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ON DERIVATIONS OF PSEUDO L-ALGEBRAS

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Abstract. In this article, the focus is on the study of derivations on two types of algebraic structures: pseudo L-algebras and pseudo CKL-algebras. For pseudo L-algebras, the notions of left and right derivations are introduced. These derivations are characterized and equivalent characterizations are given. Additionally, the concepts of identity and ideal derivations are defined based on the notion of derivations in pseudo L-algebras. It is proven that any identity derivation is also an ideal derivation. However, an example is provided to demonstrate that not all ideal derivations are identity derivations. Moreover, it is shown that ideal left derivations in pseudo L-algebras are idempotent. The article also introduces the notion of fixed point sets in pseudo L-algebras and investigates some properties associated with them. Moving on to pseudo CKL-algebras, various properties of derivations in these structures are studied. The relationship between pseudo CKL-algebras and pseudo BCK-algebras is established, and it is proven that any pseudo CKL-algebra is also a pseudo BCK-algebra. Conversely, an example is provided to show that not all pseudo BCK-algebras are pseudo CKL-algebras. Additionally, it is demonstrated that the contractive derivation of a pseudo CKL-algebra is an identity derivation. We introduce the definition of a pre-ideal and also introduce the definition of a non-empty subset I in pseudo L-algebra, which is d-invariant, and prove that every pre-ideal I in pseudo CKL-algebra is d-invariant, where d is a derivation. Overall, the article explores derivations in pseudo L-algebras and pseudo CKL-algebras, providing definitions, characterizations, and examples to illustrate various properties and relationships between these algebraic structures.

Keywords: Pseudo L-algebra, Pseudo CKL-algebre, Derivation, Identity, Ideal derivation, Fixed point.
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1. Introduction

The decomposition theorem on square-free solutions of the quantum Yang-Baxter equation, presented by German mathematician Wolfgang Rump in 2005 [18], states that certain solutions to the equation can be decomposed into square-free elements. In 2008, W. Rump published an article [19] in which he introduced the concept of L-algebra. According to the article, an L-algebra

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is a set X equipped with a binary operation that satisfies the following condition: if the left-multiplication mapping is a bijection, then for any elements x, y, and z in X, the equation $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$ holds. Furthermore, the article suggests that the set X, satisfying the aforementioned condition, contains the solutions of the Yu quantum Yang-Baxter equation. In the literature [18] L-algebra was introduced and studied by the above equations contains only one operation and satisfies three axioms. W. Rump in [19] pointed out that the Hilbert algebra, the positive implicative BCK-algebra, the (pseudo)MV-algebra are special L-algebras. It can be seen that L-algebra is a kind of basic algebraic structure, and the research about L-algebra has become a hot issue in the study of logic algebra in the last ten years, which has attracted many scholars at home and abroad [20]. According to W. Rump's work cited in reference [21], L-algebras are equipped with a group structure, which serves as a foundation for further study on L-algebras. In 2019, Y. Wu, J. Wang and others constructed an effect algebra with the negation of an L-algebra, referred to as an orthogonal complement algebra, in reference [26]. They demonstrated that every lattice-ordered effect algebra can be associated with an L-algebra that has the same orthogonal complement. In 2020, Y. Wu established the axioms of orthogonal modular L-algebras (OL-algebras for short) in reference [27]. Additionally, they provided a sufficient condition for an L-algebra to be classified as a Boolean algebra. In addition to the previous research, Holland's theorems were proven using self-similar closure theory on OL-algebras. This application of self-similar closure theory demonstrates its usefulness in studying the properties of OL-algebras. Rump and X. Zhang further refined the theory related to L-effect algebra, as described in reference [22]. They also investigated the relationship between L-effect algebra and quantum set generation under specific conditions. This exploration suggests a potential connection between L-effect algebra and the generation of quantum sets. J. Wang, Y. Wu, Y. Yang, and others focused on the connection between L-effect algebra and basic algebra. They also refined the relationship between L-algebra and basic logic algebra. This investigation suggests that there may be significant connections and implications between L-effect algebra, L-algebra, basic algebra, and basic logic algebra. X.L. Xin et al. discovered that the residue lattice, BL-algebra, MTL-algebra, and constant algebra have corresponding pseudo-structures. However, they found that there is no pseudo-structure of L-algebra, as mentioned in reference [12]. This finding highlights a distinct characteristic of L-algebra compared to these other algebraic structures. It seems that there is ongoing research and exploration in the field of L-algebra, with various researchers refining theories, exploring connections with other algebraic structures, and investigating the absence of certain pseudo-structures in L-algebra. The investigation conducted by X.L. Xin and others in 2022 focuses on the relationship between the pseudo-structure of L-algebra and the pseudo-structures of other algebras. They have generalized the structure of L-algebra and introduced a new concept called Pseudo L-algebra [23]. The purpose of this investigation is to understand how the pseudo-structure of L-algebra can be extended or adapted to other algebraic structures.

The definition of derivations from the analytic theory was introduced in 1957 by Posner [2] to a prime ring $(R, +, \cdot)$ as a map $d : R \to R$ satisfying the conditions $d(a+b) = d(a)+d(b)$ and $d(a \cdot b) = d(a) \cdot b + a \cdot d(b)$, for all $a, b \in R$. Since the derivation proved to be useful for studying the properties of algebraic systems, this definition has been defined and studied by many authors for the cases of lattices [4,13,24,25] and algebras of fuzzy logic: MV-algebras [5,14,17], BCI-algebras [6,7,28], commutative residuated lattice [16], BCC-algebras [1,15], BE-algebras [11], basic algebras [8] and pseudo-MV algebras [9]. In particular, in 2023, J.T. Wang, P.F. He and Y.H. She published an extremely interesting article [10], in which they further studied the derivation in MV-algebra, they mainly got every MV-algebra is isomorphic to the direct product of the fixed point set of Boolean additive derivations and that of their adjoint derivations. In addition, they also got the fixed point set of Boolean additive derivations and that of their adjoint derivations in MV-algebras are isomorphic and so on, all of which are very important for the study of derivation. The derivation concept has proven to be useful in studying the properties of these algebraic systems.

The purpose of this article is to discuss the concept of derivations on a pseudo L-algebra and analyze their properties. We start by defining left and right derivations on a pseudo L-algebra and providing an equivalent characterization for each. Additionally, we introduce the notions of identity derivation and ideal derivation. We prove that any arbitrary identity derivation is also an ideal derivation. However, it is important to note that an ideal derivation may not necessarily be an identity derivation. Furthermore, we introduce the concept of fixed point sets of pseudo CKL-algebraic derivation and explore their properties. In particular, we focus on the case of derivations on pseudo CKL-algebras. We demonstrate that a contractive derivation on a pseudo CKL-algebra is an identity derivation. Lastly, one of the main results of this article is the proof that the preideal I is d-invariant, where d is a derivation.

2. Preliminaries

The section contains fundamental results about L-algebra and pseudo L-algebra.

Definition 2.1. [26] An L-algebra is an algebra $(L, \to, 1)$ of type $(2, 0)$ satisfying
$(L1)$ $a \to a = a \to 1 = 1, 1 \to a = a$;
$(L2)$ $(a \to b) \to (a \to c) = (b \to a) \to (b \to c)$;
$(L3)$ $a \to b = b \to a = 1$ implies $a = b$;
for all $a, b, c \in L$.

Definition 2.2. [19] Let $(L, \to, 1)$ be an L-algebra
(1) If L satisfies condition K: $a \to (b \to a) = 1$, then L is called a KL-algebra;

(2) If L satisfies condition C: $(a \to (b \to c)) \to (b \to (a \to c)) = 1$, then L is called a CL-algebra. It follows that in any L-algebra L satisfies condition $a \to (b \to c) = b \to (a \to c)$, for all $a, b, c \in L$.

Definition 2.3. [23] A pseudo L-algebras is an algebra $(L, \to, \rightsquigarrow, 1)$ with two binary operations $\to, \rightsquigarrow$ and one constant 1 such that: for all $a, b, c \in L$
$(PL1)$ $1 \to a = a = 1 \rightsquigarrow a, a \to 1 = 1$;
$(PL2)$ $a \to a = 1$;
$(PL3)$ $(a \to b) \to (a \to c) = (b \to a) \to (b \to c)$;
$(PL4)$ $(a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = (b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c)$;
$(PL5)$ $a \to b = b \to a = 1 \Rightarrow a = b$;
$(PL6)$ $a \to b = 1$ iff $a \rightsquigarrow b = 1$.

Every pseudo L-algebra satisfying $a \to b = a \rightsquigarrow b$ for all $a, b \in L$ is an L-algebra. A pseudo L-algebra is said to be proper if it is not an L-algebra.

Example 2.4. *Let $L = \{a, b, c, 1\}$, where $a \leq b \leq c \leq 1$. Define the operations $\to$ and $\rightsquigarrow$ using the following tables*

$\to$	a	b	c	1
a	1	1	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

$\rightsquigarrow$	a	b	c	1
a	1	1	1	1
b	c	1	1	1
c	a	b	1	1
1	a	b	c	1

Then $L = \{a, b, c, 1\}$ is a proper pseudo L-algebra (since $b \to a = a$, $b \rightsquigarrow a = c$, it follows that $b \to a \neq b \rightsquigarrow a$, hence L is not an L-algebra).

Remark 2.5. [23] Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra. Then we can see that the reducts $(L, \to, 1)$ and $(L \rightsquigarrow, 1)$ of $(L, \to, \rightsquigarrow, 1)$ are both L-algebras.

Proposition 2.6. *[23] Let L be a pseudo L-algebra. Define a binary relation "$\leq$" as follows*

$$a \leq b \Leftrightarrow a \to b = 1 \Leftrightarrow a \rightsquigarrow b = 1$$

Then "$\leq$" is a partial order on L .

Lemma 2.7. *[23] Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra. If $a \leq b$, then $c \to a \leq c \to b$ and $c \rightsquigarrow a \leq c \rightsquigarrow b$ for all $a, b, c \in L$.*

Definition 2.8. [3] A pseudo-BCK-algebra(more precisely, reversed left-pseudo-BCK algebra) is a structure $\mathcal{A}$=$(A, \leq, \to, \rightsquigarrow, 1)$ where $\leq$ is a binary relation on A, $\to$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A satisfying, for all $x, y, z \in A$, the axioms
$(psBCK_1)$ $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;
$(psBCK_2)$ $x \leq (x \to y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \to y$;
$(psBCK_3)$ $x \leq x$;
$(psBCK_4)$ $x \leq 1$;

$(psBCK_5)$ if $x \leq y$ and $y \leq x$, then $x = y$;
$(psBCK_6)$ $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$.

Since the partial order " $\leq$ " is determined by any of the two "arrows", we can eliminate " $\leq$ " from the signature and denote a pseudo-BCK algebra by $\mathcal{A}$=$(A, \to, \rightsquigarrow, 1)$.

3. The derivations in pseudo L-algebras

The main purpose of this section is to give the definition of left and right derivations in pseudo L-algebra and discuss the relationship between left and right derivations and also to discuss the relationship between identity and ideal derivations.

Remark 3.1. A pseudo L-algebra L is bounded if there is an element $0 \in L$ such that $0 \leq a$ for all $a \in L$.

Example 3.2. *Let* L=$\{0, a, 1\}$ *such that* $0 < a < 1$ *and operation* $\to$ *and* $\rightsquigarrow$ *are defined as follows*

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

$\rightsquigarrow$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

We can easily check that L *is a bounded pseudo L-algebra.*

Definition 3.3. Let L be a bounded pseudo L-algebra. We define two negations $^-$ and $^\sim$, for all $a \in L$, $a^- = a \to 0$, $a^\sim = a \rightsquigarrow 0$.

Remark 3.4. If L is a bounded pseudo L-algebra, then $1^- = 0 = 1^\sim$ and $0^- = 1 = 0^\sim$.

Definition 3.5. A pseudo L-algebra L is said to be a pseudo KL-algebra if it satisfies K condition $a \leq b \to a$ and $a \leq b \rightsquigarrow a$ for all $a, b \in L$.

Example 3.6. *Let* $L = \{a, b, c, 1\}$, *such that* $a, b < c < 1$, a *and* b *are incomparable and operations* $\to$ *and* $\rightsquigarrow$ *by the following two tables*

$\to$	a	b	c	1
a	1	b	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

$\rightsquigarrow$	a	b	c	1
a	1	c	1	1
b	a	1	1	1
c	a	b	1	1
1	a	b	c	1

We can check that L *is a pseudo KL-algebra .*

Proposition 3.7. *Let* L *be a pseudo KL-algebra. If* $a \leq b$, *then* $b \to c \leq a \to c$ *and* $b \rightsquigarrow c \leq a \rightsquigarrow c$ *for all* $a, b, c \in L$.

Proof. Let $a, b, c \in L$ such that $a \leq b$, we get $a \to b = 1, a \rightsquigarrow b = 1$. Since L satisfies K condition, by $(PL3)$ and $(PL4)$, we get $b \to c \leq (b \to a) \to (b \to$

$c) = (a \to b) \to (a \to c) = a \to c$ and $b \rightsquigarrow c \leq (b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c) = (a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c) = a \rightsquigarrow c$, i.e., $b \to c \leq a \to c$ and $b \rightsquigarrow c \leq a \rightsquigarrow c$. □

Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra. Denote $a \sqcup_1 b = (a \to b) \rightsquigarrow b$ and $a \sqcup_2 b = (a \rightsquigarrow b) \to b$, for any $a, b \in L$. In order to inscribe derivation on pseudo L-algebra with $\sqcup_1$ and $\sqcup_2$, we give lemmas and properties of the binary operation $\sqcup_1$ and $\sqcup_2$ on a pseudo L-algebra.

Example 3.8. *In Example 3.6, we can get the $\sqcup_1$ operation and the $\sqcup_2$ operation as follows*

$\sqcup_1$	a	b	c	1
a	a	1	c	1
b	1	b	c	1
c	1	1	c	1
1	1	1	1	1

$\sqcup_2$	a	b	c	1
a	a	b	c	1
b	1	b	c	1
c	1	1	c	1
1	1	1	1	1

Lemma 3.9. *In any pseudo L-algebra L the following hold for all $a, b \in L$*
(1) $1 \sqcup_1 a = 1 \sqcup_2 a = a \sqcup_1 1 = a \sqcup_2 1 = 1$;
(2) *If $a \leq b$, then $a \sqcup_1 b = b$, $a \sqcup_2 b = b$;*
(3) $a \sqcup_1 a = a \sqcup_2 a = a$.

Proof. The proof is straightforward. □

A pseudo L-algebra L is said to be $\sqcup_1$-commutative($\sqcup_2$-commutative) if $a \sqcup_1 b = b \sqcup_1 a (a \sqcup_2 b = b \sqcup_2 a)$, for all $a, b \in L$.

Definition 3.10. Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra. A mapping $d_l : L \to L$ is called a left derivation if it satisfies the following conditions for all $a, b \in L$

$$(pld1)\ d_l(a \to b) = (a \to d_l(b)) \sqcup_2 (d_l(a) \to b).$$
$$(pld2)\ d_l(a \rightsquigarrow b) = (a \rightsquigarrow d_l(b)) \sqcup_1 (d_l(a) \rightsquigarrow b).$$

Definition 3.11. Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra. A mapping $d_r : L \to L$ is called a right derivation if it satisfies the following conditions for all $a, b \in L$

$$(pld3)\ d_r(a \to b) = (d_r(a) \to b) \sqcup_2 (a \to d_r(b)).$$
$$(pld4)\ d_r(a \rightsquigarrow b) = (d_r(a) \rightsquigarrow b) \sqcup_1 (a \rightsquigarrow d_r(b)).$$

Let L be a pseudo L-algebra. Denote
$PLD^{(l)}(L)$ the set of all left derivations on L;
$PLD^{(r)}(L)$ the set of all right derivations on L;
$PLD(L) = PLD^{(l)}(L) \cap PLD^{(r)}(L)$ the set of all left derivations and right derivations on L.

If a mapping d on L is both a left derivation and a right derivation, then we call d a derivation on L. In what follows we will denote dx instead of $d(x)$.

Example 3.12. *In Example 3.2, we know that L is a pseudo L-algebra . Now we define a map* $d : L \to L$ *as follows: for all* $x \in L$,

$$d(x) = \begin{cases} 1, & x = a, 1 \\ 0, & x = 0. \end{cases}$$

Then it is easy to verify that d is a left derivation, not a right derivation , because $d(a \rightsquigarrow 0) = d(a) = 1$, $(da \rightsquigarrow 0) \sqcup_1 (a \rightsquigarrow d0) = (1 \rightsquigarrow 0) \sqcup_1 (a \rightsquigarrow 0) = 0 \sqcup_1 a = (0 \to a) \rightsquigarrow a = 1 \rightsquigarrow a = a$, $1 \neq a$, *therefore, d is not a right derivation on pseudo L-algebra.*

Example 3.13. *In Example 2.4, we know that L is a pseudo L-algebra . Now we define a map* $d : L \to L$ *as follows: for all* $x \in L$,

$$d(x) = \begin{cases} c, & x = b, c \\ 1, & x = 1, a. \end{cases}$$

Then it is easy to verify that d is a right derivation, not a left derivation , because $d(1 \to b) = d(b) = c$, $(1 \to db) \sqcup_2 (d1 \to b) = c \sqcup_2 b = (c \rightsquigarrow b) \to b = b \to b = 1$, $c \neq 1$, *therefore, d is not a left derivation on pseudo L-algebra.*

Example 3.14. *In Example 3.6, we know that L is a pseudo L-algebra . Now we define a map* $d : L \to L$ *as follows: for all* $x \in L$,

$$d(x) = \begin{cases} 1, & x = b, c, 1 \\ a, & x = a. \end{cases}$$

Then it is easy to verify that d is a derivation on pseudo L-algebra.

Example 3.15. *Let* $L = \{0, a, b, c, 1\}$ *be a lattice, where* $0 < a < b$, $c < 1$, *b and c are incomparable. Define the operations* $\to$ *and* $\rightsquigarrow$ *using the following tables*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	0	a	1	c	1
c	0	a	c	1	1
1	0	a	b	c	1

Then $(L, \to, \rightsquigarrow, 1)$ *is a pseudo L-algebra. Consider the mapping d:* $L \to L$ *is given in the table below*

x	0	a	b	c	1
$d(x)$	0	a	b	c	1

We can see that d are derivation.

Example 3.16. *[23] Let* $L=\{0, a, b, c, 1\}$ *be a lattice such that* $0 < a < b, c < 1$, *b and c are incomparable and operation* $\to$ *and* $\rightsquigarrow$ *are defined as follows*

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	a	a	1	1
1	0	a	b	c	1

Then $(L, \rightarrow, \rightsquigarrow, 1)$ is an pseudo L-algebra. Now we define a map $d : L \rightarrow L$ as follows: for all $x \in L$,

$$d(x) = \begin{cases} 1, & x = 0, a, b, 1 \\ a, & x = c. \end{cases}$$

Then it is easily checked that $d(b \rightarrow c) = d(c) = a$, $(b \rightarrow d(c)) \sqcup_2 (d(b) \rightarrow c) = (b \rightarrow a) \sqcup_2 (1 \rightarrow c) = c \sqcup_2 c = c$, $a \neq c$, therefore, d is not a derivation on pseudo L-algebra.

Example 3.17. *In Example 3.16, we know that L is a pseudo L-algebra. Now we define a map $d : L \rightarrow L$ as follows: for all $x \in L$,*

$$d(x) = \begin{cases} 1, & x = a, b, c, 1 \\ 0, & x = 0. \end{cases}$$

We can get that d is a derivation.

Remark 3.18. Let L be a pseudo L-algebra, if $1_L, Id_L$: $L \rightarrow L$, defined by $1_L(x) = 1$ and $Id_L(x) = x$ for all $x \in L$, then $1_L, Id_L$ are both derivations, of which Id_L we call the identity derivation.

Proposition 3.19. *Any derivation d of a pseudo L-algebra satisfies $d1 = 1$.*

Proof. Since d is a derivation on a pseudo L-algebra, then d is both a left derivation and a right derivation.
When d is a left derivation we have
$d1 = d(1 \rightarrow 1) = (1 \rightarrow d1) \sqcup_2 (d1 \rightarrow 1) = d1 \sqcup_2 1 = 1$;
$d1 = d(1 \rightsquigarrow 1) = (1 \rightsquigarrow d1) \sqcup_1 (d1 \rightsquigarrow 1) = d1 \sqcup_1 1 = 1$;
When d is a right derivation we have
$d1 = d(1 \rightarrow 1) = (d1 \rightarrow 1) \sqcup_2 (1 \rightarrow d1) = 1 \sqcup_2 d1 = 1$;
$d1 = d(1 \rightsquigarrow 1) = (d1 \rightsquigarrow 1) \sqcup_1 (1 \rightsquigarrow d1) = 1 \sqcup_1 d1 = 1$.
Therefore, any derivation d of a pseudo L-algebra satisfies $d1 = 1$. □

Proposition 3.20. *Let L be a pseudo L-algebra. Then the following hold for all $x \in L$*
(1) if $d_l \in PLD^{(l)}(L)$, then $d_l x = d_l x \sqcup_1 x = d_l x \sqcup_2 x$;
(2) if $d_r \in PLD^{(r)}(L)$, then $d_r x = x \sqcup_1 d_r x = x \sqcup_2 d_r x$.

Proof. By Proposition 3.19, we get $d_l 1 = 1 = d_r 1$. Then
(1) $d_l(x) = d_l(1 \rightarrow x) = (1 \rightarrow d_l(x)) \sqcup_2 (d_l(1) \rightarrow x) = d_l(x) \sqcup_2 x$;
$d_l(x) = d_l(1 \rightsquigarrow x) = (1 \rightsquigarrow d_l(x)) \sqcup_1 (d_l(1) \rightsquigarrow x) = d_l(x) \sqcup_1 x$.

(2) $d_r(x) = d_r(1 \to x) = (d_r(1) \to x) \sqcup_2 (1 \to d_r(x)) = x \sqcup_2 d_r(x)$.
$d_r(x) = d_r(1 \rightsquigarrow x) = (d_r(1) \rightsquigarrow x) \sqcup_1 (1 \rightsquigarrow d_r(x)) = x \sqcup_1 d_r(x)$. □

Theorem 3.21. *Let L be a pseudo L-algebra, d be a derivation on L. If the pseudo L-algebra L is commutative of $\sqcup_1$ and commutative of $\sqcup_2$, then the left derivation is equal to the right derivation.*

Proof. Since L is $\sqcup_1$-commutative and $\sqcup_2$-commutative, we can get $a \sqcup_1 b = b \sqcup_1 a$, $a \sqcup_2 b = b \sqcup_2 a$.
$d(a \to b) = (a \to db) \sqcup_2 (da \to b) = (da \to b) \sqcup_2 (a \to db)$, $d(a \rightsquigarrow b) = (a \rightsquigarrow db) \sqcup_1 (da \rightsquigarrow b) = (da \rightsquigarrow b) \sqcup_1 (a \rightsquigarrow db)$.

Hence, we can get if L is commutative of $\sqcup_1$ and commutative of $\sqcup_2$. Then the left and right derivations are equal. □

Definition 3.22. Let L be a pseudo L-algebra and d be a derivation.
(1) d is called isotone derivation provided that $x \leq y$ implies $d(x) \leq d(y)$ for all $x, y \in L$;
(2) d is called a contractive derivation provided that $d(x) \leq x$ for all $x \in L$.
In particular, if d is both isotone and contractive, we call d the ideal derivation.

Proposition 3.23. *Let L be a pseudo L-algebra and d_l be a contractive left derivation. Then $d_l x = x$.*

Proof. Since d_l is a contractive left derivation. We can get $d_l x \leq x$, then $d_l x \rightsquigarrow x = 1$. $d_l x = d_l(1 \to x) = (1 \to d_l x) \sqcup_2 (d_l 1 \to x) = d_l x \sqcup_2 x = (d_l x \rightsquigarrow x) \to x = x$;
Similarly, we can get $d_l x = d_l(1 \rightsquigarrow x) = x$.
Hence $d_l x = x$. □

Remark 3.24. If L is a pseudo L-algebra and d is a derivation, we may not be able to get $dx \leq x$.

Example 3.25. *Consider the pseudo L-algebra L from Example 3.6. Then $L = \{a, b, c, 1\}$ is a pseudo L-algebra. Now we define a derivation d as follows*

$$d(x) = \begin{cases} b, & x = a, b, c \\ 1, & x = 1 \end{cases}$$

We can check that $da \leq a$ is not true.

Example 3.26. *Let $L = \{0, m, n, x, y, 1\}$. The Cayley tables for the operations $\to$ and $\rightsquigarrow$ are shown below.*

$\to$	0	m	n	x	y	1
0	1	1	1	1	1	1
m	y	1	y	1	y	1
n	m	m	1	1	1	1
x	0	m	y	1	y	1
y	m	m	x	x	1	1
1	0	m	n	x	y	1

$\rightsquigarrow$	0	m	n	x	y	1
0	1	1	1	1	1	1
m	n	1	n	1	n	1
n	m	m	1	1	1	1
x	0	m	y	1	y	1
y	m	m	x	x	1	1
1	0	m	n	x	y	1

We can check that L is a pseudo L-algebra. Now we define a derivation d as follows

$$d(x) = \begin{cases} 0, & x = 0, m, n, x \\ n, & x = y \\ 1, & x = 1. \end{cases}$$

We can check that d is an ideal derivation.

Theorem 3.27. *Identity derivation is an ideal derivation, but the reverse may not be true.*

Proof. we give a counter-example. In Example 3.26, we get that d is an ideal derivation where $dm = 0 \neq m$, so we know that d is not an identity derivation. □

Definition 3.28. Let L be a pseudo L-algebra and d be a derivation. Then d on L is said to be idempotent if $d^2 = d$, where $d^2(x) = d(d(x))$ or $d^2 = d \circ d$ for all $x \in L$.

Example 3.29. *Let $L = \{0, a, b, c, 1\}$ be a lattice, where $0 < a, b < c < 1$, a and b are incomparable. Define the operations $\to$ and $\rightsquigarrow$ using the following tables*

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	c	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(L, \to, \rightsquigarrow, 1)$ is a pseudo L-algebra. Consider the maps $d_1, d_2, d_3 \colon L \to L$ given in the table below

x	0	a	b	c	1
$d_1(x)$	0	a	b	c	1
$d_2(x)$	1	a	b	1	1
$d_3(x)$	1	1	1	c	1

We can get $d_1(x)$, $d_2(x)$ and $d_3(x)$ are idempotent derivations.

In Example 3.26, we know that d is a derivation, but $d^2y = d(dy) = dn = 0, dy = n$, $d^2y \neq dy$, so we can check that d is not an idempotent derivation.

Corollary 3.30. *Let L be a pseudo L-algebra. If d_l is a contractive left derivation, then d_l is idempotent derivation.*

Proof. According to Proposition 3.23, we get $d_l x = x$. Hence, we get $d_l(d_l x) = d_l x$ for all $x \in L$. □

Proposition 3.31. *Let L be a pseudo L-algebra and d be a derivation on L. Then $d(x \to dx) = d(x \rightsquigarrow dx) = d(dx \to x) = d(dx \rightsquigarrow x) = 1$, for all $x \in L$.*

Proof. For all $x \in L$, since d is a derivation of L, then we have
$d(x \to d(x)) = (x \to d(d(x)) \sqcup_2 (d(x) \to d(x)) = (x \to d(d(x)) \sqcup_2 1 = 1$;
$d(x \to d(x)) = (d(x) \to d(x)) \sqcup_2 (x \to d(d(x))) = 1 \sqcup_2 (x \to d(d(x)) = 1$;
Similarly, we can get $d(x \rightsquigarrow d(x)) = d(d(x) \to x) = d(d(x) \rightsquigarrow x) = 1$.

Hence $d(x \to dx) = d(x \rightsquigarrow dx) = d(dx \to x) = d(dx \rightsquigarrow x) = 1$, for all $x \in L$. □

Proposition 3.32. *Let L be a pseudo L-algebra and d be a derivation. If $x, y \in Fix_d(L)$, then $x \to y \in Fix_d(L), x \rightsquigarrow y \in Fix_d(L)$, where $Fix_d(L) = \{x \in L | dx = x\}$.*

Proof. Since $x, y \in Fix_d(L)$, we have $dx = x, dy = y$.
$d(x \to y) = (x \to dy) \sqcup_2 (dx \to y) = (x \to y) \sqcup_2 (x \to y) = x \to y$; $d(x \to y) = (dx \to y) \sqcup_2 (x \to dy) = (x \to y) \sqcup_2 (x \to y) = x \to y$. Similarly, we can get $d(x \rightsquigarrow y) = x \rightsquigarrow y$.

Hence, $x \to y \in Fix_d(L)$, $x \rightsquigarrow y \in Fix_d(L)$. □

Example 3.33. *In Example 3.29, we can easily check that $Fix_{d_1}(L) = L$, $Fix_{d_2}(L) = \{a, b, 1\}$, $Fix_{d_3}(L) = \{c, 1\}$.*

Corollary 3.34. *Let L be a pseudo L-algebra and d be a derivation. Then $Fix_d(L)$ is a subalgebra of L.*

Proof. According to Proposition 3.32. □

Proposition 3.35. *Let L be a pseudo L-algebra and d be a derivation. If $x, y \in Fix_d(L)$, then $x \sqcup_1 y \in Fix_d(L)$, $x \sqcup_2 y \in Fix_d(L)$.*

Proof. Let $x, y \in Fix_d(L)$, we have $dx = x$, $dy = y$. By Proposition 3.32, we get $d(x \to y) = x \to y$, $d(x \rightsquigarrow y) = x \rightsquigarrow y$. Then we get

$d(x \sqcup_1 y) = d((x \to y) \rightsquigarrow y) = ((x \to y) \rightsquigarrow dy) \sqcup_1 (d(x \to y) \rightsquigarrow y) = ((x \to y) \rightsquigarrow y) \sqcup_1 ((x \to y) \rightsquigarrow y) = (x \to y) \rightsquigarrow y = x \sqcup_1 y$, $d(x \sqcup_1 y) = d((x \to y) \rightsquigarrow y) = (d(x \to y) \rightsquigarrow y) \sqcup_1 (x \to y) \rightsquigarrow dy) = ((x \to y) \rightsquigarrow y) \sqcup_1 ((x \to y) \rightsquigarrow y) = (x \to y) \rightsquigarrow y = x \sqcup_1 y$. Similarly, we can get $d(x \sqcup_2 y) = x \sqcup_2 y$.

Hence $x \sqcup_1 y \in Fix_d(L)$, $x \sqcup_2 y \in Fix_d(L)$. □

Proposition 3.36. *Let L be a pseudo L-algebra with $\sqcup_1$-commutative and $\sqcup_2$-commutative and d be a derivation. If $x \in Ker(d)$ and $x \leq y$, then we have $y \in Ker(d)$, where $Ker(d) = \{x \in L | dx = 1\}$ for all $x, y \in L$.*

Proof. Let $x \in Kerd$ and $x \leq y$. Then $d(x) = 1$ and $x \to y = x \rightsquigarrow y = 1$.

$d(y) = d(1 \to y) = d((x \rightsquigarrow y) \to y) = d(x \sqcup_2 y) = d(y \sqcup_2 x) = d((y \rightsquigarrow x) \to x) = ((y \rightsquigarrow x) \to dx) \sqcup_2 (d(y \rightsquigarrow x) \to x) = ((y \rightsquigarrow x) \to 1) \sqcup_2 (d(y \rightsquigarrow x) \to x) = 1 \sqcup_2 (d(y \rightsquigarrow x) \to x) = 1$.

$d(y) = d(1 \to y) = d((x \rightsquigarrow y) \to y) = d(x \sqcup_2 y) = d(y \sqcup_2 x) = d((y \rightsquigarrow x) \to x) = (d(y \rightsquigarrow x) \to x) \sqcup_2 (y \rightsquigarrow x) \to dx) = (d(y \rightsquigarrow x) \to x) \sqcup_2 (y \rightsquigarrow x) \to 1) = (d(y \rightsquigarrow x) \to x) \sqcup_2 1 = 1$. Similarly, we can get $d(y) = d(1 \rightsquigarrow y) = 1$.

Hence $y \in Kerd$. This completes the proof. □

Proposition 3.37. *Let L be a pseudo L-algebra with $\sqcup_1$-commutative and $\sqcup_2$-commutative and d be a derivation. If $x \in Kerd$, then we have $x \sqcup_1 y \in Kerd$, $x \sqcup_2 y \in Kerd$ for all $y \in L$.*

Proof. d is a derivation and $x \in Kerd$. Then $d(x) = 1$. Hence we have for all $y \in L$,

$d(x \sqcup_1 y) = d(y \sqcup_1 x) = d((y \to x) \rightsquigarrow x) = ((y \to x) \rightsquigarrow dx) \sqcup_1 (d(y \to x) \rightsquigarrow x) = ((y \to x) \rightsquigarrow 1) \sqcup_1 (d(y \to x) \rightsquigarrow x) = 1 \sqcup_1 (d(y \to x) \rightsquigarrow x) = 1$.

$d(x \sqcup_1 y) = d(y \sqcup_1 x) = d((y \to x) \rightsquigarrow x) = (d(y \to x) \rightsquigarrow x) \sqcup_1 ((y \to x) \rightsquigarrow dx) = (d(y \to x) \rightsquigarrow x) \sqcup_1 ((y \to x) \rightsquigarrow 1) = (d(y \to x) \rightsquigarrow x) \sqcup_1 1 = 1$. Similarly, we can get $d(x \sqcup_2 y) = 1$.

Hence, $x \sqcup_1 y \in Kerd$, $x \sqcup_2 y \in Kerd$, for all $y \in L$. □

Proposition 3.38. *Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo L-algebra and d be a derivation. Then the following hold for all $x, y \in L$*
(1) *if $y \in Ker(d)$ then $x \sqcup_1 y, x \sqcup_2 y \in Ker(d)$;*
(2) *if $y \in Ker(d)$ then $x \to y, x \rightsquigarrow y \in Ker(d)$.*

Proof. (1) Let $y \in Ker(d)$, we have $dy = 1$. Then: $d(x \sqcup_1 y) = d((x \to y) \rightsquigarrow y) = ((x \to y) \rightsquigarrow dy) \sqcup_1 (d(x \to y) \rightsquigarrow y) = ((x \to y) \rightsquigarrow 1) \sqcup_1 (d(x \to y) \rightsquigarrow y) = 1 \sqcup_1 (d(x \to y) \rightsquigarrow y) = 1$;
$d(x \sqcup_1 y) = d((x \to y) \rightsquigarrow y) = (d(x \to y) \rightsquigarrow y) \sqcup_1 ((x \to y) \rightsquigarrow dy) = (d(x \to y) \rightsquigarrow y) \sqcup_1 ((x \to y) \rightsquigarrow 1) = (d(x \to y) \rightsquigarrow y) \sqcup_1 1 = 1$. Similarly, we can get $d(x \sqcup_2 y) = 1$.

Hence $x \sqcup_1 y$, $x \sqcup_2 y \in Ker(d)$.

(2) Let $y \in Ker(d)$, we have $dy = 1$. It follows that $d(x \to y) = (x \to dy) \sqcup_2 (dx \to y) = (x \to 1) \sqcup_2 (dx \to y) = 1 \sqcup_2 (dx \to y) = 1$;
$d(x \to y) = (dx \to y) \sqcup_2 (x \to dy) = (dx \to y) \sqcup_2 (x \to 1) = (dx \to y) \sqcup_2 1 = 1$. Similarly, we can get $d(x \rightsquigarrow y) = 1$.

Thus $x \to y, x \rightsquigarrow y \in Ker(d)$. □

Proposition 3.39. *Let L be a pseudo L-algebra and d be an isotone derivation. If $x \leq y$ and $x \in Kerd$, then $y \in Kerd$.*

Proof. Let $x \leq y$ and $x \in Kerd$. They we have $d(x) = 1$, and so $1 = d(x) \leq d(y)$, which implies $d(y) = 1$. □

Proposition 3.40. *Let L be a pseudo L-algebra and d be a derivation on L. Then $Ker(d)$ is a subalgebra of L.*

Proof. Since $d1 = 1$, it follows that $1 \in Ker(d)$. Let $x, y \in Ker(d)$, i.e. $dx = dy = 1$. Then we have $d(x \to y) = (x \to dy) \sqcup_2 (dx \to y) = (x \to 1) \sqcup_2 (1 \to y) = 1 \sqcup_2 y = 1$, hence $d(x \to y) = 1$, i.e. $x \to y \in Ker(d)$. Similarly, $x \rightsquigarrow y \in Ker(d)$, thus $Ker(d)$ is a subalgebra of L. □

4. Derivations of pseudo CKL-algebras

In this section, we mainly study the correlation properties of derivation of pseudo CKL-algebras, and get the relationship between contractive derivation and identity derivation of pseudo CKL-algebras.

Definition 4.1. A pseudo L-algebra L is said to be pseudo CKL-algebra if it satisfies condition: $a \to (b \rightsquigarrow c) = b \rightsquigarrow (a \to c)$ for all $a, b, c \in L$.

Proposition 4.2. *Let L be a pseudo CKL-algebra, then the following hold for all $a, b, c \in L$.*
(1) $a \leq a \sqcup_1 b$ *and* $a \leq a \sqcup_2 b$;
(2) *If L is bounded, then* $a \leq a^{-\sim}$ *and* $a \leq a^{\sim -}$;
(3) *If L is bounded, then* $a \to b^{-\sim} = b^{-} \rightsquigarrow a^{-}, a \rightsquigarrow b^{\sim -} = b^{\sim} \to a^{\sim}$;
(4) *L is a pseudo KL-algebra;*
(5) $a \to b \leq (b \to c) \rightsquigarrow (a \to c)$ *and* $a \rightsquigarrow b \leq (b \rightsquigarrow c) \to (a \rightsquigarrow c)$;
(6) *If L is bounded, then* $a \to b \leq b^{-} \rightsquigarrow a^{-}, a \rightsquigarrow b \leq b^{\sim} \to a^{\sim}$;
(7) *If* $a \leq b \rightsquigarrow c$, *then* $b \leq a \to c$; *If* $a \leq b \to c$, *then* $b \leq a \rightsquigarrow c$.

Proof. (1) Let $a, b \in L$. Since L is pseudo CKL-algebra, we have $a \to ((a \to b) \rightsquigarrow b) = (a \to b) \rightsquigarrow (a \to b) = 1$ and $a \rightsquigarrow ((a \rightsquigarrow b) \to b) = (a \rightsquigarrow b) \to (a \rightsquigarrow b) = 1$, so $a \leq (a \to b) \rightsquigarrow b = a \sqcup_1 b$ and $a \leq (a \rightsquigarrow b) \to b = a \sqcup_2 b$.

(2) By (1), taking $b = 0$, we have $a \leq a^{-\sim}$ and $a \leq a^{\sim -}$ for all $a \in L$.

(3) Let $a, b \in L$, by pseudo CKL-algebra definition, we have $a \to b^{-\sim} = a \to (b^{-} \rightsquigarrow 0) = b^{-} \rightsquigarrow a^{-}$ and $a \rightsquigarrow b^{\sim -} = a \rightsquigarrow (b^{\sim} \to 0) = b^{\sim} \to (a \rightsquigarrow 0) = b^{\sim} \to a^{\sim}$.

(4) Let $a, b \in L$, we have $a \to (b \rightsquigarrow a) = b \rightsquigarrow (a \to a) = b \rightsquigarrow 1 = 1$ and $a \rightsquigarrow (b \to a) = b \to (a \rightsquigarrow a) = b \to 1 = 1$, that is $a \leq b \rightsquigarrow a$ and $a \leq b \to a$, hence L is a pseudo KL-algebra.

(5) Let $a, b, c \in L$. Since L is pseudo CKL-algebra, by (4), we have $(a \to b) \to ((b \to c) \rightsquigarrow (a \to c)) = (b \to c) \rightsquigarrow ((a \to b) \to (a \to c)) = (b \to c) \rightsquigarrow ((b \to a) \to (b \to c)) = 1$, hence $a \to b \leq (b \to c) \rightsquigarrow (a \to c)$. Similarly, we get $(a \rightsquigarrow b) \rightsquigarrow ((b \rightsquigarrow c) \to (a \rightsquigarrow c)) = (b \rightsquigarrow c) \to ((a \rightsquigarrow b) \rightsquigarrow (a \rightsquigarrow c)) = (b \rightsquigarrow c) \to ((b \rightsquigarrow a) \rightsquigarrow (b \rightsquigarrow c)) = 1$, hence we get $a \rightsquigarrow b \leq (b \rightsquigarrow c) \to (a \rightsquigarrow c)$.

(6) By (5), taking $c = 0$, we have $a \to b \leq b^{-} \rightsquigarrow a^{-}$ and $a \rightsquigarrow b \leq b^{\sim} \to a^{\sim}$ for all $a, b \in L$.

(7) Since $a \leq b \rightsquigarrow c$, then $a \to (b \rightsquigarrow c) = b \rightsquigarrow (a \to c) = 1$. Hence $b \leq a \to c$. Similarly, we can get that if $a \leq b \to c$, then $b \leq a \rightsquigarrow c$. □

Remark 4.3. Every pseudo CKL-algebra is a pseudo BCK-algebra, but the reverse may not be true.

Proof. According to the definition of pseudo L-algebra and Proposition4.2(5). We get every pseudo CKL-algebra is a pseudo-BCK algebra.

In turn, we give a counterexample below □

Example 4.4. *[12] Let $A = \{o_1, a_1, b_1, c_1, o_2, a_2, b_2, c_2, 1\}$ with $o_1 < a_1$, $b_1 < c_1 < 1$ and a_1, b_1 incomparable, $o_2 < a_2$, $b_2 < c_2 < 1$ and a_2, b_2 incomparable. Assume that any element of the set $\{o_1, a_1, b_1, c_1\}$ is incomparable with any element of the set $\{o_2, a_2, b_2, c_2\}$. Consider the operations $\rightarrow$, $\rightsquigarrow$ given by the tables*

$\rightarrow$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	o_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	a_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	o_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	c_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	o_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

$\rightsquigarrow$	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1
o_1	1	1	1	1	o_2	a_2	b_2	c_2	1
a_1	b_1	1	b_1	1	o_2	a_2	b_2	c_2	1
b_1	o_1	a_1	1	1	o_2	a_2	b_2	c_2	1
c_1	o_1	a_1	b_1	1	o_2	a_2	b_2	c_2	1
o_2	o_1	a_1	b_1	c_1	1	1	1	1	1
a_2	o_1	a_1	b_1	c_1	b_2	1	b_2	1	1
b_2	o_1	a_1	b_1	c_1	b_2	c_2	1	1	1
c_2	o_1	a_1	b_1	c_1	b_2	c_2	b_2	1	1
1	o_1	a_1	b_1	c_1	o_2	a_2	b_2	c_2	1

Then $\mathcal{A}$=$(A, \leq, \rightarrow, \rightsquigarrow, 1)$ is a proper pseudo-BCK algebra. Since $(a_1 \rightarrow b_1) \rightarrow (a_1 \rightarrow o_1) = b_1 \rightarrow o_1 = a_1$, $(b_1 \rightarrow a_1) \rightarrow (b_1 \rightarrow o_1) = a_1 \rightarrow a_1 = 1$, $a_1 \neq 1$, we get pseudo-BCK algebra $\mathcal{A}$ that is not a pseudo CKL-algebra.

Proposition 4.5. *Let L be a pseudo CKL-algebra. Then $a \sqcup_1 b$ and $a \sqcup_2 b$ are upper bound $\{a, b\}$, respectively, for any $a, b \in L$.*

Proof. By Proposition4.2(1), we get $a \leq a \sqcup_1 b$ and $a \leq a \sqcup_2 b$ for all $a, b \in L$. Since L is a pseudo CKL-algebra. Then for any $a, b \in L$, we get

$$b \rightarrow (a \sqcup_1 b) = b \rightarrow ((a \rightarrow b) \rightsquigarrow b) = (a \rightarrow b) \rightsquigarrow (b \rightarrow b) = (a \rightarrow b) \rightsquigarrow 1 = 1,$$
$$b \rightsquigarrow (a \sqcup_2 b) = b \rightsquigarrow ((a \rightsquigarrow b) \rightarrow b) = (a \rightsquigarrow b) \rightarrow (b \rightsquigarrow b) = (a \rightsquigarrow b) \rightarrow 1 = 1,$$

i.e.,$b \le a \sqcup_1 b, b \le a \sqcup_2 b$.

Hence, $a \sqcup_1 b$ and $a \sqcup_2 b$ are upper bound $\{a, b\}$, respectively, for any $a, b \in L$. □

Example 4.6. *In Example 3.6, we know that L is a pseudo KL-algebra.* $a \sqcup_2 b = (a \rightsquigarrow b) \to b = c \to b = b$. *We can get an upper bound that* $a \sqcup_2 b$ *is not* $\{a, b\}$.

Proposition 4.7. *Let L be a pseudo CKL-algebra,*
(1) *If L is* $\sqcup_1$*-commutative, then* $x \sqcup_1 y$ *is the l.u.b.*$\{x, y\}$*, for all* $x, y \in L$*;*
(2) *If L is* $\sqcup_2$*-commutative, then* $x \sqcup_2 y$ *is the l.u.b.*$\{x, y\}$*, for all* $x, y \in L$.

Proof. (1) Let $x, y \in L$. According to Proposition 4.5, $x \sqcup_1 y$ is an upper bound $\{x, y\}$. Let z be another upper bound $\{x, y\}$, i.e. $x \le z$ and $y \le z$. We will prove that $x \sqcup_1 y \le z$. Indeed, applying Lemma 3.9(2) and taking into consideration that L is satisfied $y \sqcup_1 z = z \sqcup_1 y$ we have

$$x \sqcup_1 y \to z = x \sqcup_1 y \to y \sqcup_1 z = x \sqcup_1 y \to z \sqcup_1 y = ((x \to y) \rightsquigarrow y) \to ((z \to y) \rightsquigarrow y).$$

According to Proposition 4.2(5) we have $(b \to c) \rightsquigarrow (a \to c) \ge a \to b, (b \rightsquigarrow c) \to (a \rightsquigarrow c) \ge a \rightsquigarrow b$ and replacing a with $z \to y$, b with $x \to y$ and c with y we get

$$((x \to y) \rightsquigarrow y) \to ((z \to y) \rightsquigarrow y) \ge (z \to y) \rightsquigarrow (x \to y) \ge x \to z.$$

Hence $x \sqcup_1 y \to z \ge x \to z = 1$(since $x \le z$). It follows that $x \sqcup_1 y \to z = 1$, thus $x \sqcup_1 y \le z$. We conclude that $x \sqcup_1 y$ is the l.u.b.$\{x, y\}$.

(2) Similar to (1), We conclude that $x \sqcup_2 y$ is the l.u.b.$\{x, y\}$. □

Let L be a pseudo CKL-algebra. If L is $\sqcup_1$-commutative and $\sqcup_2$-commutative, we denote $a \vee b = a \sqcup_1 b = a \sqcup_2 b$ (since $a \sqcup_1 b$ and $a \sqcup_2 b$ are the l.u.b.$\{a, b\}$).

Corollary 4.8. *Let L be a pseudo CKL-algebra. Then*
(1) *If L is* $\sqcup_1$*-commutative, then* $x \sqcup_1 y \le x \sqcup_2 y, y \sqcup_2 x$ *for all* $x, y \in L$*;*
(2) *If L is* $\sqcup_2$*-commutative, then* $x \sqcup_2 y \le x \sqcup_1 y, y \sqcup_1 x$ *for all* $x, y \in L$.

Proof. (1) According to Proposition 4.5, $x \sqcup_2 y$ and $y \sqcup_2 x$ are upper bounds $\{x, y\}$. By Proposition 4.7, $x \sqcup_1 y$ is the l.u.b. $\{x, y\}$, thus $x \sqcup_1 y \le x \sqcup_2 y, y \sqcup_2 x$.

(2) Similar to (1). □

Corollary 4.9. *Let* $(L, \to, \rightsquigarrow, 1)$ *be a pseudo CKL-algebra with* $\sqcup_1$*-commutative and* $\sqcup_2$*-commutative and d be a derivation. Then the following hold for all* $a, b \in L$,
(*cpcld*1) $d(a \to b) = (a \to d(b)) \vee (d(a) \to b)$*;*
(*cpcld*2) $d(a \rightsquigarrow b) = (a \rightsquigarrow d(b)) \vee (d(a) \rightsquigarrow b)$.

Proof. Since $(L, \to, \rightsquigarrow, 1)$ be a pseudo CKL-algebra with $\sqcup_1$-commutative and $\sqcup_2$-commutative, we can get $a \vee b = a \sqcup_1 b = b \sqcup_1 a = a \sqcup_2 b = b \sqcup_2 a$.

According to Definition 3.10 and 3.11, we get: $d(a \to b) = (a \to db) \sqcup_2 (da \to b) = (da \to b) \sqcup_2 (a \to db) = (a \to db) \vee (da \to b)$, $d(a \rightsquigarrow b) = (a \rightsquigarrow db) \sqcup_1 (da \rightsquigarrow b) = (da \rightsquigarrow b) \sqcup_1 (a \rightsquigarrow db) = (a \rightsquigarrow db) \vee (da \rightsquigarrow b)$.

Therefore, $d(a \to b) = (a \to db) \vee (da \to b)$, $d(a \rightsquigarrow b) = (a \rightsquigarrow db) \vee (da \rightsquigarrow b)$. □

Proposition 4.10. *Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo CKL-algebra and d be a derivation. Then: $x \leq d(x)$ for all $x \in L$.*

Proof. Since $(L, \to, \rightsquigarrow, 1)$ is a pseudo CKL-algebra and d is a derivation. Then for all $x \in L$
$x \rightsquigarrow dx = x \rightsquigarrow d(1 \to x) = x \rightsquigarrow ((1 \to dx) \sqcup_2 (d1 \to x)) = x \rightsquigarrow (dx \sqcup_2 x) = x \rightsquigarrow ((dx \rightsquigarrow x) \to x) = (dx \rightsquigarrow x) \to (x \rightsquigarrow x) = (dx \rightsquigarrow x) \to 1 = 1$;
$x \rightsquigarrow dx = x \rightsquigarrow d(1 \to x) = x \rightsquigarrow ((d1 \to x) \sqcup_2 (1 \to dx)) = x \rightsquigarrow (x \sqcup_2 dx) = x \rightsquigarrow ((x \rightsquigarrow dx) \to dx) = (x \rightsquigarrow dx) \to (x \rightsquigarrow dx) = 1$;
Similarly, we can get $x \to dx = 1$.

Hence $x \leq dx$ for all $x \in L$. □

Theorem 4.11. *Let L be a pseudo CKL-algebra and d be an contractive derivation. We get that d is an identity derivation.*

Proof. By Proposition 4.10 and Definition 3.22(2), we can get $dx = x$. □

Proposition 4.12. *Let L be a pseudo CKL-algebra and d be a derivation. Then*
(1) *if L is $\sqcup_1$-commutative and d is isotone, then $dx \sqcup_1 dy \leq d(x \sqcup_1 y)$, for all $x, y \in L$;*
(2) *if L is $\sqcup_2$-commutative and d is isotone, then $dx \sqcup_2 dy \leq d(x \sqcup_2 y)$, for all $x, y \in L$.*

Proof. (1) From $x \leq x \sqcup_1 y$ and $y \leq x \sqcup_1 y$ we get $dx \leq d(x \sqcup_1 y)$ and $dy \leq d(x \sqcup_1 y)$. Hence $d(x \sqcup_1 y)$ is an upper bound $\{dx, dy\}$. Since L is $\sqcup_1$-commutative, according to Proposition 4.7, $dx \sqcup_1 dy$ is the lower upper bound $\{dx, dy\}$. Hence $dx \sqcup_1 dy \leq d(x \sqcup_1 y)$, for all $x, y \in L$.

(2) Similar to (1). □

Proposition 4.13. *Let L be a pseudo CKL-algebra and let $d_r \in PLD^{(r)}(L)$. Then the following holds*
(a) *$d_r x \to y \leq d_r x \to d_r y \leq x \to d_r y = d_r(x \to y)$,for all $x, y \in L$.*
(b) *$d_r x \rightsquigarrow y \leq d_r x \rightsquigarrow d_r y \leq x \rightsquigarrow d_r y = d_r(x \rightsquigarrow y)$,for all $x, y \in L$.*

Proof. (a) By Proposition 4.10, we get $x \leq d_r x$ and $y \leq d_r y$, using Proposition 3.7 and Lemma 2.7, we have $d_r x \to y \leq d_r x \to d_r y$, $d_r x \to d_r y \leq x \to d_r y$. It follows that: $d_r x \to y \leq d_r x \to d_r y \leq x \to d_r y = 1 \to (x \to d_r y) = ((d_r x \to y) \rightsquigarrow (x \to d_r y)) \to (x \to d_r y) = (d_r x \to y) \sqcup_2 (x \to d_r y) = d_r(x \to y)$.
(b) Similar to (a). □

Proposition 4.14. *Let L be a pseudo CKL-algebra and d_1, d_2 are idempotent and isotone derivations and $d_1 \leq d_2$. (i.e.$d_1x \leq d_2x$, for all $x \in L$). Then $d_2 \circ d_1 = d_2$.*

Proof. Let $x \in L$. By Proposition 4.10, $d_2x \leq d_2d_1x = (d_2 \circ d_1)(x)$, so $d_2 \leq d_2 \circ d_1$. Moreover, since $d_1x \leq d_2x$ we have $d_2d_1x \leq d_2d_2x = d_2x$, i.e. $d_2 \circ d_1 \leq d_2$.
Hence $d_2 \circ d_1 = d_2$. □

Definition 4.15. Let L be a pseudo L-algebra and I be a non-empty subset of L. Then I is called a preideal of L if
$(I1)$ $1 \in I$;
$(I2)$ If $x \in I$ and $x \to y \in I$ or $x \rightsquigarrow y \in I$, then $y \in I$.

Example 4.16. *In Example 3.26, we can easily check that $I = \{1, m, n\}$ is a preideal of L.*

Theorem 4.17. *Let L be a pseudo CKL-algebra and d be a derivation. Then every $Fix_d(L)$ is an preideal.*

Proof. (i) Since $d1 = 1$, we get $1 \in Fix_d(L)$;
(ii) According to Proposition 4.10, we get $y \leq dy$, $x \to y \leq x \to dy$.
Let $x \in Fix_d(L)$, $x \to y \in Fix_d(L)$
$x \to y = d(x \to y) = (x \to dy) \sqcup_2 (dx \to y) = (x \to dy) \sqcup_2 (x \to y) = x \to dy$;
When $x = 1$, we have $1 \to y = 1 \to dy$, $y = dy$.
Hence, $y \in Fix_d(L)$. We can get every $Fix_d(L)$ is an ideal. □

Definition 4.18. Let L be a pseudo L-algebra. A non-empty subset I of L is said to be a d-invariant if $d(I) \subseteq I$ where $d(I) = \{d(x) \mid x \in I\}$.

Example 4.19. *Let L be a pseudo L-algebra and d be a derivation. We can easily check that L is a d-invariant.*

Example 4.20. *In Example 3.29, we can easily check that $I = \{a, b, 1\}$ is a d-invariant, where d is $d_3(x)$.*

Theorem 4.21. *Let L be a pseudo CKL-algebra and d be a derivation. Then every preideal I is a d-invariant.*

Proof. Let I be a perideal of L and $y \in d(I)$. Then $y = d(x)$ for some $x \in I$. By Proposition 4.10, $x \leq d(x)$, we can get $x \to d(x) = x \rightsquigarrow d(x) = 1$. Then $x \to y = x \to d(x) = 1 \in I$, $x \rightsquigarrow y = x \rightsquigarrow d(x) = 1 \in I$, which implies $y \in I$. Thus $d(I) \subseteq I$. Hence I is a d-invariant. □

Proposition 4.22. *Let d_r be an isotone right derivation of a pseudo CKL-algebra. If d_r is idempotent, then $Ker(d_r)$ is a preideal of L.*

Proof. Clearly, $1 \in Ker(d_r)$. Let $x \in Ker(d_r)$ and $x \to y \in Ker(d_r)$. Then we have $1 = d_r(x \to y) = x \to d_ry$, $1 = d_r(x \rightsquigarrow y) = x \rightsquigarrow d_ry$ from Proposition 4.13(1), which means $x \leq d_ry$. Hence we get $1 = d_rx \leq d_r^2y = d_ry$, i.e., $d_ry = 1$. This implies $y \in Ker(d_r)$. This completes the proof. □

Theorem 4.23. *Let L be a pseudo L-algebra and d be a derivation on L. Then the following are equivalent*
(a) *$d(x \to y) = dx \to y$ or $d(x \rightsquigarrow y) = dx \rightsquigarrow y$ for all $x, y \in L$;*
(b) *d is the identity derivation.*

Proof. $(a) \Rightarrow (b)$ Indeed, $dx = d(1 \to x) = d1 \to x = 1 \to x = x$ for all $x \in L$, i.e. $dx = x$. Similarly for $dx = d(1 \rightsquigarrow x) = d1 \rightsquigarrow x = 1 \rightsquigarrow x = x$ for all $x \in L$, i.e. $dx = x$. Hence, d is the identity derivation.
$(b) \Rightarrow (a)$ Obviously. □

Proposition 4.24. *Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo KL-algebra with negation and d be an isotone derivation on L. Then we have: if $x \leq y$, then $d(x^-) \geq d(y^-)$, $d(x^\sim) \geq d(y^\sim)$ for all $x, y \in L$.*

Proof. If $x \leq y$, then $x \to 0 \geq y \to 0$. Since d is an isotone derivation, therefore, $d(x^-) \geq d(y^-)$; If $x \leq y$, then $x \rightsquigarrow 0 \geq y \rightsquigarrow 0$. Since d is an isotone derivation, therefore, $d(x^\sim) \geq d(y^\sim)$. □

Theorem 4.25. *Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo CKL-algebra with negation and d be a derivation on L. Then the following are aquivalent*
(a) *d is an ideal derivation and $d^2 = d$;*
(b) *$d(x) \to d(y) = d(x) \to y, d(x) \rightsquigarrow d(y) = d(x) \rightsquigarrow y$, for all $x, y \in L$*

Proof. $(a) \Rightarrow (b)$ According to Proposition 4.11, we can get $dx = x$. Therefore, (b) is true.
$(b) \Rightarrow (a)$ Assume that $d(x) \to d(y) = d(x) \to y, d(x) \rightsquigarrow d(y) = d(x) \rightsquigarrow y$, for all $x, y \in L$. Then $d(x) \to d(x) = d(x) \to x = 1, d(x) \rightsquigarrow d(x) = d(x) \rightsquigarrow x = 1$, which implies $d(x) \leq x$ for all $x \in L$. Thus d is contractive. Moreover, for all $x, y \in L$, let $x \leq y$, we have $d(x) \leq x \leq y$. It follows that $d(x) \to y = d(x) \to d(y) = 1, d(x) \rightsquigarrow y = d(x) \rightsquigarrow d(y) = 1$, which implies $d(x) \leq d(y)$. Hence d is isotone. Therefore, d is an ideal derivation on L. Finally, by Proposition 4.11, we get $d(x) = x$ for all $x \in L$. Therefore, $d^2 = d$. □

Proposition 4.26. *Let $(L, \to, \rightsquigarrow, 1)$ be a pseudo CKL-algebra and d be an ideal derivation. Then we have: for all $x, y \in L$, if $y \leq x^-$, then $dx \leq y^\sim$; if $y \leq x^\sim$, then $dx \leq y^-$.*

Proof. For all $x, y \in L$, assume that $y \leq x^-$. By Proposition 4.2(2) and Proposition 3.7, we get $x \leq x^{-\sim} \leq y^\sim$. According to Proposition 4.10 and d is an ideal derivation, $dx \leq dy^\sim = y^\sim$; Similarly, we can get $dx \leq dy^- = y^-$. □

5. **Conclusion**

The article focuses on the concept of derivations on pseudo L-algebras, which serve as a valuable tool for studying pseudo L-algebraic structure. By utilizing derivations on pseudo L-algebras, the article establishes the connection between the identity derivation and the ideal derivation. It is proven that while the identity derivation is indeed the ideal derivation, the reverse is not true.

Additionally, the article presents several findings regarding the derivation of pseudo CKL-algebras, which contribute to the understanding and analysis of pseudo L-algebras. These results offer practical insights for further research on pseudo L-algebras. The article acknowledges that the current definition of derivation for pseudo L-algebras is provided only for a specific case. However, when we replace $\sqcup_1$ with $\sqcup_2$ in $(pld1)$ and $\sqcup_2$ with $\sqcup_1$ in $(pld2)$, we will obtain a new definition of the left derivative. Similarly, we can also obtain a new definition of the right derivative. So, can this new definition of the left and right derivatives better study pseudo L-algebras? This would also be meaningful. In recent years, many scholars have devoted themselves to combining various logical algebras with derivation theory. For example, BCI-algebra, BCC-algebra, BCK-algebra, etc. We have found that due to the different properties of algebras themselves, the different properties of algebraic derivations are also presented. As a classical type of logical algebra, pseudo L-algebra can serve as the research foundation for other logical algebras. Currently, there are still many derivations in logical algebras that have not been considered. Therefore, in the future, we can try to study derivations in other logical algebras.

6. Author Contributions

All authors contributed to this article .

7. Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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9. Ethical considerations

We declare that we have complied with the ethical standards for publishing articles in this journal.

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11. Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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