup of G. Also, H is a Hall subgroup of G.

Looking at the Frobenius group G as a transitive permutation group on the set Ω with $|\Omega| = n$, $H = G_\alpha$ for some $\alpha \in \Omega$. Then $|\Omega| = [G : H]$. The number of orbits of H on Ω is called the rank of G, denoted by $r = rank(G)$. Each nontrivial H-orbit has size $|H|$, and there are $s = \frac{n-1}{|H|}$ such orbits. Therefore,

$$rank(G) = 1 + s = 1 + \frac{n-1}{|H|}.$$

If $r \leq 3$, then $K \leq G$ by using elementary group theory [11]. The proof utilizes the fact that H is a Hall subgroup of G, and for every prime p dividing $|K| = n = [G : H]$, the Sylow p-subgroups of G are contained in K. Thus, for small rank, a consequence of the Sylow theorem implies $K \leq G$. In particular, if $[G : H]$ is a prime power n, then K is a Sylow subgroup of G.

4. Properties of the Frobenius Kernel

Suppose G is a Frobenius group with complement H and kernel K. Assume $K \trianglelefteq G$, $G = HK$, and G has a unique kernel. If K is solvable, then H is nilpotent. Thompson showed that K is always nilpotent. Any subgroup of H of order p^2 or pq, where p and q are distinct primes, is cyclic. If $P \in Syl_p(H)$, where $p \neq 2$, then P is cyclic. If $p = 2$, then P is cyclic or generalized quaternion. K has an automorphism without fixed points. If $|H|$ is even, then K is abelian.

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