Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

WEIGHTED DIFFERENTIATION COMPOSITION OPERATORS ON THE $Q_K(p,q)$ SPACES AND THEIR ESSENTIAL NORMS

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Article type: Research Article

(Received: 22 January 2024, Received in revised form 10 July 2024)

(Accepted: 09 August 2024, Published Online: 15 August 2024)

Abstract. In this paper, firstly we obtain characterization for boundedness of the weighted differentiation composition operator from $Q_K(p,q)$ space into weighted Zygmund space. Then we give an estimation for the essential norm of such an operator on the mentioned spaces. As an application, we present a characterization for the compactness of the above operator.

Keywords: Boundedness, Compactness, Essential norm, Zygmund type space.
2020 MSC: Primary 47B38, 30H30.

1. Introduction

Let $\mathbb{D}$ be unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. The class of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\| = \sup \frac{\left|f\left(e^{i(\theta+h)}\right) + f\left(e^{i(\theta-h)}\right) - 2f\left(e^{i\theta}\right)\right|}{h} < \infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial\mathbb{D}$ and $h > 0$, is denoted by $\mathcal{Z}$ and called Zygmund space. The Closed Graph Theorem together [4, Theorem 5.3] implies that $f \in \mathcal{Z}$ if and only if

$$\sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) |f''(z)| < \infty.$$

The space $\mathcal{Z}$ is a Banach space with the following norm

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) |f''(z)|.$$

Let μ be a weight, which means that a positive continuous function on $\mathbb{D}$. The weighted Zygmund space $\mathcal{Z}_\mu$ is defined as the space of all analytic functions f on $\mathbb{D}$ for which $\sup_{z\in\mathbb{D}} \mu(z)|f''(z)| < \infty$ and the norm on this space is

$$\|f\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + \mu(z)|f''(z)|.$$

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https://doi.org/10.22103/jmmr.2024.22850.1569

Publisher: Shahid Bahonar University of Kerman
How to cite: M. Hassanlou, E. Abbasi, A. Ebrahimi, *Weighted differentiation composition operators on the $Q_K(p,q)$ spaces and their essential norms*, J. Mahani Math. Res. 2025; 14(1): 107-119.

When $\mu(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, we have $\mathcal{Z}^\alpha$ which is known as Zygmund type space and in the special case if $\alpha = 1$ then $\mathcal{Z}^1 = \mathcal{Z}$. For more information on these spaces and operators on them see for instance [7,10–12,17,18,26,27], and the related references therein.

Let $p > 0$, $q > -2$ and $K : [0,\infty) \longrightarrow [0,\infty)$ be a nondecreasing continuous function. The space $\mathcal{Q}_K(p,q)$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{\mathcal{Q}_K(p,q)} = |f(0)| + \sup_{\xi\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \left(1-|z|^2\right)^q K(g(z,\xi))\, dA(z) < \infty,$$

where dA is the normalized Lebesgue area measure in $\mathbb{D}$, $g(z,\xi) = \log \frac{1}{|\varphi_\xi(z)|}$, and $\varphi_\xi(z) = \frac{\xi - z}{1-\bar{\xi} z}$. For $p \geq 1$, $\mathcal{Q}_K(p,q)$ with the norm $\|f\|_{\mathcal{Q}_K(p,q)}$ becomes a Banach space, see [14–16, 20, 22], for more details regarding $\mathcal{Q}_K(p,q)$ spaces. Following [20], we assume that the following condition holds

$$\int_0^1 \left(1-r^2\right)^q \ K(-\log\ r) r\ dr < \infty,$$

since otherwise $Q_K(p,q)$ consists only of constant functions.

With different choices on K, p, q, many classical functions spaces such as Bloch space, Hardy space, BMOA, Q_s and Q_K can be obtained, see [2, 3, 6, 19, 21, 23]. If $K(x) = x^s$, $s \geq 0$, we get $F(p,q,s)$ spaces. About Bloch spaces, there is a useful discussion. If $f \in \mathcal{Q}_K(p,q)$ then $f \in \mathcal{B}^{\frac{q+2}{p}}$ and

$$\|f\|_{\mathcal{B}^{\frac{q+2}{p}}} \leq C\|f\|_{\mathcal{Q}_K(p,q)},$$

where $\mathcal{B}^\alpha$, $\alpha > 0$, denotes the Bloch type space (or α-Bloch space), see [20]. We need for the following fact about the functions in $\mathcal{B}^\alpha$ (see [24]):

$$\|f\|_{\mathcal{B}^\alpha} \approx \sum_{i=0}^n |f^{(i)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n} |f^{(n+1)}(z)|, \tag{1}$$

where $n \in \mathbb{N}$. Moreover $Q_K(p,q) = \mathcal{B}^{\frac{q+2}{p}}$ if and only if

$$\int_0^1 (1-r^2)^{-2} K(-\log r) r dr < \infty. \tag{2}$$

A function h with derivative $|h'(z)|^p = |1-z|^{-q-2}$ represents an extremal growth in $\mathcal{B}^{\frac{q+2}{p}}$ and by Lemma 2.7 [9], there exist parameters p, q and K such that $h \in Q_K(p,q) \subsetneq \mathcal{B}^{\frac{q+2}{p}}$. Also taking $K(t) = t^s$, $0 < s < 1$ and $q > -s-1$ imply that $Q_K(p,q)$ coincides with the non-trivial $F(p,q,s)$ space.

For an analytic self-mapping φ on $\mathbb{D}$, the composition operator C_φ is defined as follows

$$C_\varphi(f)(z) = f(\varphi(z)), \qquad f \in H(\mathbb{D}).$$

The weighted composition operator is defined by $uC_\varphi f(z) = u(z) f(\varphi(z))$, $u \in H(\mathbb{D})$, which is a combination of composition and multiplication operators. For

$m \in \mathbb{N}$, the weighted differentiation composition operator $D^m_{u,\varphi}$ is as follows

$$D^m_{u,\varphi} f(z) = u(z) f^{(m)}(\varphi(z)).$$

This kind of operator includes some classical operators like D, DC_φ, $C_\varphi D$ and so on. The above operator has been investigated by many authors, see, e.g. [5, 7, 11, 17, 18, 22, 25–27].

Boundedness and compactness of integral-type operator from $Q_K(p,q)$ spaces to α-Bloch space and Zygmund-type spaces investigated in [15] and [16], respectively. The essential norm of generalized weighted composition operators from the Bloch-type spaces to the weighted Zygmund spaces estimated in [1]. Also essential norm of Stevic-Sharma operator from $Q_K(p,q)$ space to Zygmund-type space approximated in [8]. Recently Manavi et al., found an estimation for the essential norm of generalized integral type operator from $Q_K(p,q)$ to Zygmund Spaces, see [13]. The motivation of the paper is to continue the line of finding an approximation for the essential norm of the operators. The application of the results is to find necessary and sufficient conditions for the compactness of the operator.

In this paper, firstly we investigate boundedness of the weighted differentiation composition operator $D^m_{u,\varphi}$, $m \in \mathbb{N}$ from $\mathcal{Q}_K(p,q)$ into $\mathcal{Z}_\mu$. Then we obtain an estimate for the essential norm of such operators. As an application we present a characterization for the compactness of the operator $D^m_{u,\varphi}$ on mentioned spaces.

For any bounded operator T between two Banach spaces X and Y, the essential norm of T is denoted by $\|T\|_{e,X\to Y}$ and is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T - S\| : \ S \text{ is a compact operator from } X \text{ to } Y\}.$$

The operator is compact if and only if $\|T\|_{e,X\to Y} = 0$.

If $m = 0$ then we have uC_φ where properties of it between $\mathcal{Q}_K(p,q)$ and $\mathcal{Z}_\mu$ are proved, among other general results, in [8]. This result is that $uC_\varphi : \mathcal{Q}_K(p,q) \to \mathcal{Z}_\mu$ is bounded if and only if

(1) $q+2 > p$, $\quad \sup_{z\in\mathbb{D}} \frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}-1}} < \infty$,

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty, \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}} < \infty$$

(2) $q+2 = p$, $\quad \sup_{z\in\mathbb{D}} \mu(z)|u''(z)| \ln \frac{e}{1-|\varphi(z)|^2} < \infty$,

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty, \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}} < \infty$$

(3) $q+2 < p$, $u \in \mathcal{Z}_\mu$,

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}}} < \infty, \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\frac{q+2}{p}+1}} < \infty$$

The author of [8] also presented some necessary and sufficient conditions for the compactness.

Throughout this paper, we assume that the following condition holds

$$\int_0^1 K(-\log r)(1-r)^{\min\{-1,q\}}\left(\log\frac{1}{1-r}\right)^{\chi_{-1}(q)} rdr<\infty,$$

where $\chi_O(x)$ is the characteristic function of the set O. For simplifying calculations, we set $\alpha=\frac{q+2}{p}$ and

$$A(u,\varphi,\alpha)=\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^\alpha},$$

$$B(u,\varphi,\alpha)=\limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u(z)|}{(1-|\varphi(z)|^2)^\alpha}.$$

All positive constants will be denoted by C which may vary from one occurrence to another. By $A\gtrsim B$ we mean there exists a constant C such that $A\geq CB$ and $A\approx B$ means that $A\gtrsim B\gtrsim A$.

The following two theorems are our main results.

Theorem A. Let $p>0$, $q>-2$, $K:[0,\infty)\to[0,\infty)$ be a nondecreasing continuous function, φ be an analytic self-map of $\mathbb{D}$ and u be an analytic function on $\mathbb{D}$. Then $D^m_{u,\varphi}:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ is bounded if and only if

$$A(u'',\varphi,\alpha+m-1)<\infty,\ A(2u'\varphi'+u\varphi'',\varphi,\alpha+m)<\infty,\ A(u\varphi'^2,\varphi,\alpha+m+1)<\infty,$$

where $\alpha=\frac{q+2}{p}$.

Theorem B. Let $p>0$, $q>-2$, $K:[0,\infty)\to[0,\infty)$ be a nondecreasing continuous function, φ be an analytic self-map of $\mathbb{D}$ and u be an analytic function on $\mathbb{D}$. If $D^m_{u,\varphi}:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ be bounded, then

$$\|D^m_{u,\varphi}\|_e\approx \max\left\{B(u'',\varphi,\alpha+m-1),B(2u'\varphi'+u\varphi'',\varphi,\alpha+m),B(u\varphi'^2,\varphi,\alpha+m+1)\right\},$$

where $\alpha=\frac{q+2}{p}$.

2. Proof of Main results

If $f\in Q_K(p,q)$ then using (1), we have (recall that $\alpha=\frac{q+2}{p}$)

$$|f^{(k)}(z)|\leq C\frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+k-1}},\quad z\in\mathbb{D},\quad k\in\mathbb{N}. \tag{3}$$

Lemma 2.1. *Let $p>0$, $q>-2$ and $\alpha=\frac{q+2}{p}$. Then the following function belongs to $Q_K(p,q)$,*

$$f_a(z)=\frac{1-|a|^2}{(1-\overline{a}z)^\alpha},\quad a,z\in\mathbb{D}.$$

Proof.

$$\begin{aligned}&\int_{\mathbb{D}} |f_a'(z)|^p(1-|z|^2)^q K(g(z,\xi))\, dA(z)\\ &= \int_{\mathbb{D}} \frac{|a|^p(1-|a|^2)^p}{|1-\overline{a}z|^{2+p+q}}(1-|z|^2)^q K(g(z,\xi))\, dA(z)\\ &\leq 2^p \int_{\mathbb{D}} \frac{(1-|z|^2)^q}{|1-\overline{a}z|^{2+q}} K(g(z,\xi))\, dA(z).\end{aligned}$$

Applying Lemma 2.7 [9], we obtain

$$\|f_a\|^p_{Q_K(p,q)} = |f_a(0)| + \sup_{\xi\in\mathbb{D}} \int_{\mathbb{D}} |f_a'(z)|^p(1-|z|^2)^q K(g(z,\xi))\, dA(z) \leq C$$

where C is a positive constant independent of p, q, K. □

The proof of the following lemma is similar to the proof of Lemma 2.2 of [13], so it is omitted.

Lemma 2.2. *Let $p > 0$, $q > -2$, $K : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function, φ be an analytic self-map of $\mathbb{D}$ and u be an analytic function on $\mathbb{D}$. If $\|\varphi\|_\infty < 1$ and $D^m_{u,\varphi} : \mathcal{Q}_K(p,q) \to \mathcal{Z}_\mu$ be bounded, then $D^m_{u,\varphi} : \mathcal{Q}_K(p,q) \to \mathcal{Z}_\mu$ is compact.*

Now we are ready to prove Theorem A.

Proof of Theorem A. Suppose that $f \in Q_K(p,q)$. Then (3) implies that

$$\begin{aligned}\mu(z)|(D^m_{u,\varphi}f)''(z)| &= \mu(z)|u''(z)f^{(m)}(\varphi(z)) + (2u'(z)\varphi'(z) + u(z)\varphi''(z))f^{(m+1)}(\varphi(z))\\ &\quad + u(z)\varphi'^2(z)f^{(m+2)}(\varphi(z))|\\ &\leq \sup_{z\in\mathbb{D}} \frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m-1}}\|f\|_{\mathcal{B}^\alpha} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}\|f\|_{\mathcal{B}^\alpha}\\ &\quad + \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\alpha+m+1}}\|f\|_{\mathcal{B}^\alpha}\\ &\leq \sup_{z\in\mathbb{D}} \frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m-1}}\|f\|_{Q_K(p,q)} + \sup_{z\in\mathbb{D}} \frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}\|f\|_{Q_K(p,q)}\\ &\quad + \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\alpha+m+1}}\|f\|_{Q_K(p,q)}.\end{aligned}$$

On the other hand

$$|(D^m_{u,\varphi}f)(0)| = |u(0)f^{(m)}(\varphi(0))| \leq \frac{|u(0)|}{(1-|\varphi(0)|^2)^{\alpha+m-1}}\|f\|_{Q_K(p,q)},$$

and

$$|(D^m_{u,\varphi}f)'(0)| \leq \frac{|u(0)|}{(1-|\varphi(0)|^2)^{\alpha+m-1}}\|f\|_{Q_K(p,q)} + \frac{|u(0)\varphi'(0)|}{(1-|\varphi(0)|^2)^{\alpha+m}}\|f\|_{Q_K(p,q)}.$$

In view of the above equations and using the definition of norm in $Q_K(p,q)$ spaces, if

$A(u'',\varphi,\alpha{+}m{-}1)<\infty,\quad A(2u'\varphi'{+}u\varphi'',\varphi,\alpha{+}m)<\infty,\quad A(u\varphi'^2,\varphi,\alpha{+}m{+}1)<\infty$

then $D^m_{u,\varphi}:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ is bounded. Now suppose that the operator is bounded. Applying the operator on the function $f(z)=z^m$ we obtain

$$\sup_{z\in\mathbb{D}}\mu(z)|u''(z)|<\infty, \tag{4}$$

and also applying on the functions z^{m+1} and z^{m+2} respectively and using triangle inequality, we have

$$\sup_{z\in\mathbb{D}}\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|<\infty,\qquad \sup_{z\in\mathbb{D}}\mu(z)|u(z)\varphi'^2(z)|<\infty. \tag{5}$$

For $0\neq a\in\mathbb{D}$, set

$$f_a(z)=\frac{1-|a|^2}{(1-\bar{a}z)^\alpha},\quad h_a(z)=\frac{(1-|a|^2)^2}{(1-\bar{a}z)^{\alpha+1}},\quad g_a(z)=\frac{(1-|a|^2)^3}{(1-\bar{a}z)^{\alpha+2}},\quad z\in\mathbb{D}$$

and define

$$F_a(z)=(\alpha+m+2)f_a(z)-\frac{\alpha(2\alpha+2n+3)}{\alpha+m}h_a(z)+\frac{\alpha(\alpha+1)}{\alpha+m}g_a(z). \tag{6}$$

Then $F_a\in Q_K(p,q)$ and $\sup_{a\in\mathbb{D}}\|F_a\|_{Q_K(p,q)}\leq C$ where C is a positive constant, Lemma 2.1. Also, $F_a^{(m)}(a)=F_a^{(m+2)}(a)=0$ and

$$F_a^{(m+1)}(a)=-\overline{a}^{n+1}\frac{\alpha(\alpha+1)\cdots(\alpha+m-1)}{(1-|a|^2)^{\alpha+m}}.$$

Since $D^m_{u,\varphi}:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ is bounded then

$$\begin{aligned}C\geq&\|D^m_{u,\varphi}F_{\varphi(a)}\|_{\mathcal{Z}_\mu}\geq\sup_{z\in\mathbb{D}}\mu(z)|(D^m_{u,\varphi}F_{\varphi(a)})''(z)|\\ \geq&\mu(a)|(D^m_{u,\varphi}F_{\varphi(a)})''(a)|\\ =&\mu(a)|\varphi(a)|^{m+1}\frac{\alpha(\alpha+1)\cdots(\alpha+m-1)|2u'(a)\varphi'(a)+u(a)\varphi''(a)|}{(1-|\varphi(a)|^2)^{\alpha+m}}.\end{aligned}$$

Fix $r\in(0,1)$. Then

$$\begin{aligned}&\sup_{|\varphi(a)|>r}\frac{\mu(a)|2u'(a)\varphi'(a)+u(a)\varphi''(a)|}{(1-|\varphi(a)|^2)^{\alpha+m}}\\ &\quad\leq\sup_{|\varphi(a)|>r}\frac{|\varphi(a)|^{m+1}}{r^{m+1}}\frac{\mu(a)|2u'(a)\varphi'(a)+u(a)\varphi''(a)|}{(1-|\varphi(a)|^2)^{\alpha+m}}<\infty.\end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned}&\sup_{|\varphi(a)|\leq r}\frac{\mu(a)|2u'(a)\varphi'(a)+u(a)\varphi''(a)|}{(1-|\varphi(a)|^2)^{\alpha+m}}\\ &\quad\leq\sup_{|\varphi(a)|\leq r}\frac{1}{(1-r^2)^{\alpha+m}}\mu(a)|2u'(a)\varphi'(a)+u(a)\varphi''(a)|<\infty.\end{aligned}$$

Last inequality and (7) yield that $A(2u'\varphi' + u\varphi'', \varphi, \alpha + m) < \infty$. To get the other conditions, the proof is similar. Use instead of F_a, the test functions defined below

$$G_a(z) = (\alpha+m+1)(\alpha+m+2)f_a(z) - 2\alpha(\alpha+m+2)h_a(z) + \alpha(\alpha+1)g_a(z),$$

$$H_a(z) = f_a(z) - \frac{2\alpha}{\alpha+m}h_a(z) + \frac{\alpha(\alpha+1)}{(\alpha+m)(\alpha+m+1)}g_a(z),$$

with the derivatives $G_a^{(m+1)}(a) = G_a^{(m+2)}(a) = 0$ and

$$G_a^{(m)}(a) = 2\overline{a}^m \frac{\alpha(\alpha+1)\cdots(\alpha+m-1)}{(1-|a|^2)^{\alpha+m-1}}$$

and $H_a^{(m)}(a) = H_a^{(m+1)}(a) = 0$ and

$$H_a^{(m+2)}(a) = 2\overline{a}^{m+2} \frac{\alpha(\alpha+1)\cdots(\alpha+m-1)}{(1-|a|^2)^{\alpha+m+1}}.$$

The proof of Theorem A is completed.

The proof of Theorem B is divided into two parts and we assume that $\|\varphi\|_\infty = 1$, otherwise is true by Lemma 2.2.

Lower bound of Theorem B. Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$. Let $F_j = F_{\varphi(z_j)}$ be functions are defined in (6). Then $\{F_j\}$ is a bounded sequence in $Q_K(p,q)$ which converges uniformly to zero on compact subsets of $\mathbb{D}$, $F_j^{(m)}(\varphi(z_j)) = F_j^{(m+2)}(\varphi(z_j)) = 0$ and

$$F_j^{(m+1)}(\varphi(z_j)) = -\overline{\varphi(z_j)}^{n+1} \frac{\alpha(\alpha+1)\cdots(\alpha+m-1)}{(1-|\varphi(z_j)|^2)^{\alpha+m}}.$$

Let $C_1 = \sup_{j\in\mathbb{N}} \|F_j\|_{Q_K(p,q)}$. Then

$$\begin{aligned}
C_1\|D^m_{u,\varphi}\|_e \geq & \limsup_{j\to\infty} \|D^m_{u,\varphi}F_j\|_{\mathcal{Z}_\mu} \\
\geq & \limsup_{j\to\infty} \mu(z_j)|(2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j))F_j^{(m+1)}(\varphi(z_j))| \\
& - \limsup_{j\to\infty} \mu(z_j)|u''(z_j)F_j^{(m)}(\varphi(z_j))| \\
& - \limsup_{j\to\infty} \mu(z_j)|u(z_j)\varphi'^2(z_j)F_j^{(m+2)}(\varphi(z_j))| \\
= & \limsup_{j\to\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+m-1)|\varphi(z_j)|^{n+1}|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)|}{(1-|\varphi(z_j)|^2)^{\alpha+m}} \\
= & \limsup_{j\to\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+m-1)|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)|}{(1-|\varphi(z_j)|^2)^{\alpha+m}}.
\end{aligned}$$

Hence,

$$B(2u'\varphi' + u\varphi'', \varphi, \alpha+m) = \limsup_{j\to\infty} \frac{|2u'(z_j)\varphi'(z_j) + u(z_j)\varphi''(z_j)|}{(1-|\varphi(z_j)|^2)^{\alpha+m}} \lesssim \|D^m_{u,\varphi}\|_e.$$

To prove

$$B(u\varphi'^2,\varphi,\alpha+m+1)\lesssim\|D^m_{u,\varphi}\|_e,\quad B(u'',\varphi,\alpha+m-1)\lesssim\|D^m_{u,\varphi}\|_e$$

use the functions $G_j=G_{\varphi(z_j)}$ and $H_j=H_{\varphi(z_j)}$ respectively. Indeed $\{G_j\}$ and $\{H_j\}$ are bounded sequences in $Q_K(p,q)$ which converge to zero uniformly on compact subsets of $\mathbb{D}$ and $G_j^{(m+1)}(\varphi(z_j))=G_j^{(m+2)}(\varphi(z_j))=0$ and

$$G_j^{(m)}(\varphi(z_j))=2\overline{\varphi(z_j)}^m\frac{\alpha(\alpha+1)\cdots(\alpha+m-1)}{(1-|\varphi(z_j)|^2)^{\alpha+m-1}}$$

and $H_j^{(m)}(\varphi(z_j))=H_j^{(m)}(\varphi(z_j))=0$ and

$$H_j^{(m+2)}(\varphi(z_j))=2\overline{\varphi(z_j)}^{m+2}\frac{\alpha(\alpha+1)\cdots(\alpha+m-1)}{(1-|\varphi(z_j)|^2)^{\alpha+m+1}}.$$

We omit the rest of the proof due to the similarity. So, we get

$$\|D^m_{u,\varphi}\|_e\gtrsim$$
$$\max\left\{B(u'',\varphi,\alpha+m-1),B(2u'\varphi'+u\varphi'',\varphi,\alpha+m),B(u\varphi'^2,\varphi,\alpha+m+1)\right\}.$$

The upper bound of Theorem B. Suppose that $\{r_j\}$ be a sequence in $(0,1)$ such that $lim_{j\to\infty}r_j=1$. Since $D^m_{u,\varphi}:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ is bounded, then it can be easily proved that $D^m_{u,r_j\varphi}:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ is bounded. Also, since $\|r_j\varphi\|_\infty<1$, then using Lemma 2.2, $D^m_{u,r_j\varphi}:\mathcal{Q}_K(p,q)\to\mathcal{Z}_\mu$ is a compact operator for any j. So

$$\|D^m_{u,\varphi}\|_e\le\|D^m_{u,\varphi}-D^m_{u,r_j\varphi}\|=\sup_{\|f\|_{Q_K(p,q)}\le1}\|(D^m_{u,\varphi}-D^m_{u,r_j\varphi})f\|_{\mathcal{Z}_\mu},\quad j\in\mathbb{N}.$$

Fix $\delta\in(0,1)$. For any f with $\|f\|_{Q_K(p,q)}\le1$ we have

$$\begin{aligned}&\|(D^m_{u,\varphi}-D^m_{u,r_j\varphi})f\|_{\mathcal{Z}_\mu}=|((D^m_{u,\varphi}-D^m_{u,r_j\varphi})f)(0)|+|((D^m_{u,\varphi}-D^m_{u,r_j\varphi})f)'(0)|\\&\quad+\sup_{z\in\mathbb{D}}\mu(z)|((D^m_{u,\varphi}-D^m_{u,r_j\varphi})f)''(z)|\\&\le|u(0)f^{(m)}(\varphi(0))-u(0)f^{(m)}(r_j\varphi(0))|+|u'(0)(f^{(m)}(\varphi(0))-f^{(m)}(r_j\varphi(0)))|\\&\quad+|u(0)\varphi'(0)(f^{(m+1)}(\varphi(0))-r_jf^{(m+1)}(r_j\varphi(0)))|\\&\quad+\sup_{z\in\mathbb{D}}\mu(z)|((D^m_{u,\varphi}-D^m_{u,r_j\varphi})f)''(z)|.\end{aligned}\tag{8}$$

If $j\to\infty$, then

$$\begin{aligned}&|u(0)f^{(m)}(\varphi(0))-u(0)f^{(m)}(r_j\varphi(0))|\to0,\\&|u'(0)(f^{(m)}(\varphi(0))-f^{(m)}(r_j\varphi(0)))|\to0,\\&|u(0)\varphi'(0)(f^{(m+1)}(\varphi(0))-r_jf^{(m+1)}(r_j\varphi(0)))|\to0.\end{aligned}$$

On the other hand

$$\sup_{z\in\mathbb{D}}\mu(z)|((D^m_{u,\varphi}-D^m_{u,r_j\varphi})f)''(z)| \le \sup_{z\in\mathbb{D}}\mu(z)|u''(z)||f^{(m)}(\varphi(z))-f^{(m)}(r_j\varphi(z))|$$
$$+\sup_{z\in\mathbb{D}}\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)||f^{(m+1)}(\varphi(z))-r_jf^{(m+1)}(r_j\varphi(z))|$$
$$+\sup_{z\in\mathbb{D}}\mu(z)|u(z)\varphi'^2(z)||f^{(m+2)}(\varphi(z))-r_j^2f^{(m+2)}(r_j\varphi(z))|.$$

We divide each of three suprema on the right hand side of the above into two parts, one in $|\varphi(z)|\le\delta$ which is a compact set and therefore using (4) and (5), we obtain

$$\limsup_{j\to\infty}\sup_{|\varphi(z)|\le\delta}\mu(z)|u''(z)||f^{(m)}(\varphi(z))-f^{(m)}(r_j\varphi(z))|=0,$$
$$\limsup_{j\to\infty}\sup_{|\varphi(z)|\le\delta}\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)||f^{(m+1)}(\varphi(z))-r_jf^{(m+1)}(r_j\varphi(z))|1=0,$$
$$\limsup_{j\to\infty}\sup_{|\varphi(z)|\le\delta}\mu(z)|u(z)\varphi'^2(z)||f^{(m+2)}(\varphi(z))-r_j^2f^{(m+2)}(r_j\varphi(z))|=0.$$

Now we compute the other part. Applying (3), we have

$$\sup_{\delta<|\varphi(z)|<1}\mu(z)|u''(z)||f^{(m)}(\varphi(z))-f^{(m)}(r_j\varphi(z))|$$
$$\le\sup_{\delta<|\varphi(z)|<1}\mu(z)|u''(z)||f^{(m)}(\varphi(z))|+\sup_{\delta<|\varphi(z)|<1}\mu(z)|u''(z)||f^{(m)}(r_j\varphi(z))|$$
$$\le C\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m-1}}+C\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|u''(z)|}{(1-|r_j\varphi(z)|^2)^{\alpha+m-1}}$$
(9)
$$\le 2C\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m-1}}.$$

In a same way one can see that

$$\sup_{\delta<|\varphi(z)|<1}\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)||f^{(m+1)}(\varphi(z))-r_jf^{(m+1)}(r_j\varphi(z))|$$
(10)
$$\le 2C\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}}$$

and

$$\sup_{\delta<|\varphi(z)|<1}\mu(z)|u(z)\varphi'^2(z)||f^{(m+2)}(\varphi(z))-r_j^2f^{(m+2)}(r_j\varphi(z))|$$
$$\le 2C\sup_{\delta<|\varphi(z)|<1}\frac{\mu(z)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\alpha+m+1}}. \tag{11}$$

From (8), (9), (10) and (11), we get

$$\begin{aligned}\|D^m_{u,\varphi}\|_e \leq \sup_{\|f\|_{\mathcal{Q}_K(p,q)}\leq 1} \|(D^m_{u,\varphi} - D^m_{u,r_j\varphi})f\|_{\mathcal{Z}_\mu} &\leq 2C \sup_{\delta<|\varphi(z)|<1} \frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m-1}} \\ &+ 2C \sup_{\delta<|\varphi(z)|<1} \frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} \\ &+ 2C \sup_{\delta<|\varphi(z)|<1} \frac{\mu(z)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\alpha+m+1}}.\end{aligned}$$

Finally letting $\delta \to 1$ implies that

$$\|D^m_{u,\varphi}\|_e \lesssim \max\left\{B(u'',\varphi,\alpha+m-1), B(2u'\varphi'+u\varphi'',\varphi,\alpha+m), B(u\varphi'^2,\varphi,\alpha+m+1)\right\}.$$

Corollary 2.3. *Let $p > 0$, $q > -2$, $K : [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function, φ be an analytic self-map of $\mathbb{D}$ and g be an analytic function on $\mathbb{D}$. Then $D^m_{u,\varphi} : \mathcal{Q}_K(p,q) \to \mathcal{Z}_\mu$ is compact if and only if*

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m-1}} = 0,$$

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha+m}} = 0,$$

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)|u(z)\varphi'^2(z)|}{(1-|\varphi(z)|^2)^{\alpha+m+1}} = 0.$$

Remark 2.4. With different choices on p, q and K we can obtain some corollaries for the compactness and boundedness of the operators mentioned in the introduction.

By using Theorem A and Corollary 2.3, we get the following example.

Example 2.5. *Let $m = 1$, $u = 1$ and $\varphi(z) = z$. Then $D^1_{u,\varphi} : \mathcal{Q}_K(1,0) \to \mathcal{Z}$ is not bounded. Since $\alpha = \frac{q+2}{p} = 2$, $A(u'',\varphi,\alpha+m-1) = A(2u'\varphi'+u\varphi'',\varphi,\alpha+m) = 0$ and*

$$A(u\varphi'^2,\varphi,\alpha+m+1) = \sup_{z\in\mathbb{D}} \frac{1}{(1-|z|^2)^3} = \infty.$$

Also when $\beta \geq 4$, then the operator $D^1_{u,\varphi} : \mathcal{Q}_K(1,0) \to \mathcal{Z}^\beta$ is bounded, because $A(u'',\varphi,\alpha+m-1) = A(2u'\varphi'+u\varphi'',\varphi,\alpha+m) = 0$ and $A(u\varphi'^2,\varphi,\alpha+m+1) = \sup_{z\in\mathbb{D}}(1-|z|^2)^{\beta-4} = 1$.

Similarly $B(u'',\varphi,\alpha+m-1) = B(2u'\varphi'+u\varphi'',\varphi,\alpha+m) = 0$ and $B(u\varphi'^2,\varphi,\alpha+m+1) = \lim_{|z|\to 1}(1-|z|^2)^{\beta-4}$. So, $D^1_{u,\varphi} : \mathcal{Q}_K(1,0) \to \mathcal{Z}^4$ is not compact and for $\gamma > 4$, $D^1_{u,\varphi} : \mathcal{Q}_K(1,0) \to \mathcal{Z}^\gamma$ is compact.

3. Conclusion

By using test functions we presented equivalence condition for boundedness and compactness of operator $D^m_{u,\varphi} : \mathcal{Q}_K(p,q) \to \mathcal{Z}_\mu$. Also an estimation for the essential norm of the operator has been found in the paper. It is worthwhile mentioned that the results can be stated for some well-known operators included in the weighted differentiation composition operator. By comparing Theorems 2.4 and 3.2 of [1] with Theorem A and Theorem B respectively, the operator $D^m_{u,\varphi} : \mathcal{Q}_K(p,q) \to \mathcal{Z}_\mu$ is bounded (compact) if and only if $D^m_{u,\varphi} : \mathcal{B}^{\frac{q+2}{p}} \to \mathcal{Z}_\mu$ is bounded (compact). By setting $\mu(z) = (1-|z|^2)^\beta$, similar results will be obtained for the operator $D^m_{u,\varphi}$ from $\mathcal{Q}_K(p,q)$ into Zygmund type space. A future work on this subject can be replacing the weighted Zygmund space by n-th weighted type space.

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