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SUPERDERIVATIONS AND JORDAN SUPERDERIVATIONS OF GENERALIZED QUATERNION ALGEBRAS

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Abstract. Let $H_{\alpha,\beta}$ be the generalized quaternion algebra over a unitary commutative ring. This paper aims to consider superderivations and Jordan superderivations of $H_{\alpha,\beta}$ and hence to obtain the superalgebra $Der_s(H_{\alpha,\beta})$ of superderivations and $Der_{Js}(H_{\alpha,\beta})$ of Jordan superderivations of $H_{\alpha,\beta}$. It turns out that on generalized quaternion algebras, any superderivation is inner. In particular, there exist Jordan superderivations that are not superderivations.

Keywords: Superalgebra, superderivation, Jordan superderivation, generalized quaternion algebra.
2020 MSC: 16H05, 16W25,17A70, 17B60.

1. Introduction

Throughout the paper, $\mathcal{R}$ will denote a commutative ring with unity and all algebras and modules will be unital over $\mathcal{R}$. Let $\mathcal{A}$ be an algebra (not necessarily associative) and $Z(\mathcal{A})$ denote the center of $\mathcal{A}$. As usual, the Lie product is denoted by $[x, y] = xy - yx$, and the Jordan product by $x \circ y = xy + yx$ for all $x, y \in \mathcal{A}$. A linear map $d : \mathcal{A} \to \mathcal{A}$ is a *derivation* (resp. *Jordan derivation*) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x \circ y) = d(x) \circ y + x \circ d(y)$) for all $x, y \in \mathcal{A}$. Every derivation is obviously a Jordan derivation but the converse is not true in general (see example in [11]). In 1957 Herstein proved that every Jordan derivation of a prime ring of characteristic not 2 is a derivation ([8, Theorem 3.1]). Later on Brešar [1] extended the result to 2-torsion-free semiprime rings. The problem of whether every Jordan derivation of a ring or algebra into itself is a derivation was discussed by many mathematicians in different rings and algebras. For example, Zhang and Yu [12] proved that any Jordan derivation of a 2-torsion free triangular algebra is a derivation. Recently, Ghahramani et al. [7] have proved this result for a quaternion ring.

Over recent years there has been considerable interest in the study of superalgebra versions of Herstein's theorem on superderivations. One of the interesting problems in the theory of derivations is characterizing algebras on which every

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derivation is inner. Many studies have been performed in this regard (see [6, 9] and the references therein for more details). In [3] the authors have shown that any superderivation on a superalgebra $M_n(\mathbb{C})$ is an inner superderivation. Also, in [2] Cheraghpoor et al. investigated the same result on trivial extensions with some prescribed conditions. This paper considers the analogous problem of innerness of superderivations of generalized quaternion algebras as a class of Lie superalgebra. We first show that every superderivation of $\mathcal{H}_{\alpha,\beta}$ is an inner superderivation and then we obtain a typical superderivation in its matrix form. Moreover, according to these two research approaches and in continuation of Herstein's research program, we describe Jordan superderivations of generalized quaternion algebra. In the case of trivial superalgebras (i.e., the odd part is 0) Jordan superderivations coincide with Jordan derivations. Clearly, every superderivation is also a Jordan superderivation. In view of Herstein's theorem one might ask whether the converse is true. A Jordan superderivation which is not a superderivation will be called a *proper Jordan superderivation.* Here we see that although every Jordan derivation is a derivation in quaternion rings [7], there exist proper Jordan superderivations on nontrivial generalized quaternion superalgebras.

2. **Preliminaries**

Throughout the paper, by an algebra, we shall mean an algebra over a fixed unital commutative ring $\mathcal{R}$. We assume without further mention that $\frac{1}{2} \in \mathcal{R}$.

A superalgebra is a $\mathbb{Z}_2$-graded algebra. This means that there exist $\mathcal{R}$-submodules $\mathcal{A}_0$ and $\mathcal{A}_1$ of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ and $\mathcal{A}_\imath \mathcal{A}_\jmath \subseteq \mathcal{A}_{\imath+\jmath}$, where indices are computed modulo 2. A superalgebra is called trivial if $\mathcal{A}_1 = 0$. Elements in $\mathcal{A}_0 \cup \mathcal{A}_1$ are said to be *homogeneous* of *degree* $\imath$ and we write $|x| = \imath$ to mean $x \in \mathcal{A}_\imath$. We say that $\mathcal{A}_0$ is the *even* part of $\mathcal{A}$ and $\mathcal{A}_1$ is the *odd* part of $\mathcal{A}$. Define a product in $\mathcal{A}_0 \cup \mathcal{A}_1$, the *Lie superproduct*, by

$$[x, y]_s = xy - (-1)^{|x||y|} yx$$

for $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$. The *Jordan superproduct* is defined by

$$x \circ_s y = \frac{1}{2}(xy + (-1)^{|x||y|} yx).$$

Extend $[x, y]_s$ and $x \circ_s y$ by bilinearity to $\mathcal{A} \times \mathcal{A}$ accordingly,

$$[x, y]_s = [x_0, y_0]_s + [x_1, y_0]_s + [x_0, y_1]_s + [x_1, y_1]_s$$

and

$$x \circ_s y = x_0 \circ_s y_0 + x_1 \circ_s y_0 + x_0 \circ_s y_1 + x_1 \circ_s y_1,$$

where $x = x_0 + x_1$, $y = y_0 + y_1$. Note that in trivial superalgebras the Lie superproduct coincides with the Lie product and the Jordan superproduct coincides with the Jordan product.

Generalized quaternions and quaternion algebras have been introduced in the last decades as tools for studying quadratic form theory. This construction is essentially a natural generalization of $\mathcal{H} = \mathcal{H}(\mathbb{R}) = \mathbb{R}\oplus\mathbb{R}i\oplus\mathbb{R}j\oplus\mathbb{R}k$, the well-known quaternion algebra over the real numbers introduced by sir Hamilton. In [10, Chapter III], Lam works with quaternion algebras with coefficients over an arbitrary field of characteristic distinct from 2. More generally, preserving Lam's notations, we will consider here quaternion algebras over a commutative ring with $1, \frac{1}{2}$.

Let $\mathcal{R}$ be a ring with unity and α, β are units. The generalized quaternion algebra denoted by $\mathcal{H}_{\alpha,\beta}$ consists of elements of the form

$$x + yi + zj + wk$$

where x, y, z, $w \in \mathcal{R}$, with componentwise addition and multiplication obeying the relations $i^2 = -\alpha$, $j^2 = -\beta$, $k^2 = -\alpha\beta$,

$$ij = -ji = k$$
$$jk = -kj = \beta i$$
$$ki = -ik = \alpha j.$$

Clearly, $\mathcal{H}_{\alpha,\beta}$ is a noncommutative associative algebra with unity and a two-sided module over $\mathcal{R}$ with basis $\{1, i, j, k\}$. Specifically, when $\alpha = \beta = 1$, we denote the algebra by $\mathcal{H}(\mathcal{R})$ in reference to Hamilton's quaternions.

By letting

$$\mathcal{A}_0 = \mathcal{R} \oplus \mathcal{R}i \quad \text{and} \qquad \mathcal{A}_1 = \mathcal{R}j \oplus \mathcal{R}k,$$

it is easily verified that $\mathcal{H}_{\alpha,\beta} = \mathcal{A}_0 \oplus \mathcal{A}_1$ becomes a superalgebra. In the following, we always consider the above-mentioned superalgebraic structure on a quaternion algebra.

The following definition will be needed throughout the paper.

For $\imath = 0, 1$, recall that a *superderivation* of degree $\imath$ is a linear map $D_\imath : \mathcal{A} \to \mathcal{A}$ such that $D_\imath(\mathcal{A}_\jmath) \subseteq \mathcal{A}_{\imath+\jmath}$ (index modulo 2) and

$$D_\imath(xy) = D_\imath(x)y + (-1)^{\imath|x|}xD_\imath(y)$$

for all $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$.

A superderivation of $\mathcal{A}$ is the sum of a superderivation of degree 0 and a superderivation of degree 1. Note that every superderivation of degree 0 is a derivation from $\mathcal{A}$ to $\mathcal{A}$. For example, for any $a_0 \in \mathcal{A}_0$ and $a_1 \in \mathcal{A}_1$, the maps $I_{a_0}(x) = [a_0, x]_s$ and $I_{a_1}(x) = [a_1, x]_s$ are superderivations of degree 0 and 1, respectively. Hence for any $a = a_0 + a_1 \in \mathcal{A}$, the linear mapping $I_a(x) = [a, x]_s$ is a superderivation, since $[a, x]_s = [a_0, x]_s + [a_1, x]_s$. The superderivation $I_a(x) = [a, x]_s$ will be called the *inner superderivation.*

The $\mathcal{R}$-module of even (odd) superderivations on $\mathcal{A}$ is a $\mathcal{R}$-submodule of $\mathcal{L}(\mathcal{A})$ (the associative algebra of linear maps from $\mathcal{A}$ to $\mathcal{A}$). We denote the direct sum of these two $\mathcal{R}$-submodules by $Der_s(\mathcal{A})$ and call it the $\mathcal{R}$-module of superderivations. The $\mathcal{R}$-module $Der_s(\mathcal{A})$ with the following bracket is a Lie superalgebra:

$$[D, D']_s = DD' - (-1)^{|D||D'|}D'D,$$

where D, D' are homogeneous superderivaions. On the other hand, the inner superderivations $Inn_s(\mathcal{A})$ form a Lie ideal. The quotient superalgebra $Der_s(\mathcal{A})/Inn_s(\mathcal{A})$ is called the *outer superderivation algebra* of $\mathcal{A}$ and is denoted by $Out_s(\mathcal{A})$.

We say that a *Jordan superderivation* of degree $\imath$ is a linear map $J_\imath\colon \mathcal{A} \to \mathcal{A}$ such that $J_\imath(\mathcal{A}_\jmath) \subseteq \mathcal{A}_{\imath+\jmath}$, where indices are calculated modulo 2 and

$$J_\imath(x \circ_s y) = J_\imath(x) \circ_s y + (-1)^{\imath|x|} x \circ_s J_\imath(y)$$

for all $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$.

We define a Jordan superderivation as the sum of a Jordan superderivation of degree 0 and a Jordan superderivaton of degree 1. In addition, we use the notation $Der_{Js}(\mathcal{A})$ for the superalgebra of Jordan superderivations from $\mathcal{A}$ to $\mathcal{A}$.

3. ***Superderivations on*** $H_{\alpha,\beta}$

Lemma 3.1. The linear map $D_0\colon H_{\alpha,\beta} \to H_{\alpha,\beta}$ is a superderivation of degree 0 if and only if $D_0(q) = [t_0, q]_s$, where $t_0 = \frac{1}{2}ai$ for some $a \in \mathcal{R}$.

Proof. Let $D_0\colon \mathcal{H}_{\alpha,\beta} \to \mathcal{H}_{\alpha,\beta}$ be a superderivation of degree 0. Then $D_0(\mathcal{A}_0) \subseteq \mathcal{A}_0$, $D_0(\mathcal{A}_1) \subseteq \mathcal{A}_1$ and

$$D_0(xy) = D_0(x)y + xD_0(y)$$

for all $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$. Clearly, $D_0(1) = 0$. Suppose that $D_0(i) = a + bi$ for some $a, b \in \mathcal{R}$. Then

$$0 = D_0(-\alpha) = D_0(i^2) = D_0(i)i + iD_0(i) = 2ai - 2\alpha b.$$

Since 2 and α are invertible in $\mathcal{R}$ we see that $a = b = 0$ and $D_0(i) = 0$. Set $D_0(j) = cj + dk$ and $D_0(k) = c'j + d'k$, where $c, d, c', d' \in \mathcal{R}$. By similar arguments as above we get $D_0(j) = dk$ and $D_0(k) = c'j$. By applying D_0 to $k = ij$, we deduce that $D_0(k) = iD_0(j)$ so $c' = -\alpha d$. By letting $d = a$, we see that $D_0(j) = ak$ and $D_0(k) = -\alpha aj$.

Now, let $q = x + yi + zj + wk$ be in $H_{\alpha,\beta}$. Then $D_0(q) = -w\alpha aj + zak$. It is strightforward to show that $D_0(q) = [t_0, q]_s$ where $t_0 = \frac{1}{2}ai$, as desired.

Conversely, if $D_0(q) = [t_0, q]_s$, where $t_0 = \frac{1}{2}ai$ for some $a \in \mathcal{R}$, it is easy to show that D_0 is a superderivation of degree 0. □

Lemma 3.2. The linear map $D_1\colon H_{\alpha,\beta} \to H_{\alpha,\beta}$ is a superderivation of degree 1 if and only if $D_1(q) = [t_1, q]_s$, where $t_1 = \frac{-1}{2}\beta^{-1}(bj + \alpha^{-1}ck)$ for some $b, c \in \mathcal{R}$.

Proof. Let $D_1\colon H_{\alpha,\beta} \to H_{\alpha,\beta}$ be a superderivation of degree 1. Then $D_1(\mathcal{A}_0) \subseteq \mathcal{A}_1$, $D_1(\mathcal{A}_1) \subseteq \mathcal{A}_0$ and

$$D_1(xy) = D_1(x)y + (-1)^{|x|} xD_1(y)$$

for all $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$. Now, let $D_1(j) = a' + b'i$ and $D_1(k) = a'' + b''i$ for suitable coefficients in $\mathcal{R}$. Since $D_1(1) = 0$, we have

$$0 = D_1(-\beta) = D_1(j^2) = D_1(j)j - jD_1(j) = 2b'k.$$

Hence, $b' = 0$ and $D_1(j) = a'$. On the other hand,

$$0 = D_1(-\alpha\beta) = D_1(k^2) = D_1(k)k - kD_1(k) = -2\alpha b'' j.$$

So $b'' = 0$ and $D_1(k) = a''$. From $\beta D_1(i) = D_1(\beta i) = D_1(jk) = D_1(j)k - jD_1(k) = a'k - a''j$ we have $D_1(i) = \beta^{-1}(-a''j + a'k)$. By letting $b = a'$ and $c = a''$ for every $q = x + yi + zj + wk \in H_{\alpha,\beta}$ we have

$$D_1(q) = y\beta^{-1}(-cj + bk) + zb + wc = (zb + wc) - y\beta^{-1}cj + y\beta^{-1}bk.$$

Therefore the assertion follows from direct computations.

Conversely, if $D_1(q) = [t_1, q]_s$ where $t_1 = \frac{-1}{2}\beta^{-1}(bj + \alpha^{-1}ck)$ for some $b, c \in \mathcal{R}$, then by direct computations, it follows that D_1 is a superderivation of degree 1. □

Theorem 3.3. Any superderivation of $\mathcal{H}_{\alpha,\beta}$ is an inner superderivation.

Proof. Suppose that D is a superderivation. According to the definition, $D = D_0 + D_1$ where D_0 is a superderivation of degree 0 and D_1 is a superderivation of degree 1. Now, from Lemmas 3.1 and 3.2, it follows that $D = I_{t_0} + I_{t_1} = I_t$ is an inner superderivation. □

Let $D_\imath$ denote a superderivation of degree $\imath$ on $\mathcal{H}_{\alpha,\beta}$. Thus, $D_\imath$ admits a matrix representation with respect to the basis $\mathcal{B}(\mathcal{H}_{\alpha,\beta})$, which is the 4×4 matrix $[D_\imath] = (d_{\imath\jmath})^T$ whose entries are defined by the following equations:

$$D_\imath(1) = d_{11} + d_{12}i + d_{13}j + d_{14}k,$$

$$D_\imath(i) = d_{21} + d_{22}i + d_{23}j + d_{24}k,$$

$$D_\imath(j) = d_{31} + d_{32}i + d_{33}j + d_{34}k,$$

$$D_\imath(k) = d_{41} + d_{42}i + d_{43}j + d_{44}k.$$

Theorem 3.4. Let D be a superderivation on the generalized quaternion algebra $\mathcal{H}_{\alpha,\beta}$. Then the matrix representation $[D]$ of D is as follows:

$$[D] = \begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & -\beta^{-1}c & 0 & -\alpha a \\ 0 & \beta^{-1}b & a & 0 \end{pmatrix}$$

where $a, b, c \in \mathcal{R}$.

Proof. Let $D = D_0 + D_1$ be a superderivation on $\mathcal{H}_{\alpha,\beta}$, where D_0 is a superderivation of degree 0 and D_1 is a superderivation of degree 1. Hence $[D] = [D_0] + [D_1]$. From Lemma 3.1, $D_0(1) = D_0(i) = 0$ so we have $d_{11} = d_{12} = d_{13} = d_{14}k = d_{21} = d_{22}i = d_{23}j = d_{24}k = 0$. Using $D_0(j) = ak$ and $D_0(k) = -\alpha aj$ for some $a \in \mathcal{R}$, we conclude that $d_{34} = a$ and $d_{43} = -\alpha a$.

Therefore,

$$[D_0] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha a \\ 0 & 0 & a & 0 \end{pmatrix}.$$

Similarly, using Lemma 3.2, $D_1(1) = 0$ and $D_1(j) = b$ for $b \in \mathcal{R}$. Hence, in $[D_1]$, we see that $d_{11} = d_{12} = d_{13} = d_{14} = d_{32} = d_{33} = d_{34} = 0$ and $d_{31} = b$. Also, $D_1(k) = c$ implies $d_{41} = c$ and $d_{42} = d_{43} = d_{44} = 0$. In addition, using $D_1(i) = -\beta^{-1}cj + \beta^{-1}bk$ we arrive at $d_{21} = d_{22} = 0$, $d_{23} = -\beta^{-1}c$ and $d_{24} = \beta^{-1}b$. Thus,

$$[D_1] = \begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & -\beta^{-1}c & 0 & 0 \\ 0 & \beta^{-1}b & 0 & 0 \end{pmatrix}$$

and the result holds. □

As a consequence of Theorem 3.4, we have the following result:

Corollary 3.5. The algebra $Der_s(\mathcal{H}_{\alpha,\beta})$ of superderivations of $\mathcal{H}_{\alpha,\beta}$ is generated by the following matrices:

$$\begin{pmatrix} 0 & 0 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & \beta^{-1}c & 0 & -\alpha a \\ 0 & \beta^{-1}b & a & 0 \end{pmatrix}$$

where $a, b, c \in \mathcal{R}$.

The algebra of superderivations $Der_s(\mathcal{H}_{\alpha,\beta})$ is generated by inner superderivations of the form given in Theorem 3.4; hence, the superderivation algebra coincides with the algebra of inner superderivations: $Der_s(\mathcal{H}_{\alpha,\beta}) = Inn_s(\mathcal{H}_{\alpha,\beta})$, and the algebra of outer superderivations $Out_s(\mathcal{H}_{\alpha,\beta})$ vanishes for any generalized quaternion algebra $\mathcal{H}_{\alpha,\beta}$.

4. *Jordan superderivations of* $H_{\alpha,\beta}$

Now, we state and prove our main theorem of Jordan superderivations on generalized quaternion algebra.

Theorem 4.1. Let $J : \mathcal{H}_{\alpha,\beta} \to \mathcal{H}_{\alpha,\beta}$ be a linear mapping. Then

(i) J is a Jordan superderivation of degree 0 if and only if $J(q) = (za + wc)j + (zb - wa)k$ for some $a, b, c \in \mathcal{R}$ and all $q = x + yi + zj + wk \in \mathcal{H}_{\alpha,\beta}$.

(ii) J is a Jordan superderivation of degree 1 if and only if $J(q) = [t_1, q]_s$ where $t_1 = -\frac{1}{2}\beta^{-1}(dj + \alpha^{-1}ek)$ for some $d, e \in \mathcal{R}$ and all $q = x + yi + zj + wk \in \mathcal{H}_{\alpha,\beta}$.

(iii) J is a Jordan superderivation if and only if $J(q) = (za + wc)j + (zb - wa)k + [t_1, q]_s$ where $t_1 = -\frac{1}{2}\beta^{-1}(dj + \alpha^{-1}ek)$ for some $a, b, c, d, e \in \mathcal{R}$ and all $q = x + yi + zj + wk \in \mathcal{H}_{\alpha,\beta}$.

Proof. (i) Let $J\colon \mathcal{H}_{\alpha,\beta} \to \mathcal{H}_{\alpha,\beta}$ be a Jordan superderivation of degree 0. Then $J(\mathcal{A}_0) \subseteq \mathcal{A}_0$, $J(\mathcal{A}_1) \subseteq \mathcal{A}_1$ and

$$J(x \circ_s y) = J(x) \circ_s y + x \circ_s J(y)$$

for all $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$. It is clear that $J(1) = 0$. Suppose that $J(i) = e + fi$ where $e, f \in \mathcal{R}$. Thus $0 = J(-\alpha) = J(i \circ_s i) = J(i) \circ_s i + i \circ_s J(i) = -2\alpha f + 2ei$. So $e = f = 0$ and hence $J(i) = 0$. Assume that $J(j) = aj + bk$ and $J(k) = cj + dk$ for some $a, b, c, d \in \mathcal{R}$. By appyling J on $\beta i = j \circ_s k$ we have

$$0 = J(j) \circ_s k + j \circ_s J(k) = \beta(a + d)i$$

So $d = -a$. From these equations, it follows that for any $q = x + yi + zj + wk \in \mathcal{H}_{\alpha,\beta}$ we have $J(q) = (za + wc)j + (zb - wa)k$.

The converse is obtained with a straightforward computation.

(ii) Assume that $J\colon \mathcal{H}_{\alpha,\beta} \to \mathcal{H}_{\alpha,\beta}$ is a Jordan superderivation of degree 1. Then $J(\mathcal{A}_0) \subseteq \mathcal{A}_1$, $J(\mathcal{A}_1) \subseteq \mathcal{A}_0$ and

$$J(x \circ_s y) = J(x) \circ_s y + (-1)^{|x|} x \circ_s J(y)$$

for all $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$. Clearly, $J(1) = 0$. Assume that $J(i) = x'j + y'k$, $d(j) = d + d'i$ and $J(k) = e + e'i$, $x', y', d, d', e, e' \in \mathcal{R}$. Thus $0 = J(i \circ_s j) = J(i) \circ_s j + i \circ_s J(j) = -\alpha d' + (d - y'\beta)i$. So $d' = 0$ and $y' = \beta^{-1}d$. Hence $J(j) = d$. Moreover, $0 = J(i \circ_s k) = J(i) \circ_s k + i \circ_s J(k) = -\alpha e' + (\beta x' + e)i$. So $e' = 0$ and $x' = -\beta^{-1}e$. Applying J on $q = x + yi + zj + wk$, it follows that $J(q) = (zd + we) - y\beta^{-1}ej + y\beta^{-1}dk$. Now, direct computation shows that,

$$J(q) = [t_1, q]_s$$

where $t_1 = -\frac{1}{2}\beta^{-1}(dj + \alpha^{-1}ek)$.

The converse is trivial.

(iii) Suppose that J is a Jordan superderivation. According to the definition, $J = J_0 + J_1$, where J_0 is a Jordan superderivation of degree 0 and J_1 is a Jordan superderivation of degree 1. From (i) and (ii), J is in the desired form.

Conversely, suppose that J has the mentioned form. Define J_0 and J_1 on $\mathcal{H}_{\alpha,\beta}$ by

$$J_0(q) = (za + wc)j + (zb - wa)k$$

and

$$J_1(q) = [t_1, q]_s.$$

So $J = J_0 + J_1$. It follows from (i) and (ii) that J_0 is a Jordan superderivation of degree 0 and J_1 is a Jordan superderivation of degree 1. Hence J is a Jordan superderivation. □

Corollary 4.2. Let $J : \mathcal{H}_{\alpha,\beta} \to \mathcal{H}_{\alpha,\beta}$ be a linear map. The following are equivalent:

(i) J is a superderivation of degree 1;
(ii) J is a Jordan superderivation of degree 1;
(iii) J is an inner superderivation of degree 1.

Proof. (i)⇒(ii): With a routine computation, it is achieved.
(ii)⇒(iii): It is clear by Theorem 4.1(ii).
(iii)⇒(i): It can be easily verified. □

We now give an example of a proper Jordan superderivation on generalized quaternion algebras.

Example 4.3. Define $J : \mathcal{H}_{\alpha,\beta} \to \mathcal{H}_{\alpha,\beta}$ by the given $J(q) = zj - wk$ for all $q = x + yi + zj + wk \in \mathcal{H}_{\alpha,\beta}$. Since $J(ij) = J(k) = -k$ while $J(i)j + iJ(j) = k$, J is a proper Jordan superderivation of degree 0.

As a consequence of Theorem 3.4, we have the following:

Corollary 4.4. The algebra $Der_{Js}(\mathcal{H}_{\alpha,\beta})$ of superderivations of $\mathcal{H}_{\alpha,\beta}$ is generated by the following matrices:

$$\begin{pmatrix} 0 & 0 & d & e \\ 0 & 0 & 0 & 0 \\ 0 & \beta^{-1}e & a & c \\ 0 & \beta^{-1}d & b & a \end{pmatrix}$$

where $a, b, c, d, e \in \mathcal{R}$.

Proof. Let J denote a Jordan superderivation on $\mathcal{H}_{\alpha,\beta}$. To obtain a matrix associated with J, it suffices to apply J on the basis $\mathcal{B}(\mathcal{H}_{\alpha,\beta})$. Thus $[J]$ is uniquely determined by $J(1), J(i), J(j)$ and $J(k)$. Now, by using Theorem 4.1(iii) we get

$$J(1) = 0,$$

$$J(i) = [t_1, i] = -\beta^{-1}ej + \beta^{-1}dk,$$

$$J(j) = aj + bk + [t_1, j] = d + aj + bk,$$

$$J(k) = cj + ak + [t_1, k] = e + cj + ak,$$

for some $a, b, c, d, e \in \mathcal{R}$. Hence, we can present J in an 4×4 matrix as follows:

$$\begin{pmatrix} 0 & 0 & d & e \\ 0 & 0 & 0 & 0 \\ 0 & \beta^{-1}e & a & c \\ 0 & \beta^{-1}d & b & a \end{pmatrix}.$$

□

5. ***Aknowledgement***

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