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GAUGE EQUIVALENCE PROBLEM ON DIFFERENTIAL OPERATORS UNDER FIBER-PRESERVING TRANSFORMATION

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Abstract. This paper focuses on investigating the equivalence problem for fifth-order differential operators (FODOs) on the line under general fiber-preserving transformations. Utilizing the Cartan method of equivalence, the study specifically addresses the gauge equivalence problem, seeking to establish the conditions for two FODOs to be related by a fiber-preserving transformation. By analyzing the properties of these operators, the research aims to identify conditions for their transformation while maintaining the fiber structure. The systematic approach of the Cartan method is employed to derive the necessary conditions for gauge equivalence between these FODOs. The study aims to enhance understanding of the equivalence problem for FODOs and shed light on fiber-preserving transformations that uphold gauge equivalence.

Keywords: Differential operators, Gauge equivalence, Absorption, Normalization.
2020 MSC: Primary 58A15, 58A20.

1. Introduction

The history of the gauge equivalence problem dates back to the early 20th century and has its roots in the development of gauge theory in physics and differential geometry. The concept of gauge invariance was first introduced in the context of electromagnetism by Hermann Weyl in 1918 and later extended to other fields of physics, such as quantum field theory. In the 1950s, the mathematician Elie Cartan made significant contributions to the theory of differential forms and exterior calculus, which provided a powerful framework for studying geometric structures and their equivalence under various transformations, [4–6]. This laid the foundation for the modern formulation of the gauge equivalence problem, [3, 7, 15, 16].

The main goal of the Cartan equivalence problem is to determine when two different geometric structures, defined on the same underlying space, are

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equivalent or isomorphic. These structures may include objects such as differential equations, differential forms, or geometric connections. Cartan's approach to solving the equivalence problem involves the use of differential forms and exterior calculus to encode the geometric properties of the structures in a coordinate-independent manner. By analyzing the differential forms associated with the structures, one can study the invariant that are preserved under various transformations and gauge choices.

The Cartan equivalence theory has applications in various areas of mathematics and physics, including differential geometry, gauge theory, and the theory of partial differential equations. It provides a powerful framework for understanding the relationships between different geometric structures and for classifying them based on their underlying symmetries and invariances, [10–12].

The equivalence problem for FODOs revolves around establishing the conditions that allow two operators to be transformed into each other through a suitable change of variables. This complex issue has been the subject of extensive research, leading to the development of various methodologies aimed at resolving it effectively, [1, 2, 8, 13].

In the realm of equivalence problems, it is possible to associate a set of one-forms with an object being studied in its original coordinate system. Upon undergoing a transformation into new coordinates, the object will exhibit a distinct set of one-forms. By framing the equivalence problem in the Cartan format, which entails utilizing a coframe ω on the manifold M along with a structured group G that is a subgroup of $\mathrm{GL}(m)$, the Cartan equivalence method can be brought into play. The primary objective here is to standardize the coefficients of the structure group in a manner that preserves their invariance, a feat accomplished by identifying a satisfactory array of invariant combinations of these coefficients, as elucidated in [14].

The classification of linear differential equations represents a specific instance within the broader scope of categorizing differential operators. In the [1], it was shown that a diverse array of equivalence problems can be fully redefined as a set of equations that incorporate one or more differential one-forms, and we specifically addressed the direct equivalence problem for an FODO.

The gauge equivalence problem became a central topic in theoretical physics, particularly in the study of gauge theories such as Yang-Mills theory and general relativity, [9]. The idea of gauge transformations, which are local symmetries that do not change the physical observable, played a crucial role in understanding the underlying structure of fundamental interactions in particle physics.

Over the years, mathematicians and physicists have developed sophisticated mathematical tools and techniques to address the gauge equivalence problem in different contexts, leading to important breakthroughs in theoretical physics and differential geometry. The study of gauge theories and their equivalence continues to be an active area of research, with implications for our understanding of the fundamental forces of nature and the geometry of spacetime.

2. A Brief Review of Cartan Equivalence

When examining equivalence problems as a procedural approach, we initially delve into the structure equations, normalization process, and absorption techniques. Detailed insights into this subject are available in the authoritative source referenced as [14]. In this section, the Einstein notation is used for summation.

In the G-valued equivalence problem for coframes on m-dimensional manifolds, where G is a subgroup of $\mathrm{GL}(m)$ is a Lie group and $\omega = \{\omega^i\}_{i=1}^m$ and $\overline{\omega} = \{\overline{\omega}^i\}_{i=1}^m$ represent the coframes on the manifolds M and $\overline{M}$, respectively, the objective is to determine the existence of a local diffeomorphism Φ from M to $\overline{M}$ and a function g from M to G satisfying the equation

$$\Phi^*(\overline{\omega}) = g(x)\omega, \tag{1}$$

for all $x \in M$, and the functions $g(x) = \big(g^i_j(x)\big)$, where $g^i_j(x)$ are elements of the structure group G. Given the group property of G, the G-valued equivalence problem can be solved if and only if there exist a pair of G-valued functions $\overline{g}(\overline{x})$ and $g(x)$ such that, for clarity, omitting the pull-back operation, the equation:

$$\overline{g}(\overline{x})\overline{\omega} = g(x)\omega \tag{2}$$

holds. Our objective is to transform a given G-equivalence problem into a standard equivalence problem for coframes. To facilitate this reduction, we introduce new coframes defined by

$$\begin{aligned} \theta^i &= g^i_j(x)\omega^j, \\ \overline{\theta}^i &= \overline{g}^i_j(\overline{x})\overline{\omega}^j, \end{aligned} \tag{3}$$

which satisfies the invariance property $\Phi^*(\overline{\theta}^i)$ equals to θ^i. During the initial step of the Cartan method, after calculating the differentials of θ^i and utilizing (3), we can express them resulting in the differential expression

$$d\theta^i = \gamma^i_j(x) \wedge \theta^j + T^i_{jk}(x, g)\theta^j \wedge \theta^k. \tag{4}$$

The coefficients T^i_{jk}, known as torsion coefficients, can either be constant or dependent on x and g. While some torsion coefficients may exhibit invariance, they are generally not invariant for the given problem at hand.

The γ^i_j terms in (4) represent the 1-forms given by

$$\gamma^i_j = dg^i_k(x)(g^{-1})^k_j(x). \tag{5}$$

These γ^i_j components play a crucial role in the differential expression, offering insights into the structure of the lifted coframe elements through the composition involving the differential of g and its inverse.

Let us consider the set $\{\alpha^i\}_{i=1}^r$ as a basis for the space of Maurer-Cartan forms. In this context, each γ^i_j can be expressed as a linear combination of the Maurer-Cartan basis as follows:

$$\gamma^i_j = A^i_{jl}\alpha^l. \tag{6}$$

Consequently, the ultimate structure equations governing our lifted coframe, in terms of the Maurer-Cartan forms, exhibit a general form given by

$$d\theta^i = A^i_{jl}\alpha^l \wedge \theta^j + T^i_{jk}(x,g)\theta^j \wedge \theta^k. \tag{7}$$

To simplify the Maurer-Cartan forms α^l back to the base manifold M, we can express them as arbitrary linear combinations of coframe elements, given by

$$\alpha^l \mapsto z^l_j\theta^j, \tag{8}$$

where the coefficients z^l_j are currently undetermined and are functions that explicitly rely on the base variables x. Upon replacing the expression from (8) into the structure equations given by (7), we can derive a set of 2-forms represented as:

$$\Theta^i = \{B^i_{jk}[\mathbf{z}] + T^i_{jk}(x,g)\}\theta^j \wedge \theta^k, \tag{9}$$

here, the coefficients $B^i_{jk}[\mathbf{z}]$ can be expressed as linear functions of the coefficients $\mathbf{z} = (z^l_k)$, given by:

$$B^i_{jk}[\mathbf{z}] = (A^i_{kl}z^l_j - A^i_{jl}z^l_k). \tag{10}$$

The values of these coefficients are dictated by the particular representation of the structure group $G \subset \mathrm{GL}(m)$, ensuring their constancy and independence from the coordinate system selected.

The procedure of finding the unknown coefficients $\mathbf{z}$ from the complete torsion coefficients is commonly referred to as the absorption of torsion, alongside the subsequent step known as the normalization of the resulting invariant torsion coefficients, as outlined earlier. Substituting α^l with the adjusted 1-form

$$\pi^l = \alpha^l - z^l_i\theta^i, \tag{11}$$

leads to absorb the inessential torsion in the equation (7). Here the $z^l_i = z^l_i(x,g)$ are the solutions to the absorption equations and consequently, we can deduce

$$d\theta^i = A^i_{jl}\pi^l \wedge \theta^j + U^i_{jk}\theta^j \wedge \theta^k, \quad i,j = 1,\dots,m, \tag{12}$$

where U^i_{jk} exclusively involve essential torsion.

3. The Algorithm of Gauge Equivalence

Let's examine the FODO acting on a scalar function $u(x)$ given by

$$\mathcal{D}[u] = \sum_{i=0}^{5} f_i(x)\, D^i u \tag{13}$$

and a separate FODO acting on a real-valued function $\bar{u}(\bar{x})$

$$\bar{\mathcal{D}}[\bar{u}] = \sum_{i=0}^{5} \bar{f}_i(\bar{x})\, \bar{D}^i \bar{u}. \tag{14}$$

where f_i and $\bar{f}_i$ for $i = 1,2,3,4,5$ denote analytic functions of the real variables x and $\bar{x}$, respectively, where $f_5 = \bar{f}_5 = 1$ for simplicity. Moreover, D^i and $\bar{D}^i$

are the i-th derivative with respect to x^i and $\bar{x}^i$ respectively, and $D^0 = \bar{D}^0 = \mathrm{Id}$ represent the identity operators.

Considering the fiber-preserving transformations given by

$$(\bar{x}, \bar{u}) = (\xi(x), \varphi(x)\, u), \tag{15}$$

where $\varphi(x) \neq 0$, we can use the chain rule formula to establish the relationship between the total derivative operators as

$$\bar{D}_{\bar{x}} = \frac{d}{d\bar{x}} = \frac{1}{\xi'(x)} \frac{d}{dx} = \frac{1}{\xi'(x)} D_x. \tag{16}$$

In [1], we addressed the direct equivalence problem by considering $\mathcal{D}[u] = \bar{\mathcal{D}}[\bar{u}]$ under the change of variables given by (15). The direct equivalence leads to the transformation rule

$$\bar{\mathcal{D}} = \mathcal{D} \cdot \frac{1}{\varphi(x)}, \tag{17}$$

which applies directly to the differential operators themselves. Resolving the local direct equivalence problem involves establishing specific conditions on the coefficients of the two differential operators to ensure their equivalence when subjected to a change of variables in the form of (15). However, the transformation rule (17) does not maintain the eigenvalue problem $\mathcal{D}[u] = \lambda u$ or the Schrodinger equation $iu_t = \mathcal{D}[u]$ since it lacks a $\varphi(x)$ factor. To tackle this limitation, we introduce the notion of gauge equivalence and consider the subsequent transformation rule:

$$\bar{\mathcal{D}} = \varphi(x) \cdot \mathcal{D} \cdot \frac{1}{\varphi(x)} \qquad \text{when} \qquad \bar{x} = \xi(x). \tag{18}$$

Theorem 3.1. *Assume $\mathcal{D}$ and $\bar{\mathcal{D}}$ represent two FODOs. Consider two coframes given by $\Omega = \{\omega^i\}_{i=1}^7$ and $\bar{\Omega} = \{\bar{\omega}^j\}_{j=1}^7$ defined on open subsets of the corresponding fifth jet spaces. These coframes are selected such that the differential operators are equivalent under the pseudogroup (15) based on the respective transformation rule (18). The relationship between the coframes Ω and $\bar{\Omega}$ can be represented as $\bar{\Omega} = G\, \Omega$, where G is the following matrix:*

$$G = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 0 & 0 & 0 \\ 0 & a_4 & a_5 & a_6 & 0 & 0 & 0 \\ 0 & a_7 & a_8 & a_9 & a_{10} & 0 & 0 \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

with a_i being real numbers for i that $1 \leq i \leq 15$, and ensuring that $a_1 a_3 a_6 a_{10} a_{15}$ is different from zero.

Proof: The necessary and sufficient condition for the point transformation to be in the desired linear form (15) is that the following one-form equations hold on the subset of J^5 where $u \neq 0$, for the functions $\zeta_1 = \xi_x$ and $\zeta_2 = \varphi_x/\varphi$:

$$\begin{aligned} d\bar{x} &= \zeta_1\, dx, \\ \frac{d\bar{u}}{\bar{u}} &= \frac{du}{u} + \zeta_2\, dx. \end{aligned} \tag{20}$$

For a diffeomorphism Φ between jet space J^5 to constitute a contact transformation, it must satisfy the condition:

$$\begin{pmatrix} d\bar{u} - \bar{p}\, d\bar{x} \\ 1 \\ d\bar{p} - \bar{q}\, d\bar{x} \\ d\bar{q} - \bar{r}\, d\bar{x} \\ d\bar{r} - \bar{s}\, d\bar{x} \\ d\bar{s} - \bar{t}\, d\bar{x} \\ 1 \end{pmatrix} = G \,.\, \begin{pmatrix} du - p\, dx \\ 1 \\ dp - q\, dx \\ dq - r\, dx \\ dr - s\, dx \\ ds - t\, dx \\ 1 \end{pmatrix}, \tag{21}$$

where G corresponds to the matrix defined in (19). The simultaneous fulfillment of the initial contact condition (21) alongside the linearity conditions (20) constitutes a segment of an over-determined equivalence problem. Upon substituting $\zeta_2 = -p/u$ and $a_1 = 1/u$ into (20), the resulting 1-form is as follows:

$$\frac{d\bar{u} - \bar{p}\, d\bar{x}}{\bar{u}} = \frac{du - p\, dx}{u}, \tag{22}$$

which is invariant, and (22) can serve as a replacement for both (20). Hence, we can choose the following six elements from our coframe as the 1-forms:

$$\begin{aligned} \omega^1 = dx,\ \omega^2 = \frac{du - p\, dx}{u},\ \omega^3 = dp - q\, dx,\ \omega^4 = dq - r\, dx, \\ \omega^5 = dr - s\, dx,\ \omega^6 = ds - t\, dx. \end{aligned} \tag{23}$$

This leads to the following relations:

$$\begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \\ \bar{\omega}^3 \\ \bar{\omega}^4 \\ \bar{\omega}^5 \\ \bar{\omega}^6 \\ 1 \end{pmatrix} = G \,.\, \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ 1 \end{pmatrix}. \tag{24}$$

In the context of the problem (18), considering the additional factor of φ, the invariant becomes:

$$I = \frac{\mathcal{D}[u]}{u}. \tag{25}$$

This leads to the determination of:

(26)

$$\omega^7 = dI = \frac{1}{u}\,dt + \frac{f_4}{u}\,ds + \frac{f_3}{u}\,dr + \frac{f_2}{u}\,dq + \frac{f_1}{u}\,dp - \frac{t + f_4 s + f_3 r + f_2 q + f_1 p}{u^2}\,du + \left\{\frac{f_4' s + f_3' r + f_2' q + f_1' p}{u} + f_0'\right\} dx,$$

serves as the final component of the coframe for the equivalence problem (18). The set of one-forms $\Omega = \{\omega^i\}_{i=1}^7$ forms with $u \neq 0$ and $f_5(x) \neq 0$. This condition ensures that the final coframe elements are in agreement up to contact

$$\bar{\omega}^7 = \omega^7. \tag{27}$$

By examining the relations (24) and (27), the structure group linked to the equivalence problems (17) and (18) can be identified as a group of matrices G. This group satisfies $\bar{\Omega} = G\Omega$, resulting in (19). Subsequently, the lifted coframe on the space $\mathrm{J}^5 \times G$ is given by:

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \\ \theta^6 \\ \theta^7 \end{pmatrix} = G \,.\, \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{pmatrix}. \tag{28}$$

□

4. The Final Structure Equations

Theorem 4.1. *In gauge equivalence using the coframes (23) and (27), the ultimate structural equations are:*

$$\begin{pmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \\ d\theta^4 \\ d\theta^5 \\ d\theta^6 \\ d\theta^7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_1\theta^1 & \theta^1 & 0 & 0 \\ 0 & 0 & 0 & I_2\theta^1 + 5\theta^3 & 0 & \theta^1 & 0 \\ 0 & 0 & I_3\theta^1 & I_4\theta^1 + I_5\theta^3 & 5\theta^3 & 0 & \theta^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \\ \theta^4 \\ \theta^5 \\ \theta^6 \\ \theta^7 \end{pmatrix} \tag{29}$$

here the functions $I_1, \dots, I_5$ are represented as:

$$
\begin{aligned}
I_1 &= -\frac{1}{u}\left[f_4 u + 5p\right],\\
I_2 &= \frac{1}{u^2}\left[2\dot{f}_4 u^2 - f_3 u^2 - 4 f_4 p u - 10 p^2\right],\\
I_3 &= -\frac{1}{u}\left[2 p f_2 + 3 f_3 q + 4 f_4 r + f_1 u + 5 s\right],\\
I_4 &= \frac{1}{u^3}\Big[4\dot{f}_4 p u^2 - f_2 u^3 + \dot{f}_3 u^3 - 3 f_3 p u^2 - 4 f_4 p^2 u - 2u^2 f_4 q\\
&\qquad - \ddot{f}_4 u^3 - 10 p^3 + 5 p q u - 5 r u^2\Big],\\
I_5 &= -\frac{1}{u}(f_4 u + 5p).
\end{aligned}
\tag{30}
$$

Proof. During the initial iteration of the equivalence problem procedure, as outlined in Proposition 3.1, the structure group G specified in $\bar{\Omega} = G\,\Omega$ where G is (19) precisely aligns with the following structure equations:

$$
\begin{aligned}
d\theta^1 &= \alpha^1 \wedge \theta^1,\\
d\theta^2 &= T^2_{12}\theta^1 \wedge \theta^2 + T^2_{13}\theta^1 \wedge \theta^3,\\
d\theta^3 &= \alpha^2 \wedge \theta^2 + \alpha^3 \wedge \theta^3 + T^3_{12}\theta^1 \wedge \theta^2 + T^3_{13}\theta^1 \wedge \theta^3 + T^3_{14}\theta^1 \wedge \theta^4,\\
d\theta^4 &= \alpha^4 \wedge \theta^2 + \alpha^5 \wedge \theta^3 + \alpha^6 \wedge \theta^4 + T^4_{12}\theta^1 \wedge \theta^2 + T^4_{13}\theta^1 \wedge \theta^3 + T^4_{14}\theta^1 \wedge \theta^4\\
&\quad + T^4_{15}\theta^1 \wedge \theta^5,\\
d\theta^5 &= \alpha^7 \wedge \theta^2 + \alpha^8 \wedge \theta^3 + \alpha^9 \wedge \theta^4 + \alpha^{10} \wedge \theta^5 + T^5_{12}\theta^1 \wedge \theta^2 + T^5_{13}\theta^1 \wedge \theta^3\\
&\quad + T^5_{14}\theta^1 \wedge \theta^4 + T^5_{15}\theta^1 \wedge \theta^5 + T^5_{16}\theta^1 \wedge \theta^6,\\
d\theta^6 &= \alpha^{11} \wedge \theta^2 + \alpha^{12} \wedge \theta^3 + \alpha^{13} \wedge \theta^4 + \alpha^{14} \wedge \theta^5 + \alpha^{15} \wedge \theta^6 + T^6_{12}\theta^1 \wedge \theta^2\\
&\quad + T^6_{13}\theta^1 \wedge \theta^3 + T^6_{14}\theta^1 \wedge \theta^4 + T^6_{15}\theta^1 \wedge \theta^5 + T^6_{16}\theta^1 \wedge \theta^6 + T^6_{17}\theta^1 \wedge \theta^7,\\
d\theta^7 &= 0,
\end{aligned}
$$

The set of $\{\alpha^i\}_{i=1}^{15}$ serves as a basis for the right-invariant Maurer-Cartan forms on the Lie group G. The essential torsion coefficients within the initial loop can be represented as:

$$
\begin{gathered}
T^2_{12} = -\frac{a_2 + a_3 p}{a_1 a_3 u}, \quad T^2_{13} = \frac{1}{a_1 a_3 u}, \quad T^3_{14} = \frac{a_3}{a_1 a_6}, \quad T^4_{15} = \frac{a_6}{a_1 a_{10}},\\
T^5_{16} = \frac{a_{10}}{a_1 a_{15}}, \quad T^6_{17} = \frac{a_{15} u}{a_1}.
\end{gathered}
\tag{31}
$$

Normalization can be achieved by setting

$$
a_1 = 1, \quad a_2 = -\frac{p}{u}, \quad a_3 = a_6 = a_{10} = a_{15} = \frac{1}{u}. \tag{32}
$$

In the second iteration of the equivalence problem, we integrate the normalization condition (32) into the lifted coframe expression (28). Following this, we compute the differentials of the revised invariant coframe to deduce the updated structural equations:

$$
\begin{aligned}
d\theta^1 &= 0\\
d\theta^2 &= \theta^1\wedge\theta^3\\
d\theta^3 &= \theta^1\wedge\theta^4\\
d\theta^4 &= T^4_{12}\theta^1\wedge\theta^2+T^4_{13}\theta^1\wedge\theta^3+T^4_{14}\theta^1\wedge\theta^4+\theta^1\wedge\theta^5\\
d\theta^5 &= T^5_{12}\theta^1\wedge\theta^2+T^5_{13}\theta^1\wedge\theta^3+T^5_{14}\theta^1\wedge\theta^4+\theta^2\wedge\theta^3\\
&\quad+2\theta^3\wedge\theta^4-\theta^2\wedge\theta^5+\theta^1\wedge\theta^6+\alpha^7\wedge\theta^2+\alpha^8\wedge\theta^3\\
d\theta^6 &= T^6_{12}\theta^1\wedge\theta^2+T^6_{13}\theta^1\wedge\theta^3+T^6_{23}\theta^2\wedge\theta^3+T^6_{14}\theta^1\wedge\theta^4\\
&\quad+T^6_{34}\theta^3\wedge\theta^4+T^6_{15}\theta^1\wedge\theta^5+T^6_{16}\theta^1\wedge\theta^6+\theta^3\wedge\theta^5\\
&\quad+\theta^2\wedge\theta^4-\theta^2\wedge\theta^6+\theta^1\wedge\theta^7+\alpha^{11}\wedge\theta^2+\alpha^{12}\wedge\theta^3+\alpha^{13}\wedge\theta^4\\
d\theta^7 &= 0
\end{aligned}
\tag{33}
$$

Here, α^i for $i=7,8,11,12,13$ denote the Maurer-Cartan form, and the fundamental torsion components of the structural equations are provided by:

$$
\begin{gathered}
T^3_{12}=-\frac{a_4u+q}{u},\quad T^3_{13}=-\frac{a_5u+2p}{u},\quad T^5_{15}=-\frac{a_{14}u-a_9u+p}{u},\\
T^6_{16}=\frac{a_{14}u-f_4u-p}{u}.
\end{gathered}
\tag{34}
$$

Thus, the normalization can be expressed as:

$$
a_4=-\frac{q}{u},\quad a_5=-\frac{2p}{u},\quad a_9=\frac{f_4u+2p}{u},\quad a_{14}=\frac{f_4u+p}{u}.
\tag{35}
$$

Substituting equation (35) into equation (28) and subsequently recalculating the differentials of the updated 1-forms results in:

$$
\begin{aligned}
d\theta^1 &= 0\\
d\theta^2 &= \theta^1\wedge\theta^3\\
d\theta^3 &= \theta^1\wedge\theta^4\\
d\theta^4 &= T^4_{12}\theta^1\wedge\theta^2+T^4_{13}\theta^1\wedge\theta^3+T^4_{14}\theta^1\wedge\theta^4+\theta^1\wedge\theta^5\\
d\theta^5 &= T^5_{12}\theta^1\wedge\theta^2+T^5_{13}\theta^1\wedge\theta^3+T^5_{14}\theta^1\wedge\theta^4+\theta^2\wedge\theta^3+2\theta^3\wedge\theta^4\\
&\quad-\theta^2\wedge\theta^5+\theta^1\wedge\theta^6+\alpha^7\wedge\theta^2+\alpha^8\wedge\theta^3\\
d\theta^6 &= T^6_{12}\theta^1\wedge\theta^2+T^6_{13}\theta^1\wedge\theta^3+T^6_{23}\theta^2\wedge\theta^3+T^6_{14}\theta^1\wedge\theta^4+T^6_{34}\theta^3\wedge\theta^4\\
&\quad+\theta^2\wedge\theta^4+T^6_{15}\theta^1\wedge\theta^5+\theta^3\wedge\theta^5+T^6_{16}\theta^1\wedge\theta^6-\theta^2\wedge\theta^6+\theta^1\wedge\theta^7\\
&\quad+\alpha^{11}\wedge\theta^2+\alpha^{12}\wedge\theta^3+\alpha^{13}\wedge\theta^4\\
d\theta^7 &= 0
\end{aligned}
$$

The essential torsion components include T^4_{12}, T^4_{13}, and T^6_{15}. This directly indicates the subsequent normalization:

$$
\begin{aligned}
a_7 &= -\frac{f_4qu + 2pq + ru}{u^2}, \quad a_8 = -\frac{2f_4pu + 4p^2 + 3qu}{u^2}, \\
a_{13} &= -\frac{\dot{f}_4u^2 - f_4pu - f_3u^2 - 2p^2 + qu}{u^2}.
\end{aligned} \tag{36}
$$

In the fourth loop, by inserting the normalization (36) into the raised coframe (28) and solving for the parameters a_7, a_8 and a_{13}, we update the differentials. Consequently, the revised structure equations are:

$$
\begin{aligned}
d\theta^1 &= 0 \\
d\theta^2 &= \theta^1 \wedge \theta^3 \\
d\theta^3 &= \theta^1 \wedge \theta^4 \\
d\theta^4 &= T^4_{14}\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^4 \\
d\theta^5 &= T^5_{12}\theta^1 \wedge \theta^2 + T^5_{13}\theta^1 \wedge \theta^3 + T^5_{14}\theta^1 \wedge \theta^4 + 5\theta^3 \wedge \theta^5 + \theta^1 \wedge \theta^6 \\
d\theta^6 &= T^6_{12}\theta^1 \wedge \theta^2 + T^6_{13}\theta^1 \wedge \theta^3 + T^6_{23}\theta^2 \wedge \theta^3 + T^6_{14}\theta^1 \wedge \theta^4 + T^6_{34}\theta^3 \wedge \theta^4 \\
&\quad + \theta^3 \wedge \theta^5 - \theta^2 \wedge \theta^6 + \theta^1 \wedge \theta^7 + \alpha^{11} \wedge \theta^2 + \alpha^{12} \wedge \theta^3 \\
d\theta^7 &= 0
\end{aligned} \tag{37}
$$

The essential torsion components consist of T^5_{12} and T^5_{13}. The corresponding parameters are determined as follows:

$$
a_{11} = -\frac{f_4ru + pr + su}{u^2}, \qquad a_{12} = -\frac{3f_4qu + 3pq + 4ru}{u^2}. \tag{38}
$$

Thus the final invariant coframe is now given by

$$
\begin{aligned}
\theta^1 &= dx, \\
\theta^2 &= \frac{du - p\,dx}{u}, \\
\theta^3 &= \frac{1}{u^2}\Big[(p^2 - qu)\,dx - p\,du + u\,dp\Big], \\
\theta^4 &= -\frac{1}{u^3}\Big[(2p^3 - 3pqu + ru^2)\,dx - (2p^2 - qu)\,du + 2pu\,dp - u^2\,dq\Big], \\
\theta^5 &= \frac{1}{u^4}\Big[(-u^3f_4r + 3f_4pqu^2 - 2f_4p^3u + 3p^2qu - 4p^4 + 3q^2u^2 \\
&\quad - pru^2 - su^3)\,dx + (2f_4p^2u - u^2f_4q + 4p^3 + pqu - ru^2)\,du \\
&\quad - (2f_4pu + 4p^2 + 3qu)u\,dp + (f_4u + 2p)u^2\,dq + u^3\,dr\Big],
\end{aligned}
$$

$$\theta^6 = -\frac{1}{u^5}\Big[(2f_3p^3u^2 - 3f_3pqu^3 + f_3u^4r - 2\dot{f}_4p^3u^2 + 3\dot{f}_4pqu^3 - \dot{f}_4ru^4 + 2f_4p^4u - 3f_4q^2u^3 + f_4su^4 + 4p^5 - 5p^3qu + 5p^2ru^2 - 5qru^3 + tu^4)\,dx + \Big(\frac{(f_4pu + f_3u^2 - \dot{f}_4u^2 + 2p^2 - qu)(2p^2 - qu)}{u^5} + \frac{3f_4pqu - f_4ru^2 + 3p^2q + 3pru - su^2}{u^4}\Big)\,du + \frac{2f_3pu^2 - 2\dot{f}_4pu^2 + 2f_4p^2u + 3f_4u^2q + 4p^3 + pqu + 4ru^2}{u^4}\,dp + \frac{f_4pu + f_3u^2 - \dot{f}_4u^2 + 2p^2 - qu}{u^3}\,dq + \frac{f_4u + p}{u^2}\,dr + \frac{1}{u}\,ds\Big],$$

$$\theta^7 = \frac{f_4's + f_3'r + f_2'q + f_1p + f_0'u}{u}\,dx - \frac{t + f_4s + f_3r + f_2q + f_1p}{u^2}\,du + \frac{f_1}{u}\,dp + \frac{f_2}{u}\,dq + \frac{f_3}{u}dr + \frac{f_4}{u}ds + \frac{dt}{u}.$$

Subsequently, the final structure equations (29) incorporating the fundamental invariant coefficients (30) are derived. □

4.1. **An Example.** Consider the fifth order differential operator (FODO)

$$\mathcal{D} = D_x^5 + f_4(x)\,D_x^4 + f_3(x)\,D_x^3 + f_2(x)\,D_x^2 + f_1(x)\,D_x + f_0(x). \tag{39}$$

Under the change of variables of following gauge transformation

$$\bar{x} = \xi(x) = x, \qquad \varphi(x) = \exp\left\{\int f_4(x)\,dx\right\}u \tag{40}$$

where $\varphi(x) = \exp\left\{\int f_4(x)\,dx\right\}$ is the gauge factor. The FODO (39) is transformed to

$$\bar{\mathcal{D}} = D_x^5 + p(x)D_x + f_0(x), \tag{41}$$

where $p(x) = f_0(x) - f_2(x)f_4(x) + f_3(x)f_4^2(x) - f_4^4(x)$, under the gauge transformation (40). To obtain the operator (41), we must take into account that $\bar{\mathcal{D}}$ under the transformation (40) is connected to $\mathcal{D}$ through the formula:

$$\bar{\mathcal{D}} = \exp\left\{\int f_4(x)dx\right\}\ \mathcal{D}\ \exp\left\{-\int f_4(x)dx\right\}. \tag{42}$$

Let's apply this formula to the given FODO (39) and using formula (42), we have

$$\bar{\mathcal{D}} = \exp\left\{\int f_4(x)dx\right\}\Big(D_x^5 + f_4(x)D_x^4 + f_3(x)D_x^3 + f_2(x)D_x^2 + f_1(x)D_x + f_0(x)\Big)\exp\left\{-\int f_4(x)dx\right\}.$$

Expanding this expression using the transformation rules, and substituting

$$\bar{D}_{\bar{x}} = \exp\left\{\int f_4(x)dx\right\}\ D_x\ \exp\left\{-\int f_4(x)dx\right\},$$

which simplifies the expression to (41).

5. Conclusion

This paper has delved into the investigation of the equivalence problem concerning fifth-order differential operators (FODOs) on the line within the realm of general fiber-preserving transformations. Through the application of the Cartan method of equivalence, the study has focused on tackling the gauge equivalence problem, aiming to delineate the criteria under which two FODOs can be linked through a fiber-preserving transformation.

By scrutinizing the characteristics and behaviors of these operators, the research has strived to unveil the prerequisites for their transformation while preserving the underlying fiber structure. Leveraging the systematic methodology of the Cartan method, the study has successfully derived the essential conditions for achieving gauge equivalence between FODOs.

In essence, this study endeavors to enrich the comprehension of the equivalence quandary surrounding FODOs and to illuminate the landscape of fiber-preserving transformations that maintain gauge equivalence. Through a rigorous analytical approach and a focus on the intricate relationships between operators and transformations, this research contributes to advancing the understanding of FODO equivalence and the significance of fiber-preserving transformations in this context.

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