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STRONGLY REGULAR RELATIONS ON REGULAR HYPERGROUPS

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Abstract. Hypergroups that have at least one identity element and where each element has at least one inverse are called regular hypergroup. In this regards, for a regular hypergroup H, it is shown that there exists a correspondence between the set of all strongly regular relations on H and the set of all normal subhypergroups of H containing S_β. More precisely, it has been proven that for every strongly regular relation ρ on H, there exists a unique normal subhypergroup of H containing S_β, such that its quotient is a group, isomorphic to H/ρ. Furthermore, this correspondence is extended to a lattice isomorphism between them.

Keywords: Normal subhypergroup, Regular hypergroup, Strongly regular relation.
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1. Introduction

The hyperstructure theory, born in 1934 with Marty's paper at the viii Congress of Scandinavian Mathematicians, was subsequently developed around the 40s with the contribution of various authors especially in France and in the United States [15]. Marty showed that the characteristics of hypergroups can be used in solving some problems of groups, algebraic functions, and rational functions. Surveys of the theory can be found in [7, 8]. A special type of equivalence relations which is called fundamental relations play important roles in the theory of algebraic hyperstructures. The fundamental relations are one of the most important and interesting concepts in algebraic hyperstructures that ordinary algebraic structures are derived from algebraic hyperstructures by them. The fundamental relation β^* on hypergroups was defined by Koskas [14], Corsini [6], Ferni [10, 11], and Vogiouklis [16]. Then D. Ferni introduced the fundamental relation γ^* which is the transitive closure of γ and is the smallest relation such that H/γ^* is an abelian group. Subsequently, fundamental relations were gradually introduced on other algebraic hyperstructures. R. Ameri et al. introduced and studied the congruence relations on multialgebras, as a general case of algebraic hyperstructures, such as hypergroups, hyperrings and etc. in [2].

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T. Vogiouklis generalized the fundamental relations in [16] to use on hyperrings and then R. Ameri et al., reformulated it based on polynomials [3]. In [1], it has been demonstrated that relations β^* and γ^* are related together in the form of $\gamma^*= \delta*\beta^*$, where δ is the congruence relation with respect to the commutator subgroup.
A. Connes and C. Consani introduced hyperrings corresponding to adele classes and studied algebraic geometry based on hyperrings [4]. Then, they utilized hyperrings to prove certain propositions in number theory [5]. J. Jun investigated tropical varieties on hyperrings [13], also in his doctoral thesis, he studied algebraic geometry on hyperstructures and proved important propositions in the field of algebraic geometry on hyperfields [12].
In the theory of algebraic hyperstructures, particularly in research related to algebraic geometry based on hyperstructural concepts, one of the most crucial question is whether the fundamental relations transfer the derived concepts back to their classical form?
On the other hand, fundamental relations usually have long and intricate formulas, posing challenges in the investigation of such issues. Since working with congruence relations is significantly easier, our goal is to examine conditions under which we can view fundamental relations as congruence relations. Now in this paper, we follow [2], to established an isomorphism between the lattices of all strongly regular relations on a hypergroups and its subhypergroups.
One of the most important applications of the results obtained in this article is that they enable us to establish a one-to-one correspondence between strongly regular relations on hyperrings and suitable hyperideals of hyperrings. Additionally, we aim to investigate similar results concerning hypermodules.

2. Preliminaries

Let $(H,\circ)$ be a semi-hypergroup. An equivalence relation ρ is called a
1- *Regular on the right (resp. on the left)*, if for all $x \in H$, from $a\rho b$, it follows that $(a\circ x)\bar{\rho}(b\circ x)$ (resp. $(x\circ a)\bar{\rho}(x\circ b)$);
2- *Strongly regular on the right (resp. on the left)*, if for all $x \in H$, from $a\rho b$, it follows that $(a\circ x)\bar{\bar{\rho}}(b\circ x)$ (resp. $(x\circ a)\bar{\bar{\rho}}(x\circ b)$);
3- *Regular (resp. strongly regular)*, if it is regular (resp. strongly regular) on the right and on the left.

Theorem 2.1 ([6]). *Let $(H,\circ)$ be a semi-hypergroup and ρ be an equivalence relation on H.*
If ρ is regular, then $H/\rho = \{\rho(h);\ h \in H\}$ is a semi-hypergroup with respect to the hyperoperation $\rho(x)\otimes\rho(y) = \{\rho(z);\ z \in x\circ y\}$.
Furthermore, if the above hyperoperation is well defined on H/ρ, then ρ is regular.

Corollary 2.2 ([6]). *If $(H,\circ)$ is a hypergroup and ρ is an equivalence relation on H, then ρ is regular (resp. strongly regular) if and only if $(H/\rho,\otimes)$ is a hypergroup (resp. group).*

Theorem 2.3 ([6]). *Let $(H, \circ)$ be a semi-hypergroup and ρ be an equivalence relation on H.*
If ρ is strongly regular, then H/ρ is a semi-group with respect to the operation $\rho(x) \otimes \rho(y) = \rho(z)$, for all $z \in x \circ y$.
Furthermore, if the above operation is well defined on H/ρ, then ρ is strongly regular.

Definition 2.4 ([9]). For all $n > 1$, we define the relations β_n and γ_n on a semi-hypergroup H, as follows:

$$a\beta_n b \Longleftrightarrow \exists (x_1, x_2, ..., x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i$$

$$a\gamma_n b \Longleftrightarrow \exists (x_1, x_2, ..., x_n) \in H^n, \sigma \in \mathbb{S}_n : \ a \in \prod_{i=1}^{n} x_i \ , \ b \in \prod_{i=1}^{n} y_{\sigma(i)}$$

and $\beta = \bigcup_{n \geq 1} \beta_n$ and $\gamma = \bigcup_{n \geq 1} \gamma_n$ where $\beta_1 = \gamma_1 = \{(x, x); x \in H\}$. Let β^*be the transitive closure of β and γ^* be the transitive closure of γ.

If H is a hypergroup then $\gamma = \gamma^*$ and $\beta = \beta^*$. Also, β^* is the smallest strongly regular relation on H and γ^* is the smallest strongly regular relation on H such that the quotient H/γ^* is a commutative group [9].

Definition 2.5 ([6]). β^* is called the fundamental equivalence relation on H and H/β^* is called the fundamental group.

Let ρ be a strongly regular relation on a hypergroup H, and σ be a congruence relation on group H/ρ, then consider:

$$(a, b) \in \sigma{*}\rho \Longleftrightarrow (\rho(a), \rho(b)) \in \sigma.$$

If δ is a congruence relation on H/ρ corresponding to the commutator subgroup of H/ρ, then $\delta{*}\rho$ is the smallest strongly regular relation containing ρ on H, such that $H/(\delta{*}\rho)$ is a commutative group. In particular, $\gamma = \delta{*}\beta$ [1].

Definition 2.6 ([9]). Let $(H, \circ)$ be a hypergroup and $(K, \circ)$ be a subhypergroup of it. We say that K is:
(i) *closed on the left (on the right)* if for all k_1, k_2 of K and x of H, from $k_1 \in x \circ k_2$ ($k_1 \in k_2 \circ x$, respectively), it follows that $x \in K$;
(ii) *invertible on the left (on the right)* if for all x, y of H, from $x \in K \circ y$ ($x \in y \circ K$), it follows that $y \in K \circ x$ ($y \in x \circ K$, respectively);
(iii) *conjugable on the left* if it is closed on the left and for all $x \in H$, there exists $x' \in H$ such that $x \circ x' \subseteq K$. Similarly, we can define the notion of conjugable on the right.
We say that K is *closed* (*invertible*, *conjugable*) if it is closed (invertible, conjugable) on the left and on the right.

It is proved that, each conjugable subhypergroup is invertible and each invertible subhypergroup is closed [9].

3. Correspondence

For a hypergroup H, denote the set of all strongly regular relations on H by $SR(H)$. Letting $S(H) = \{S_\rho; \rho \in SR(H)\}$, where $S_\rho := \{s \in H; \rho(s) = e_{H/\rho}\}$. Since H/ρ is a group, then there exists $h \in H$ such that $\rho(h) = e_{H/\rho}$. Hence, $S_\rho \neq \emptyset$. Also, denote the set of all subhypergroups of H containing S_β by (S_β), and the set of all normal subhypergroups of H containing S_β by $N(S_\beta)$. Clearly, if H is a group then $S_\beta = \{e\}$ and $S(H) = N(S_\beta)$, because on groups, the set of strongly regular relations becomes to the set of regular relations (congruence relations) that is in one to one correspondence with the set of normal subgroups.

Theorem 3.1. *If H is a hypergroup, then $S(H) \subseteq N(S_\beta)$.*

Proof. Let $S_\rho \in S(H)$ and $x \in S_\rho$. Then $x \circ S_\rho \subseteq S_\rho$ and $S_\rho \circ x \subseteq S_\rho$. If $s \in S_\rho$, since $H \circ x = x \circ H = H$, then there exist $h, h' \in H$ such that $s \in h \circ x \cap x \circ h'$ and $e_{H/\rho} = \rho(s) = \rho(h \circ x) = \rho(h) \otimes \rho(x) = \rho(h) = \rho(h')$. So $h, h' \in S_\rho$ and $S \subseteq x \circ S_\rho \cap S_\rho \circ x$. Therefore S_ρ is a subhypergroup of H. Now consider $\rho \in SR(H)$ and $x \in H$. So there is an $x' \in H$ such that $\rho(x \circ x') = \rho(x) \otimes \rho(x') = e_{H/\rho}$. Since $\rho(S_\rho) = e_{H/\rho}$ then $x \circ S_\rho \circ x' \subseteq S_\rho$ and therefore $x \circ S_\rho \circ x' \circ x \subseteq S_\rho \circ x$. But $x' \circ x \subseteq S_\rho$ and hence $x \circ S_\rho \subseteq S_\rho \circ x$. Similarly, it can be proved that $S_\rho \circ x \subseteq x \circ S_\rho$ and therefore the equality holds. Also, since $\beta \subseteq \rho$ then $S_\beta \subseteq S_\rho$. □

Example 3.2. *Consider the hypergroup $(H, \circ)$, where $H = \{0, 1\}$ and*

$\circ$	0	1
0	0	1
1	1	H

Hence $S_{\beta_H} = H$ and $S(H) = \{H\}$. Now consider $K = S_3 \times H$, then $S(S_3) = \{\{e\}, A_3, S_3\}$, $S_{\beta_K} = \{e\} \times H$ and $(S_{\beta_K}) = \{\{e\} \times H, (\tau_1) \times H, (\tau_2) \times H, (\tau_3) \times H, A_3 \times H, S_3 \times H\}$, $N(S_{\beta_K}) = \{\{e\} \times H, A_3 \times H, S_3 \times H\}$. It is easy to verify that the congruence relation modulo of each element of $N(S_{\beta_K})$ is strongly regular relation on K. Hence $N(S_{\beta_K}) \subseteq S(K)$ and $N(S_{\beta_K}) = S(K)$.

Proposition 3.3. *If H is a hypergroup and $S_\rho \in S(H)$, then S_ρ is conjugable.*

Proof. Let $s_1, s_2 \in S_\rho$ and $h \in H$ such that $s_1 \in h \circ s_2$. Then $e_{H/\rho} = \rho(s_1) = \rho(h \circ s_2) = \rho(h) \otimes \rho(s_2) = \rho(h)$, and hence $h \in S_\rho$. If $h \in H$, then there exists an element such as $\rho(k)$ in group H/ρ which $\rho(h) \otimes \rho(k) = e_{H/\rho}$. Thus $h \circ k \subseteq S_\rho$. □

Let $(H, \circ)$ be a hypergroup and $E_H = \{e \in H;\ x \in e \circ x \cap x \circ e, \forall x \in H\}$. For every $x \in H$, set $C_H(x) = \{y \in H;\ E_H \cap (y \circ x \cap x \circ y) \neq \emptyset\}$. If $C_L(x) = \{y \in H;\ E_H \cap y \circ x \neq \emptyset\}$ and $C_R(x) = \{y \in H;\ E_H \cap x \circ y \neq \emptyset\}$, then $C_H(x) = C_L(x) \cap C_R(x)$. Also $y \in C_L(x)$ if and only if $x \in C_R(y)$, and it is clear that $E_H \subseteq S_\rho$ for every $\rho \in SR(H)$.

Definition 3.4. A hypergroup H is called a regular hypergroup if $E_H \neq \emptyset$ and $C_H(x) \neq \emptyset$ for every $x \in H$. A regular hypregroup H is called a strongly regular hypergroup, if $C_H(x) = \{x^{-1}\}$ for every $x \in H$.

If H is a strongly regular hypergroup, then $|E_H| = 1$. Because for every $e_1, e_2 \in E_H$; $e_1 \in e_2 \circ e_1$ and $e_2 \in e_2 \circ e_2$. So $e_1, e_2 \in C_H(e_2)$ and $e_1 = e_2$.

Lemma 3.5. *Let H be a regular hypregroup. Then $\rho(x) = x \circ S_\rho$ for every $S_\rho \in S(H)$ and $x \in H$.*

Proof. If $z \in \rho(x)$, then $x^{-1} \circ z \subseteq S_\rho$ and $z \in x \circ (x^{-1} \circ z) \subseteq x \circ S_\rho$. So, $\rho(x) \subseteq x \circ S_\rho$. Conversely, if $z \in x \circ S_\rho$, then $z \in x \circ h$, for some $h \in S_\rho$, such that $\rho(z) = \rho(x)$ and $z \in \rho(x)$, then $x \circ S_\rho \subseteq \rho(x)$. □

Consider a subhypergroup K of a regular hypergroup H, and $S_\beta \subseteq K$. Let $a, b \in K$ and $h \in H$ such that $a \in h \circ b$. Then $\beta(a) = \beta(z)$ for each $z \in h \circ b$, and by Lemma 3.5, $a \circ S_\beta = h \circ b \circ S_\beta$. Thus $a \circ S_\beta \circ K = h \circ b \circ S_\beta \circ K$ and $K = h \circ K$. Since H is a regular hypergroup then $E_H \neq \emptyset$ and since $E_H \subseteq S_\beta \subseteq K$ then there exists $e \in E_H$ such that $h \in h \circ e \subseteq h \circ K \subseteq K$. Therefore $h \in K$. Also, there is a element such as $\beta(h')$ in group H/β such that $\beta(h) \otimes \beta(h') = \beta(z) = e_{H/\beta}$ for each $z \in h \circ h'$. Hence $h \circ h' \subseteq S_\beta \subseteq K$. Thus K is conjugable on the left (resp. invertible on the left). Similarly it is proved that K is conjugable on the right and therefore conjugable.
Also, if $(x_1, ..., x_n) \in H^n$ such that $\prod_{i=1}^n x_i \cap K \neq \emptyset$ and $a \in \prod_{i=1}^n x_i \cap K$ then $\beta(a) = \beta(\prod_{i=1}^n x_i)$. Therefore, $a \circ S_\beta = (\prod_{i=1}^n x_i) \circ S_\beta$, and

$$a \circ S_\beta \circ K = (\prod_{i=1}^n x_i) \circ S_\beta \circ K \Rightarrow K = (\prod_{i=1}^n x_i) \circ K.$$

Let $x \in \prod_{i=1}^n x_i$, then there are $y_1, y_2 \in K$ such that $y_1 \in x \circ y_2$. Since K is conjugable, then it is closed and therefore $x \in K$. Thus $\prod_{i=1}^n x_i \subseteq K$ and K is complete part.

If $S \in N(S_\beta)$ then $\{(x, y) \in H^2;\ x \circ S = y \circ S\}$ is called the congruence relation modulo S and, the following theorem can be stated about the elements of $N(S_\beta)$.

Theorem 3.6. *If H is a regular hypregroup, then the congruence relation modulo of each element of $N(S_\beta)$ is strongly regular.*

Proof. Let $x, y, z \in H$ and $S \in N(S_\beta)$. If $x \circ S = y \circ S$, then $(z \circ x) \circ S = (z \circ y) \circ S$ and since $S_\beta \subseteq S$ then $(z \circ x) \circ S_\beta \circ S = (z \circ y) \circ S_\beta \circ S$. By Lemma 3.5, for every $r \in z \circ x$ and $t \in z \circ y$, one has $z \circ x \circ S_\beta = r \circ S_\beta$ and $z \circ y \circ S_\beta = t \circ S_\beta$. So, $r \circ S = t \circ S$ and the congruence relation modulo of S is strongly regular on the left. Since S is normal, from $x \circ S = y \circ S$ it concludes $x \circ z \circ S = y \circ z \circ S$. Similarly it is proved that the congruence relation modulo of S is strongly regular on the right. □

Theorem 3.7. *If H is a regular hypregroup, then $SR(H) \cong N(S_\beta) = S(H)$.*

Proof. Consider the maps $f : SR(H) \to N(S_\beta)$ and $g : N(S_\beta) \to SR(H)$, defined by $f(\rho) = S_\rho$ and $g(S) = \rho_S := \{(x, y);\ x \circ S = y \circ S\}$. f and g are well defined by Theorem 3.1 and Theorem 3.6, respectively. Since $E_H \subseteq S_\beta \subseteq S$ then $h \in S$ if and only if $h \circ S = S$, for every $h \in H$. Hence

$$\begin{aligned} f \circ g(S) &= f\big(\{(x, y);\ x \circ S = y \circ S\}\big) \\ &= \{h \in H;\ (x \circ S)(h \circ S) = x \circ S,\ \forall x \in H\} \\ &= \{h \in H;\ x \circ h \circ S = x \circ S,\ \forall x \in H\}. \end{aligned}$$

Since S is complete part and $E_H \subseteq S_\beta \subseteq S$ then $(x^{-1} \circ x) \cap S \neq \emptyset$ and $x^{-1} \circ x \subseteq S$. So if $x \circ h \circ S = x \circ S$ then $x^{-1} \circ x \circ h \circ S = x^{-1} \circ x \circ S$ and $h \circ S = S$. Therefore $f \circ g(S) = \{h \in H;\ h \circ S = S\} = S$. By Lemma 3.5

$$g \circ f(\rho) = g(S_\rho) = \{(x, y);\ x \circ S_\rho = y \circ S_\rho\} = \{(x, y);\ \rho(x) = \rho(y)\} = \rho.$$

□

Theorem 3.8. *Let $\rho \in SR(H)$ and $H/S_\rho := \{h \circ S_\rho;\ h \in H\}$. Then $H/\rho \cong H/S_\rho$ is a group isomorphism.*

Proof. Since $S_\beta \subseteq S_\rho$, then $x \circ S_\rho\ .\ y \circ S_\rho = (x \circ y) \circ S_\rho = z \circ S_\rho$ for every $z \in x \circ y$. Thus the hyperopration on hypergroup H/S_ρ reduces to an opration, and hence H/S_ρ is a group. Therefore, $H/\rho \to H/S_\rho$ by $\rho(x) \mapsto x \circ S_\rho$ is isomorphism because $\rho(x) = x \circ S_\rho$ for every $x \in H$. □

Proposition 3.9. *If H is a (resp. strongly) regular hypregroup and $S \in (S_\beta)$ then S is conjugable and (resp. strongly) regular.*

Proof. Consider $s_1, s_2 \in S$ and $h \in H$ such that $s_1 \in h \circ s_2$. Then $\beta(s_1) = \beta(h \circ s_2)$ and by Lemma 3.5, $s_1 \circ S_\beta \circ S = (h \circ s_2) \circ S_\beta \circ S$, and hence $S = h \circ S$. Also, from $E_H \subseteq S_\beta \subseteq S$ it concluded that $h \in S$. Also, there exisrs $h^{-1} \in H$ such that $E_H \cap h \circ h^{-1} \neq \emptyset$. So $h \circ h^{-1} \subseteq S_\beta \subseteq S$.
Since $E_H \subseteq S$ then $E_H \subseteq E_S$ and if $x \in S$, then there exists $x' \in H$ such that $E_S \cap (x \circ x') \neq \emptyset$. So, $e \in x \circ x'$ for some $e \in E_S$, and because S is conjugable then $x' \in S$. Therefore, $C_S(x) \neq \emptyset$.

□

Remark 3.10. The converses of Propositions 3.3 and 3.9 are not correct, because if H is a strongly regular hypergroup, then $S = \{e\} \times H$ is a conjugable and strongly regular subhypergroup of $H \times H$ but $S \notin S(H \times H)$.

Let $S \in (S_\gamma)$ and $x \in H$. Then there exists $x' \in H$ such that $\gamma(x) \otimes \gamma(x') = \gamma(y) = \gamma(z) = e_{H/\gamma}$ for each $y \in x \circ x'$ and $z \in x' \circ x$. Hence for every $s \in S$ one has

$$(x \circ s \circ x') \circ S_\gamma = (x' \circ s \circ x) \circ S_\gamma = (s \circ x \circ x') \circ S_\gamma = s \circ S_\gamma.$$

Thus $(x \circ s \circ x') \circ S_\gamma \circ S = (x' \circ s \circ x) \circ S_\gamma \circ S = s \circ S_\gamma \circ S = S$. Since S is regular and $E_H \subseteq S$ then $x \circ s \circ x' \cup x' \circ s \circ x \subseteq S$. So, $x \circ S \circ x' \cup x' \circ S \circ x \subseteq S$ and since $x \circ x' \cup x' \circ x \subseteq S_\gamma \subseteq S$, then $x \circ S \subseteq S \circ x$ and $S \circ x \subseteq x \circ S$. Therefore:

$$(S_\gamma) = N(S_\gamma) \subseteq N(S_\beta).$$

4. Lattice perspective

From now on, all hypergroups are strongly regular hypergroups unless otherwise stated. For a hypergroup H, $(S(H), \subseteq)$ and $(SR(H), \subseteq)$ are posets and $S(H) = \{S_\theta;\ \theta \in SR(H)\} = N(S_\beta)$, $SR(H) = \{\theta_S;\ S \in N(S_\beta)\}$, where θ_S is the congruence relation modulo S.
Let $\Delta_H = \{(h,h); h \in H\}$ and $\nabla_H = H^2$. Then $\nabla_H, \beta_H \in SR(H)$. For every $\rho, \sigma \in SR(H)$ put $\rho \vee \sigma = \rho \cup (\rho \circ \sigma) \cup (\rho \circ \sigma \circ \rho) \cup (\rho \circ \sigma \circ \rho \circ \sigma) \cup ...$ where $\rho \circ \sigma = \{(x,y) \in H^2;\ \exists z \in H \ni (x,z) \in \rho, (z,y) \in \sigma\}$. Also, $\rho \cap \sigma \in SR(H)$ and $\bigcap_{\rho \in SR(H)} \rho = \beta_H$.
If L is a lattice and $a \in L$, then $I(a) := \{b \in L;\ b \leqslant a\}$ is the principal ideal generated by a and $F(a) := \{b \in L;\ a \leqslant b\}$ is the filter generated by a.

Lemma 4.1. *If $\rho, \sigma \in SR(H)$, then $\rho \vee \sigma \in SR(H)$.*

Proof. Clearly, $\rho \vee \sigma$ is the smallest equivalence relation containing ρ and σ. Let $(x,y) \in \rho \vee \sigma$ and $z \in H$. Then there are $n \in \mathbb{N}$ and $z_1, ..., z_n \in H$ such that $z_1 = x$, $z_n = y$ and $(z_i, z_{i+1}) \in \rho \cup \sigma$, for every $1 \leq i \leq n-1$. Without loss of generality assume that $(x, z_2) \in \rho$, $(z_2, z_3) \in \sigma$,..., $(z_{n-1}, y) \in \rho$. So

$$z \circ x \ \bar{\bar{\rho}}\ z \circ z_2 \ \bar{\bar{\sigma}}\ z \circ z_3 \ ... \ z \circ z_{n-1} \ \bar{\bar{\rho}}\ z \circ y.$$

Hence for every $t_1 \in z \circ x$ and $t_n \in z \circ y$ there are $t_2, ..., t_{n-1} \in H$, where $t_1\ \rho\ t_2\ \sigma\ t_3\ ...\ t_{n-1}\ \rho\ t_n$ and $t_k \in z \circ z_k$, for every $2 \leq k \leq n-1$. Thus $(t_1, t_n) \in \rho \vee \sigma$ and $z \circ x\ \overline{\overline{\rho \vee \sigma}}\ z \circ y$. Similarly, one can prove that $x \circ z\ \overline{\overline{\rho \vee \sigma}}\ y \circ z$. □

Theorem 4.2. *Let $(H, \circ)$ be a hypergroup. Then $S(H)$ and $SR(H)$ are complete lattices.*

Proof. Let $A \subseteq SR(H)$ then by Lemma 4.1;

$$\bigvee_{\rho \in A} \rho \ \in SR(H)\ ,\ \bigcap_{\rho \in SR(H)} \rho = \beta \ \in SR(H).$$

If $S, T \in S(H)$, then $S \circ T$ is the smallest normal subhypergroup of H such that $S \cup T \subseteq S \circ T$. So $S \vee T = S \circ T$. Also $\bigcap_{S \in S(H)} S = S_\beta \in S(H)$. □

Proposition 4.3. *The lattices $S(H)$ and $SR(H)$ are isomorph.*

Proof. Consider the function f introduced in Theorem 3.7 and suppose $\rho, \sigma \in SR(H)$ and $S, T \in S(H)$ where $\rho \subseteq \sigma$ and $S \subseteq T$. Hence $f(\rho) = S_\rho \subseteq S_\sigma = f(\sigma)$ and if $(x,y) \in H^2$ be such that $x \circ S = y \circ S$ then $x \circ S \circ T = y \circ S \circ T$. Therefore, $x \circ T = y \circ T$ and $f^{-1}(S) \subseteq f^{-1}(T)$. □

Therefore $S_\rho \circ S_\sigma = S_{\rho \vee \sigma}$, $S_\rho \cap S_\sigma = S_{\rho \cap \sigma}$, $\theta_S \vee \theta_T = \theta_{S \circ T}$ and $\theta_S \cap \theta_T = \theta_{S \cap T}$. Where $S, T \in S(H)$, $\rho, \sigma \in SR(H)$ and $\theta_S = \{(x,y) \in H^2;\ x \circ S = y \circ S\}$. Also $\bigvee_{\rho \in SR(H)} \rho = \nabla_H$, $\bigcap_{\rho \in SR(H)} \rho = \beta_H$, $\bigvee_{S \in S(H)} S = H$ and $\bigcap_{S \in S(H)} S = S_\beta$.

Theorem 4.4. *Let $f : H \to K$ be a good homomorphism of strongly regular hypergroups and $\rho \in SR(H)$, such that $S_\rho \subseteq Kerf$. Then there is a unique good homomorphism $\bar{f} : H/\rho \to K$ where $\bar{f}(\rho(h)) = f(h)$ such that $Ker\bar{f} = \rho(Kerf)$ and $Imf = Im\bar{f}$. Moreover, f is isomorphism if and only if f is onto and $S_\rho = Kerf$.*

Proof. If $x, y \in H$ and $\rho(x) = \rho(y)$, then $x \circ y^{-1} \subseteq S_\rho \subseteq Kerf$. Therefore, $f(x \circ y^{-1}) = f(x)f(y)^{-1} = e_K$ and $f(x) = f(y)$. Clearly, $Im\bar{f} = Imf$, $Ker\bar{f} = \{\rho(x);\ x \in Kerf\} = \rho(Kerf)$ and $\bar{f}$ is unique. Thus $\bar{f}(\rho(x) \otimes \rho(y)) = \bar{f}(\rho(x \circ y)) = f(x \circ y) = f(x)f(y)$ and $\bar{f}$ is one to one if and only if $ker\bar{f} = \rho(Kerf) = \{S_\rho\}$ if and only if $Kerf = S_\rho$.

□

Corollary 4.5. *Let $f : H \to K$ be a good homomorphism of strongly regular hypergroups and $\rho \in SR(H)$, $\sigma \in SR(K)$ such that $f(S_\rho) \leqslant S_\sigma$. Then there is a unique homomorphism $\bar{f} : H/\rho \to K/\sigma$ where $\bar{f}(\rho(h)) = \sigma(f(h))$. Moreover, $\bar{f}$ is a isomorphism if and only if $Imf \vee S_\rho = K$ and $f^{-1}(S_\sigma) \subseteq S_\rho$.*

Proof. If $\rho(x) = \rho(y)$, then $(f(x), f(y)) \in f(\rho) \subseteq \sigma$ and $\sigma(f(x)) = \sigma(f(y))$. Thus $\bar{f}$ is welldefined. Also $\bar{f}(\rho(x) \otimes \rho(y)) = \bar{f}(\rho(x \circ y)) = \sigma(f(x \circ y)) = \sigma(f(x)f(y)) = \sigma(f(x)) \otimes \sigma(f(y)) = \bar{f}(\rho(x))\bar{f}(\rho(y))$. Let $\bar{f}$ be an isomorphism, then f is onto and $Imf \vee S_\sigma = K$. Since $Ker\bar{f} = S_\rho$ then $\{\rho(x);\ \bar{f}(\rho(x)) = S_\sigma\} = \{\rho(x);\ \sigma(f(x)) = S_\sigma\} = \{x \circ S_\rho;\ f(x) \in S_\sigma\} = S_\rho$. Thus, $f^{-1}(S_\sigma) \subseteq S_\rho$. If $f^{-1}(S_\sigma) \subseteq S_\rho$ and $Imf \vee S_\sigma = K$, then $Ker\bar{f} = S_\rho$ and $S_\sigma \subseteq Imf$. Therefore, $Imf = K$ and $\bar{f}$ is onto. □

If H is a strongly regular hypergroup and A, B, X and Y are lattices of subhypergroups of H, normal subhypergroups of H, equivalence relations on H and strongly regular relations on H, respectively, then $A \cong X$ and $B \cong Y$.

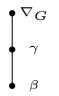
5. Examples

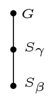
Example 5.1. *Let $G = \{p, q, x_1, x_2, x_3, x_4, x_5\}$ and consider $(G, \circ)$ as the following table*

$\circ$	p	q	x_1	x_2	x_3	x_4	x_5
p	$\{p,q\}$	$\{p,q\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$
q	$\{p,q\}$	$\{p,q\}$	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$
x_1	$\{x_1\}$	$\{x_1\}$	$\{p,q\}$	$\{x_4\}$	$\{x_5\}$	$\{x_2\}$	$\{x_3\}$
x_2	$\{x_2\}$	$\{x_2\}$	$\{x_5\}$	$\{p,q\}$	$\{x_4\}$	$\{x_3\}$	$\{x_1\}$
x_3	$\{x_3\}$	$\{x_3\}$	$\{x_4\}$	$\{x_5\}$	$\{p,q\}$	$\{x_1\}$	$\{x_2\}$
x_4	$\{x_4\}$	$\{x_4\}$	$\{x_3\}$	$\{x_1\}$	$\{x_2\}$	$\{x_5\}$	$\{p,q\}$
x_5	$\{x_5\}$	$\{x_5\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1\}$	$\{p,q\}$	$\{x_4\}$

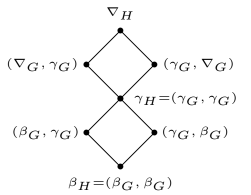
It is routine to see that $\beta = \Delta_G \cup \{(p, q), (q, p)\}$ and $G/\beta \cong \mathbb{S}_3$. Since β is the smallest strongly regular relation and, also strongly regular relations and regular relations are the same on groups. Then $SR(H) = \{\beta, \gamma, \nabla_G\}$, where

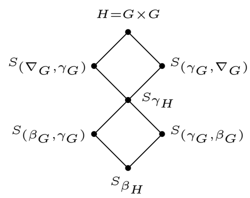
$\gamma = \beta \cup \{(x_i, x_j);\ i, j \in \{1,2,3\}, i \neq j\} \cup \{(x_r, x_s);\ r, s \in \{4,5\}, r \neq s\}$, $G/\gamma \cong \mathbb{Z}_2$. Also $S_\beta = \{p, q\}$ and $S_\gamma = \{p, q, x_4, x_5\}$.



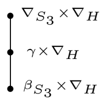


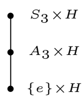
Now consider the hypergroup $H = G \times G$. Then $\big((h_1, h_2), (k_1, k_2)\big) \in \beta_H$ if and only if $(h_1, k_1), (h_2, k_2) \in \beta_G$, similarly for γ_H. Also $S_{\gamma_H} = S_{\gamma_G} \times S_{\gamma_G}$ and $S_{\beta_H} = \{(p,p), (q,q), (p,q), (q,p)\}$.





Example 5.2. *In Example 3.2 we have the following correspondence:*





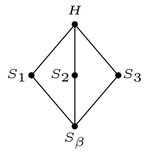
Example 5.3. *Let $H = Z_2 \times Z_2 \times Z_2$ and $\circ$ be the following hyperoperation:*

$\circ$	e	p	q	r	s	t	u	v
$e = (0,0,0)$	$\{e\}$	$\{e,p\}$	$\{q\}$	$\{q,r\}$	$\{s\}$	$\{s,t\}$	$\{u\}$	$\{u,v\}$
$p = (0,0,1)$	$\{e,p\}$	$\{p\}$	$\{q,r\}$	$\{r\}$	$\{s,t\}$	$\{t\}$	$\{u,v\}$	$\{v\}$
$q = (1,0,0)$	$\{q\}$	$\{q,r\}$	$\{e\}$	$\{e,p\}$	$\{u\}$	$\{u,v\}$	$\{s\}$	$\{s,t\}$
$r = (1,0,1)$	$\{q,r\}$	$\{r\}$	$\{e,p\}$	$\{p\}$	$\{u,v\}$	$\{v\}$	$\{s,t\}$	$\{t\}$
$s = (0,1,0)$	$\{s\}$	$\{s,t\}$	$\{u\}$	$\{u,v\}$	$\{e\}$	$\{e,p\}$	$\{q\}$	$\{q,r\}$
$t = (0,1,1)$	$\{s,t\}$	$\{t\}$	$\{u,v\}$	$\{v\}$	$\{e,p\}$	$\{p\}$	$\{q,r\}$	$\{r\}$
$u = (1,1,0)$	$\{u\}$	$\{u,v\}$	$\{s\}$	$\{s,t\}$	$\{q\}$	$\{q,r\}$	$\{e\}$	$\{e,p\}$
$v = (1,1,1)$	$\{u,v\}$	$\{v\}$	$\{s,t\}$	$\{t\}$	$\{q,r\}$	$\{r\}$	$\{e,p\}$	$\{p\}$

Since H is commutative,

$$\beta = \gamma = \Delta \cup \{(e,p), (p,e), (q,r), (r,q), (s,t), (t,s), (u,v), (v,u)\}$$

and $S_\gamma = S_\beta = \{e, p\}$. Let $S_1 = \{e,p,q,r\}$, $S_2 = \{e,p,r,s\}$ and $S_3 = \{e,p,u,v\}$, so $H/S_\beta \cong V_4$ and $H/S_1 \cong H/S_2 \cong H/S_3 \cong \mathbb{Z}_2$.



Example 5.4. *Let $H = \{e, a, b, c\}$ and $\circ$ be the following hyperoperation:*

$\circ$	e	a	b	c
e	$\{e, b\}$	$\{a, c\}$	$\{e, b\}$	$\{a, c\}$
a	$\{a, c\}$	$\{e, b\}$	$\{a, c\}$	$\{e, b\}$
b	$\{e, b\}$	$\{a, c\}$	$\{e, b\}$	$\{a, c\}$
c	$\{a, c\}$	$\{e, b\}$	$\{a, c\}$	$\{e, b\}$

Since H is commutative, $\beta = \gamma = \Delta \cup \{(e, b), (b, e), (a, c), (c, a)\}$ and $S_\gamma = S_\beta = \{e, b\}$. Also $H/\beta \cong \mathbb{Z}_2$ and

6. Conclusions

In this paper we considered hypergroups, as a generalization of groups. In this paper we study the lattices of strong regular relations on a fixed hypergroup H, and has been proved there is a lattice isomorphisms between strong regular congruences relations of H and its normal subhypergroups. This paper was provided a good introduction for study the relationship between strong congruence relations and subhyperstructures, for othere classes of hyperstructures such as hyperrings and hypermodules.
Our next goal will be to generalize the one-to-one correspondence presented in this article to arbitrary hypergroups. Additionally, we aim to examine the correspondence between the strongly regular relations whose quotient is a cyclic, nilpotent or idempotent group and the corresponding normal subhypergroups, from the perspective of lattice theory.

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