Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

INTERVAL SHRINKAGE ESTIMATION OF TWO-PARAMETER EXPONENTIAL DISTRIBUTION WITH RANDOM CENSORED DATA

A. Soori, P. Nasiri, M. Jabbari Nooghabi ✉, F. Hormozinejad, and M. Ghalani

Article type: Research Article
(Received: 29 November 2023, Received in revised form 25 June 2024)
(Accepted: 18 August 2024, Published Online: 24 August 2024)

Abstract. The use of the two-parameter exponential distribution model in fitting survival and reliability analysis data in the presence of censored random data has recently attracted the attention of a large number of authors. Considering the importance of the model, its parameter estimation is discussed using the method of moment, maximum likelihood and shrinkage estimation. To present the interval shrinkage estimator, it is first proved that the moment estimators are asymptotically unbiased and the interval shrinkage estimator performs better compared to other estimators. Finally, using two real data sets and statistical criteria, the goodness of fit of the model is compared with censored random data based on parameter estimation methods.

Keywords: Two-parameter exponential distribution, Random censoring, Interval shrinkage estimation, Goodness of fit, Mean squared error.
2020 MSC: Primary 62-XX, 62Fxx, 62F15.

1. Introduction

The two-parameter exponential distribution is one of the important statistical distributions, which has wide application in the fields of reliability, waiting time, lifetime of electrical equipment, clinical trials, biology and so on. Davis [6] in 1952 used this distribution to test life span data, followed by [3] and [7] to fit statistical data. Considering the importance of this distribution in life span data, researchers have provided different methods to estimate the parameters of two parametric exponential distribution. For more information on the application of maximum likelihood, moment, least squared error and Bayesian methods, you can refer to [14]. A class of contraction estimators for parameters of two-parameter exponential distribution by examining the contraction estimator of location and scale parameters are presented by [12]. For more details, one can refer to [4], [15], [18], [1], and [16]. A random variable X has a two-parameter exponential distribution if it has a density function as:

✉ jabbarinm@um.ac.ir, ORCID: 0000-0002-5636-2209

https://doi.org/10.22103/jmmr.2024.22595.1543
Publisher: Shahid Bahonar University of Kerman
How to cite: A. Soori, P. Nasiri, M. Jabbari Nooghabi, F. Hormozinejad, M. Ghalani, *Interval shrinkage estimation of two-parameter exponential distribution with random censored data*, J. Mahani Math. Res. 2025; 14(1): 121-136.

$$f(x;\mu,\theta)=\frac{1}{\theta}e^{-\frac{(x-\mu)}{\mu}} \quad , \quad x>\mu,\theta>0, \tag{1}$$

such that θ is the scale parameter and μ is the location parameter or the warranty period before the first failure, and its corresponding distribution function is given by:

$$F(x;\mu,\theta)=1-\frac{1}{\theta}e^{-\frac{(x-\mu)}{\theta}} \quad , \quad x>\mu,\theta>0.$$

Krishna and Goel [11] discussed the parameters of two parametric exponential distribution considering random censored data and showed that the Bayesian estimator performs better compared to the maximum likelihood estimator. Bayesian parameters of the two-parameter exponential distribution using the linear transformation of the reliability function are estimated by [5]. Baloui et al. [2] presented the efficiency of contraction estimators of the Pareto-Rayleigh distribution shape parameter and showed that the proposed contraction estimator performs better compared to other estimators. Also, Hussein et al. [10] discussed the estimation of the parameters of the generalized exponential distribution with the presence of censored data.

Random censoring is introduced by [8] and he showed that in the production process, an item or observation may be randomly removed from the production process at different times. For example, in clinical trials, a patient may drop out or die before the end of the treatment period with prior notice. Similarly, in reliability, to save time and money, an item may be completely excluded from the process of pre-failure testing, depending on the type of censoring. This type of censorship has been investigated by [13]. In the second part of the article, two-parameter exponential distribution model with random censored data is presented. In the third part, the parameters of the model are discussed using the maximum likelihood and moment methods and it is proved that the moment estimators are asymptotically unbiased. In the fourth part, interval shrinkage estimation is presented. In the fifth and sixth parts, the study of simulation and goodness of fit by using two real data sets are presented, respectively.

2. Models

In this section, a two-parameter exponential distribution model with randomly censored data is presented. For this purpose, suppose a random sample $X_1, X_2, ..., X_n$ from the two-parameter exponential distribution

$$f(x;\mu,\theta)=\frac{1}{\theta}e^{-\frac{(x-\mu)}{\theta}}, \quad x>\mu>0,\theta>0, \tag{2}$$

and random variables $T_1, T_2, ..., T_n$ censoring times with density function

$$f(t;\mu,\lambda)=\frac{1}{\lambda}e^{-\frac{(x-\mu)}{\lambda}}, \quad t>\mu,\lambda,\theta>0. \tag{3}$$

Assuming that the random variables X_i and T_i are independent of each other, the random variable Y_i

$$Y_i = \min(X_i, T_i), \quad i = 1, 2, ..., n. \tag{4}$$

For the actual occurrence time of observation i-th, it is clear that $Y_i = X_i$ if $X_i \leq T_i$ and $Y_i = T_i$ if $X_i > T_i$, to obtain the density function of the random variable Y_i, the indicator variable D_i is considered as follows:

$$D_i = \begin{cases} 1 & if \quad X_i \leq T_i \\ 0 & if \quad X_i > T_i \end{cases} \quad , \quad i = 1, 2, ..., n. \tag{5}$$

If $p = P(X_i \leq T_i)$ is considered the chance of success, then D_i follows a Bernoulli distribution with parameter p therefore, its density function is equal to:

$$P(D = d) = p^d (1-p)^{(1-d)}, \quad d = 0, 1, 0 \leq p \leq 1. \tag{6}$$

If the random variables X and T are survival time and censoring time with density functions (2) and (3), respectively, the probability of failure of an item before censoring is equal to:

$$\begin{aligned} P(D=1) = P(X \leq T) &= \int_{x<t}\int f(x,t)\ dt\ dx \\ &= \int_\mu^\infty \int_x^\infty f(x,t)\ dt\ dx = \int_\mu^\infty \left[\int_x^\infty f(x|\mu,\theta) f(t|\mu,\lambda) dt\right] dt \\ &= \int_\mu^\infty f(x|\mu,\theta) \left[\int_x^\infty f(x|\mu,\theta)\right] = \int_\mu^\infty f(x|\mu,\theta)[1 - F_T(x)]dx \\ &= \int_\mu^\infty \frac{1}{\theta} e^{\frac{-(x-\mu)}{\theta}} \left(e^{\frac{-(x-\mu)}{\lambda}}\right) dx = \int_\mu^\infty \frac{1}{\theta} e^{-\left(\frac{1}{\theta}+\frac{1}{\lambda}\right)(x-\mu)} dx \\ &= \frac{1}{\theta}\int_\mu^\infty e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)(x-\mu)} dx = \frac{1}{\theta}\left[-\frac{\lambda\theta}{\lambda+\theta} e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)(x-\mu)}\Big|_\mu^\infty\right] \\ &= \frac{1}{\theta}\left(\frac{\lambda\theta}{\lambda+\theta}\right) = \frac{\lambda}{\lambda+\theta}, \end{aligned} \tag{7}$$

where the parameter λ is the threshold or warranty period. As we known that the independence of X and T implies the independence of D and Y, in which case the joint density function of the random variables D and Y is equal to:

$$
\begin{aligned}
(8)\quad f_{Y,D}(y,d;\mu,\theta,\lambda)&=[f_X(y;\mu,\theta)(1-F_T(y;\mu,\lambda))][f_T(y;\mu,\mu)(1-F_X(y;\mu,\theta))]\\
&=\begin{cases} f_X(y;\mu,\theta)(1-F_T(y;\mu,\lambda)), & d=1,\\ f_T(y;\mu,\lambda)(1-F_X(y;\mu,\theta)), & d=0\end{cases}\\
&=\begin{cases} \frac{1}{\theta}e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)(y-\mu)} & y>\mu, d=1,\\ \frac{1}{\lambda}e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)(y-\mu)} & y>\mu, d=0.\end{cases}
\end{aligned}
$$

According to relation (8), the marginal density functions Y and D are equal to:

$$
\begin{aligned}
(9)\qquad f_Y(y;\mu,\theta,\lambda)] &= \sum_{d=0}^{1} f_{Y,D}(y,d;\mu,\theta,\lambda)\\
&= \frac{1}{\theta}e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)(y-\mu)} + \frac{1}{\lambda}e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)(y-\mu)}\\
&= \frac{\lambda+\theta}{\lambda\theta}e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)(y-\mu)}, y>\mu,
\end{aligned}
$$

and

$$
\begin{aligned}
(10)\qquad P(D=d) &= \begin{cases} \frac{1}{\lambda}\int_{\mu}^{\infty} e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)(y-\mu)}dy, & d=1,\\ \frac{1}{\theta}\int_{\mu}^{\infty} e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)(y-\mu)}dy, & d=0\end{cases}\\
&= \begin{cases} \frac{\lambda}{\lambda+\theta}, & d=1,\\ \frac{\theta}{\lambda+\theta}, & d=0.\end{cases}\quad .
\end{aligned}
$$

3. Estimation of the parameters

In this section, the parameters of the two-parameter exponential distribution with randomly censored data are estimated by the maximum likelihood and moments methods, and it is shown that the moment estimators are asymptotically unbiased.

3.1. Estimation of parameters by maximum likelihood method.

Suppose that for a random sample $(Y_i, D_i), i = 1, 2, , n$ from equation (8), in this case, the likelihood function is equal to:

$$
\begin{aligned}
L(\mu,\theta,\lambda) &= \prod_{i=1}^{n} f_{Y,D}(y_i,d_i)\\
&= \left(\frac{1}{\theta}\right)^{\sum_{i=1}^{n} d_i}\left(\frac{1}{\lambda}\right)^{n-\sum_{i=1}^{n} d_i} e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)\sum_{i=1}^{n}(y_i-\mu)}.
\end{aligned}
$$

If the $Y_{(1)}, Y_{(2)}, , Y_{(n)}$ ordinal statistics are variables, then we can write:

(11)

$$L(\mu,\theta,\lambda)=\left(\frac{1}{\theta}\right)^{\sum_{i=1}^{n}d_i}\left(\frac{1}{\lambda}\right)^{n-\sum_{i=1}^{n}d_i}e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)\sum_{i=1}^{n}(y_i-\mu)}$$
$$=\left(\frac{1}{\theta}\right)^{\sum_{i=1}^{n}d_i}\left(\frac{1}{\lambda}\right)^{n-\sum_{i=1}^{n}d_i}e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)[\sum_{i=1}^{n}(y_{(1)}-\mu)+\sum_{i=1}^{n}(y_{(i)}-y_{(1)})}$$
$$=\left(\frac{1}{\theta}\right)^{\sum_{i=1}^{n}d_i}\left(\frac{1}{\lambda}\right)^{n-\sum_{i=1}^{n}d_i}e^{-n\left(\frac{\lambda+\theta}{\lambda\theta}\right)(y_{(1)}-\mu)}e^{-\left(\frac{\lambda+\theta}{\lambda\theta}\right)[\sum_{i=1}^{n}(y_{(i)}-y_{(1)})}.$$

Let $l(\mu,\theta,\lambda)$ be the logarithm of the relation (11), then it can be written:

$$l(\mu,\theta,\lambda)=-\sum_{i=1}^{n}d_i\log(\theta)-(n-\sum_{i=1}^{n}d_i)\log(\lambda) - n\left(\frac{\lambda+\theta}{\lambda\theta}\right)(Y_{(1)}-\mu)-\left(\frac{\lambda+\theta}{\lambda\theta}\right)\sum_{i=1}^{n}(Y_{(i)}-Y_{(1)}). \tag{12}$$

Then, the estimator is equal to:

$$\hat{\mu}=Y_{(1)}=min(X_1,X_2,,X_n). \tag{13}$$

By replacing equation (13) with equation (12), we have

$$l(\hat{\mu},\theta,\lambda)=-\sum_{i=1}^{n}d_i\log(\theta)-\left(n-\sum_{i=1}^{n}d_i\right)\log(\lambda)-\frac{\lambda+\theta}{\lambda\theta}\sum_{i=1}^{n}\left(Y_{(i)}-Y_{(1)}\right). \tag{14}$$

Now to estimate parameters of θ and λ, we can write

$$\frac{\partial l(\hat{\mu},\theta,\lambda)}{\partial\lambda}=-\frac{1}{\theta}\sum_{i=1}^{n}d_i+\frac{1}{\theta^2}\sum_{i=1}^{n}\left(Y_{(i)}-Y_{(1)}\right)=0, \tag{15}$$

and

$$\frac{\partial l(\hat{\mu},\theta,\lambda)}{\partial\theta}=-\frac{1}{\lambda}\left(n-\sum_{i=1}^{n}d_i\right)+\frac{1}{\lambda^2}\sum_{i=1}^{n}\left(Y_{(i)}-Y_{(1)}\right)=0. \tag{16}$$

By using the equation (15) and (16), the estimators of $\hat{\theta}$ and $\hat{\lambda}$ are equal to

$$\hat{\theta}=\frac{\sum_{i=1}^{n}\left(Y_{(i)}-Y_{(1)}\right)}{\sum_{i=1}^{n}d_i}, \tag{17}$$

and

$$\hat{\lambda}=\frac{\sum_{i=1}^{n}\left(Y_{(i)}-Y_{(1)}\right)}{n-\sum_{i=1}^{n}d_i}, \tag{18}$$

respectively.

3.2. **Estimation of parameters by method of Moment.** In this section, parameters λ, θ and μ of two-parameter exponential distribution with random censored data are estimated by the moment method. For this purpose, according to the equations (9) and (10), we can write

$$E(Y) = \int_{\mu}^{\infty} y \frac{\lambda+\theta}{\lambda\theta} e^{-\frac{\lambda+\theta}{\lambda\theta}(y-\mu)} dy.$$

Let we consider $Y = u + \mu$, $u = Y - \mu$, and $du = dy$, then

$$\begin{aligned} E(Y) &= \frac{\lambda+\theta}{\lambda\theta} \int_0^{\infty} e^{-\frac{\lambda+\theta}{\lambda\theta} u(u+\mu)} du \\ &= \frac{\lambda+\theta}{\lambda\theta} \int_0^{\infty} u e^{-\frac{\lambda+\theta}{\lambda\theta} u} du + \frac{(\lambda+\theta)\mu}{\lambda\theta} \int_0^{\infty} e^{-\frac{\lambda+\theta}{\lambda\theta} u} du \\ &= \frac{\lambda\theta}{\lambda+\theta} + \mu, \end{aligned} \tag{19}$$

and

$$E(Y^2) = \int_{\mu}^{\infty} y^2 f_Y(y|\mu,\theta,\lambda) dy = \frac{\lambda+\theta}{\lambda\theta} \int_{\mu}^{\infty} y^2 e^{-\frac{\lambda+\theta}{\lambda\theta}(y-\mu)} dy.$$

Let

$$\frac{\lambda+\theta}{\lambda\theta}(y-\mu) = u, \quad \frac{\lambda\theta}{\lambda+\theta} u + \mu = y, \quad \text{and} \quad \frac{\lambda\theta}{\lambda+\theta} du = dy.$$

Then

$$E(Y^2) = \frac{\lambda+\theta}{\lambda\theta} \left[\int_0^{\infty} \left(\frac{\lambda\theta}{\lambda+\theta} u + \mu \right)^2 e^{-u} \frac{\lambda\theta}{\lambda+\theta} du \right] \tag{20}$$

$$\begin{aligned} &= \int_0^{\infty} \left(\frac{\lambda\theta}{\lambda+\theta} \right)^2 u^2 e^{-u} du + \mu^2 \int_0^{\infty} e^{-u} du + 2\mu \left(\frac{\lambda\theta}{\lambda+\theta} \right) \int_0^{\infty} u e^{-u} du \\ &= \left(\frac{\lambda\theta}{\lambda+\theta} \right)^2 \int_0^{\infty} u^2 e^{-u} du + \mu^2 + 2\mu \left(\frac{\lambda\theta}{\lambda+\theta} \right) \\ &= 2 \left(\frac{\lambda\theta}{\lambda+\theta} \right)^2 + \mu^2 + 2\mu \left(\frac{\lambda\theta}{\lambda+\theta} \right), \end{aligned}$$

and

$$V(Y) = E(Y^2) - (E(Y))^2 = \left(\frac{\lambda\theta}{\lambda+\theta}\right)^2 + \mu^2 + 2\mu\left(\frac{\lambda\theta}{\lambda+\theta}\right) - \left(\frac{\lambda\theta}{\lambda+\theta}\right)^2 - \mu^2 - 2\mu\left(\frac{\lambda\theta}{\lambda+\theta}\right) = \left(\frac{\lambda\theta}{\lambda+\theta}\right)^2. \tag{21}$$

Also

$$E(D) = \sum_{d=0}^{1} P(D=d) = \frac{\lambda}{\lambda+\theta}. \tag{22}$$

From the equality of moments, we can write

$$\bar{d} = \frac{\lambda}{\lambda+\theta}, \quad \bar{y} = \frac{\lambda\theta}{\lambda+\theta} + \mu, \quad \text{and} \quad S_y^2 = \left(\frac{\lambda\theta}{\lambda+\theta}\right)^2.$$

As a result, the moment estimators of the parameters are equal to

$$\hat{\mu} = \bar{y} - S_y, \quad \hat{\theta} = \frac{S_y}{\hat{y}}, \quad \text{and} \quad \hat{\lambda} = \frac{S_y}{1-\bar{d}},$$

where $S_y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$.

Here, we show that the moment estimators are asymptotically unbiased. For this purpose, the following lemma is presented.

Lemma 3.1. *Let $W_1 = \frac{1}{n}\sum_{i=1}^{n} D_i$ and $W_2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$. If we consider $\nu_1 = E(W_1)$ and $\nu_2 = E(W_2)$ then the estimators of θ, λ, and μ are unbiased.*

Proof.

$$\nu_1 = E(W_1) = E\left(\frac{1}{n}\sum_{i=1}^{n} D_i\right) = E(D) = \frac{\lambda}{\lambda+\theta}, \tag{23}$$

$$\begin{aligned}\nu_2 = E(W_2) &= E\left(\frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right) = \frac{1}{n}E\left(\sum_{i=1}^{n} Y_i^2 - \bar{Y}^2\right)\\ &= E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^2\right) - E(\bar{Y}^2) = E(Y^2) - \left[V(\bar{Y}) + (E(\bar{Y}))^2\right]\\ &= E(Y^2) - V\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) - (E(\bar{Y}))^2 = V(Y) - \frac{1}{n}V(Y)\end{aligned}$$

$$= \left(\frac{n-1}{n}\right)V(Y) = \frac{n-1}{n}\left(\frac{\lambda\theta}{\lambda+\theta}\right)^2. \tag{24}$$

As we know $\hat{\theta} = \frac{S_y}{\bar{y}}$ or $\hat{\theta} = \frac{\sqrt{w_2}}{\sqrt{w_1}} = f(w_1, w_2)$, now with Taylor's expansion, the function $f(w_1, w_2)$ around (ν_1, ν_2) can be written as:

$$
\begin{aligned}
(25) \qquad f(w_1, w_2) &= f(\nu_1, \nu_2) + (w_1 - \nu_1)\frac{\partial f(w_1, w_2)}{\partial w_1}\Big|_{w_1=\nu_1}^{w_2=\nu_2} \\
&+ (w_2 - \nu_2)\frac{\partial f(w_1, w_2)}{\partial w_2}\Big|_{w_1=\nu_1}^{w_2=\nu_2} + ...
\end{aligned}
$$

By using the equation (23) and (24), the following equation is obtained.

$$
\begin{aligned}
(26) \qquad E(\hat{\theta}) = f(v_1, v_2) &= \lim_{n\to 0} \frac{\sqrt{\nu_2}}{\nu_1} = \lim_{n\to\infty} \frac{\sqrt{\frac{n-1}{n}}\left(\frac{\lambda\theta}{\lambda+\theta}\right)}{\frac{\lambda}{\lambda+\theta}} \\
&= \lim_{n\to\infty} \sqrt{\frac{n-1}{n}}\theta = \theta.
\end{aligned}
$$

Similarly, it can be shown as

$$
(27) \qquad E(\hat{\lambda}) = \lambda.
$$

□

Lemma 3.2. *Let* $W_3 = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}$ *and* $S_y^2 = \frac{1}{n}\sum((Y_i - \bar{Y})^2$*, then the estimator* $\hat{\mu} = \bar{Y} - S_y$ *is unbiased.*

Proof. Let $W_3 = \bar{Y}$, we know that

$\nu_3 = E(W_3) = E(\bar{Y}) = \frac{\lambda\theta}{\lambda+\theta} + \mu$, hence $\hat{\mu} = w_3 - \sqrt{w_2} = f(w_2, w_3)$. Now we expand $f(w_2, w_3)$ around the ν_2 and ν_3 as

$$
\begin{aligned}
(28) \qquad \hat{\mu} = f(w_2, w_3) &= f(\nu_2, \nu_3) + (w_2 - \nu_2)\frac{\partial f(w_2, w_3)}{\partial w_2}\Big|_{w_3=\nu_3}^{w_2=\nu_2} \\
&+ (w_3 - \nu_3)\frac{\partial f(w_2, w_3)}{\partial w_3}\Big|_{w_3=\nu_3}^{w_2=\nu_2} + ...,
\end{aligned}
$$

and

$$
\begin{aligned}
(29) \qquad E(\hat{\mu}) = f(\nu_2, \nu_3) &= \lim_{n\to 0}\left[\frac{\lambda\theta}{\lambda+\theta} + \mu - \sqrt{\frac{n-1}{n}}\frac{\lambda}{\lambda+\theta}\right] \\
&= \mu + \lim_{n\to\infty}\left[1 - \sqrt{\frac{n-1}{n}}\right]\frac{\lambda\theta}{\lambda+\theta} = \mu.
\end{aligned}
$$

□

4. Interval shrinkage estimation of scale parameter

In this section, shrinkage estimation, which is a novel combination of classical estimation and initial guess about the parameter(s), is discussed. Many articles have been presented on shrinkage estimation, including [17] and [9]. To introduce the shrinkage estimator, if $\hat{\theta}$ is the classical estimator and θ_g the initial guess of the research θ, the shrinkage estimator is defined as follows.

$$\hat{\theta}_{sh} = \theta_g + \omega(\hat{\theta} - \theta_g), \tag{30}$$

where ω called shrinkage factor and its value can be obtained by solving the second power of the error.

$$\omega = \frac{(\theta - \theta_g)^3 + (\theta - \theta_g)E(\hat{\theta} - \theta)}{MSE(\hat{\theta}) + (\theta - \theta_g)^2}, \tag{31}$$

while $\hat{\theta}$ is unbiased estimator and its value is equal to:

$$\omega = \frac{(\theta - \theta_g)^2}{V(\hat{\theta}) + (\theta - \theta_g)^2}. \tag{32}$$

By inserting the equation (32) in the equation (30) $\hat{\theta}_{sh}$ is given by

$$\hat{\theta}_{sh} = \theta_g + \frac{(\theta - \theta_g)^2}{V(\hat{\theta}) + (\theta - \theta_g)^2}(\hat{\theta} - \theta_g). \tag{33}$$

Thompson [17] introduced a shrinkage estimator by using interval $\theta \in (\theta_0, \theta_1)$, in which the mean of point shrinkage estimators with equal weights was considered, based on $\tilde{\theta} \in (\theta_0, \theta_1)$ which the interval shrinkage estimator is

$$\breve{\theta}(\theta) = \int_{\theta_0}^{\theta_1} \frac{\omega\hat{\theta}(1 - \omega)\breve{\theta}}{\theta_1 - \theta_0} d\breve{\theta}, \tag{34}$$

such that

$$\breve{\theta}(\theta) = \hat{\theta} + \sqrt{V(\hat{\theta})}\frac{\theta - \hat{\theta}}{\theta_1 - \theta_0}\left[\arctan\left(\frac{\theta_1 - \theta_0}{\sqrt{V(\hat{\theta})}}\right) - \arctan\left(\frac{\theta_0 - \theta}{\sqrt{V(\hat{\theta})}}\right)\right] + \left[\frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)}\log\left(\frac{V(\hat{\theta}) + (\theta_1 - \theta)^2}{V(\hat{\theta}) + (\theta_0 - \theta)^2}\right)\right]. \tag{35}$$

For more information, see [9]. Now considering the estimator $\hat{\theta}$ is an unbiased estimator, it can be easily shown that the mean and variance of the relation (35) are respectively equal to

$$E\left(\breve{\theta}(\theta)\right) = \theta + \frac{V(\hat{\theta})}{2(\theta_1 - \theta)} \log\left(\frac{V(\hat{\theta}) + (\theta_1 - \theta)^2}{V(\hat{\theta}) + (\theta_0 - \theta)^2}\right), \tag{36}$$

and

(37)

$$V\left(\breve{\theta}(\theta)\right) = V(\hat{\theta})\left[1 - \frac{V(\hat{\theta})}{2(\theta_1 - \theta)}\left(\arctan\left(\frac{\theta_1 - \theta}{\sqrt{V(\hat{\theta})}}\right) - \arctan\left(\frac{\theta_0 - \theta}{\sqrt{V(\hat{\theta})}}\right)\right)\right]^2.$$

According to the equation (36), the shrinkage estimator of θ according to the prior information of the interval is equal to:

$$\breve{\theta}(\hat{\theta}) = \hat{\theta} + \frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)} \log\left(\frac{V(\hat{\theta}) + (\theta_1 - \hat{\theta})^2}{V(\hat{\theta}) + (\theta_0 - \hat{\theta})^2}\right). \tag{38}$$

Based on the method of moment estimator, the shrinkage interval estimator of λ and μ are respectively equal to

$$\tilde{\lambda}(\hat{\lambda}) = \hat{\lambda} + \frac{V(\hat{\lambda})}{2(\lambda_1 - \lambda_0)} \log\left(\frac{V(\hat{\lambda}) + (\lambda_1 - \hat{\lambda})^2}{V(\hat{\lambda}) + (\lambda_0 - \hat{\lambda})^2}\right), \tag{39}$$

and

$$\tilde{\mu}(\hat{\mu}) = \hat{\mu} + \frac{V(\hat{\mu})}{2(\mu_1 - \mu_0)} \log\left(\frac{V(\hat{\mu}) + (\mu_1 - \hat{\mu})}{V(\hat{\mu}) + (\mu_0 - \hat{\mu})}\right). \tag{40}$$

5. **Simulation study**

In this section, the estimators of two-parameter exponential distribution parameters with the presence of censored data are compared in different ways. For this purpose, the set of parameter $\mu = 2, \lambda = 2, \theta = 1$, and $\mu = 2, \lambda = 2, \theta = 3$ is considered.

For samples with size $n = 10(5)40$, the mean and mean squared error of the estimator's method of moment, maximum likelihood and interval shrinkage method after repeating 1000 times for each sample size are given in Tables 1 and 2. According to the results, the mean square errors (MSE) of the estimators decreases with the increase of the sample size. According to the results for comparing the estimation methods, it can be said that the maximum likelihood estimation works better than the method of moment estimation and interval shrinkage estimation works better than the maximum likelihood method. Meantime according to Tables 1 and 2, with the increase of the parameter value of θ, the average mean square error increases for all three estimators.

Table 1. Estimation and mean square error of estimators when $\theta = 1, \lambda = 2$, and $\mu = 2$.

n	parameter	Moment estimation		Maximum likelihood estimation		Shrinkage estimation	
		Estimation	MSE	Estimation	MSE	Estimation	MSE
10	μ	2.317113	0.125903	2.335388	0.116306	2.341326	0.116532
	λ	2.274854	2.739427	2.214608	2.348476	2.43623	0.190654
	θ	0.992973	0.247011	0.963034	0.175645	1.111158	0.012526
15	μ	2.309594	0.115737	2.316371	0.102259	2.329877	0.108822
	λ	2.322206	3.189533	2.303996	2.982427	2.425764	0.181495
	θ	0.983842	0.16413	0.97311	0.114194	1.070684	0.005143
20	μ	2.30032	0.10506	2.304933	0.094089	2.320473	0.102711
	λ	2.219349	1.601299	2.209544	1.525711	2.382302	0.146995
	θ	0.981542	0.113258	0.973218	0.07355	1.047433	0.002366
25	μ	2.295702	0.101464	2.297387	0.08916	2.312635	0.097756
	λ	2.108702	1.145142	2.09617	0.893871	2.327801	0.10822
	θ	0.975762	0.09811	0.97407	0.06517	1.047812	0.002318
30	μ	2.294028	0.097513	2.290754	0.084926	2.310657	0.096517
	λ	2.049086	0.688551	2.062744	0.602353	2.266807	0.071413
	θ	0.982276	0.078492	0.987318	0.054392	1.03642	0.001378
35	μ	2.287785	0.093547	2.290001	0.084491	2.307848	0.094782
	λ	2.087394	0.613197	2.084292	0.544688	2.242856	0.05938
	θ	0.985531	0.066994	0.981967	0.041345	1.02783	0.000891
40	μ	2.286698	0.090814	2.286671	0.082441	2.305246	0.09318
	λ	2.089641	0.499496	2.089328	0.409052	2.21748	0.047914
	θ	0.984964	0.05502	0.985351	0.037757	1.02487	0.000658

6. Real data

In this section, two real data sets are considered for the goodness of fit of the two-parameter exponential distribution with random censored data, and the results of the goodness of fit of the distribution according to the estimation methods are given in Tables 3 and 4. The results of the model goodness of fit test for two sets of data are shown in Figures 1 and 2, respectively.

A) To check the efficiency of the interval shrinkage estimator compared to other estimators, the real data set of front disc brake pad life for 40 cars of one model per 1000 kms is reported as follows (for more information, see [11]).

59.0	38.4	41.0	56.4	81.3	62.4	45.3	36.7
42.2	51.6	34.4	22.7	22.6	40.0	38.8	50.2
48.8	81.7	61.5	53.6	50.7	42.8	102.5	42.7
80.6	64.5	73.1	28.4	46.9	49.0	33.8	59.8
31.7	33.9	50.6	56.7	86.2	45.1	52.1	54.2

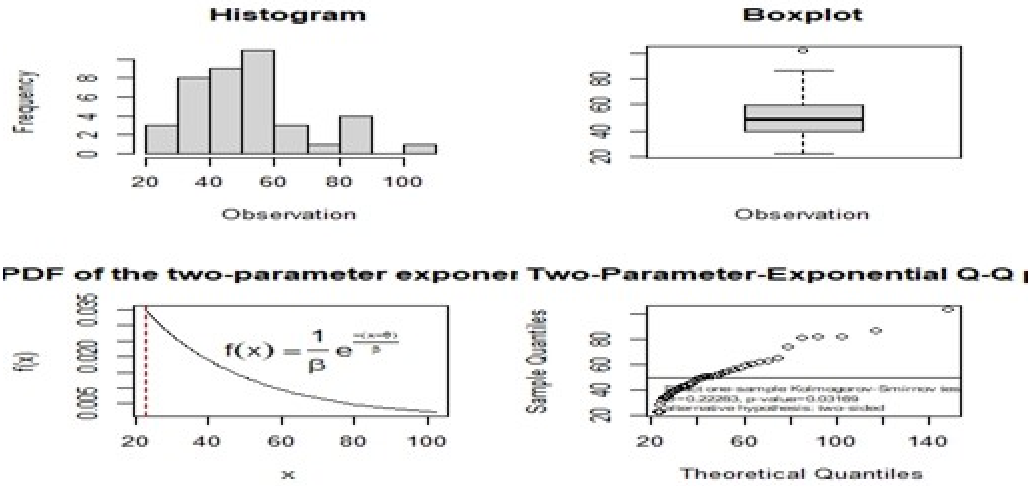

FIGURE 1. Goodness of fit test for data set A.

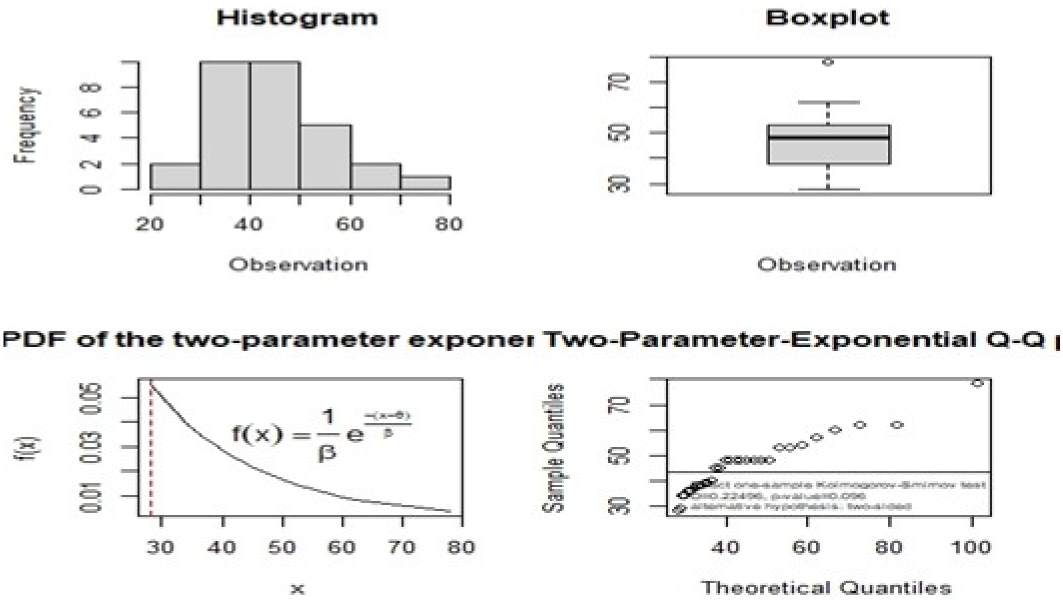

FIGURE 2. Goodness of fit test for data set B.

TABLE 2. Estimation and mean square error of estimators when $\theta = 3, \lambda = 2$, and $\mu = 2$.

n	parameter	Moment estimation		Maximum likelihood estimation		Shrinkage estimation	
		Estimation	MSE	Estimation	MSE	Estimation	MSE
10	μ	1.874458	0.10785	1.904156	0.023345	1.953233	0.002213
	λ	2.03016	1.553577	1.972584	1.14118	2.280913	0.079013
	θ	3.380464	7.681631	3.294459	5.742346	3.472715	0.2238
15	μ	1.848516	0.09263	1.8597	0.026287	1.919659	0.006486
	λ	1.992021	0.740508	1.976078	0.535022	2.205831	0.042634
	θ	3.284536	4.854192	3.245031	4.010476	3.421264	0.178193
20	μ	1.832254	0.075461	1.840418	0.028848	1.90314	0.009395
	λ	1.98552	0.529794	1.969198	0.378364	2.17071	0.029253
	θ	3.115323	2.024189	3.09939	1.717343	3.376788	0.142541
25	μ	1.819863	0.078799	1.831903	0.030732	1.888911	0.012348
	λ	1.974714	0.443821	1.952721	0.292834	2.152161	0.023684
	θ	3.121785	1.513801	3.093973	1.135917	3.33197	0.110757
30	μ	1.823525	0.070251	1.820688	0.033742	1.883398	0.013603
	λ	1.988249	0.336573	1.996682	0.24723	2.13752	0.019348
	θ	3.131693	1.475449	3.130799	1.084067	3.312506	0.098673
35	μ	1.810801	0.072619	1.816926	0.034872	1.875824	0.01544
	λ	2.00728	0.342903	1.99347	0.210612	2.117548	0.014107
	θ	3.043152	1.014162	3.034095	0.825918	3.286303	0.082193
40	μ	1.802614	0.070537	1.810058	0.03697	1.872768	0.016198
	λ	2.038157	0.277723	2.026293	0.190631	2.107601	0.012125
	θ	3.038173	0.852317	3.017779	0.626902	3.250628	0.063194

The estimation of the parameters by different methods along with the estimated value of the likelihood function and the Akaike and Bayesian information criterion are given in Table 3.

B) According to part A, the results of the goodness of fit of the data set of the lifetime of 30 passenger car tires of one of the brands made in Iran by month are reported as follows. A number of them are out of order due to flaking and cracking from the side due to improper baking and materials within the warranty period as censor data is marked in yellow. The estimation of the parameters by different methods along with the estimated value of the likelihood function and the Akaike information criterion $2k - 2\log(\hat{L})$ and Bayesian information criterion $k\log(n) - 2\log(\hat{L})$ are given in Table 4.

45	48	60	34	48	38	28	36
29	37	48	40	39	45	54	39
34	53	62	48	48	48	57	62
48	53	36	78	38	48		

Table 3. Estimation of distribution parameters according to estimation methods and Akaike information criterion for data set A.

parameter	Moment estimation	Maximum likelihood estimation	Interval shrinkage estimation
μ	33.7421	22.61001	35.62680
λ	65.0612	167.5001	68.31434
θ	23.8558	34.48529	23.47103
$-l(\mu, \theta, \lambda)$	197.225	178.0762	173.4112
Akaike information criterion	400.450	360.1524	352.8224
Bayesian information criterion	399.256	358.9602	351.6284

Table 4. Estimation of distribution parameters according to estimation methods and Akaike information criterion for data set B.

parameter	Moment estimation	Maximum likelihood estimation	Interval shrinkage estimation
μ	35.08413	28.00123	36.72793
λ	13.17904	20.84615	13.17904
θ	95.89521	3135.500	115.8952
$-l(\mu, \theta, \lambda)$	128.6023	113.7966	109.3064
Akaike information criterion	263.2064	233.5932	224.6128
Bayesian information criterion	262.0124	232.3992	223.4188

According to the results of Tables 3 and 4 and using the Akaike and Bayesian information criterion, the interval shrinkage estimator performs better compared to the maximum likelihood and moment estimators. Therefore, it is recommended to use the interval shrinkage estimator to estimate the parameters of the two-parameter exponential distribution with randomized censored data.

7. Discussion

In presenting different estimators to check their efficiency in estimating statistical distribution parameters using statistical criteria, the aim is to introduce efficient estimators. In this article, to estimate the parameters of two-parameter exponential distribution with random censored data, maximum likelihood, moment and interval shrinkage methods are discussed for parameter estimation. So that the aim was to show the superiority of the interval shrinkage estimator compared to other estimators, and the simulation results and the goodness of fit of the two-parameter exponential distribution with random censored data using real data show this superiority. It is worth noting that interval use of the parameter space leads to improvement of the accuracy of the interval shrinkage estimator, which is another advantage of the interval shrinkage estimator.

8. Acknowledgment

The authors of the article are grateful to the respected referees and the editor of the journal for their constructive suggestions to improve this article.

References

[1] Ahmad, SP, & Bhat, BA. (2010). Posterior Estimates of Two-Parameter Exponential Distribution using SPLUS Software. Journal of Reliability and Statistical Studies, 3(2) 27–34.

[2] Baloui, M., Deiri, AE, Hormozinejad, F., & Jamkhane, G. (2021). Efficiency of some shrinking estimators of Pareto-Rayleigh distribution shape parameter. Autumn and Winter Journal of Statistical Sciences, 15(2), 407–427.

[3] Bartholomew, DJ. (1957). A problem in life testing. J. Amer, Statist. Assoc., 52(279), 350–355.

[4] Balakrishnan, N., & Sundhu, RA. (1996). Best Linear Unbiased and Maximum Likelihood Estimation for Exponential Distributions Under General Progressive Type-II Censored Samples. Sankhya: The Indian Journal of Statistics, Series B, Part 1, 58(1), 1–9.

[5] Ban Ghanim, RS., AL-Ani, AL-Rassam, RS., & Rashed, SN. (2020). Bayesian Estimation for Two-Parameter Exponential Distribution Using Linear Transformation of Reliability Function, 8(1), 242–247.

[6] Davis, DJ. (1952). The analysis of some failure data. J. Amer, Statist. Assoc., 47, 133–150.

[7] Epstein, B. (1958). Exponential distribution and its role in life testing. Industrial Quality Control., 15, 4–6.

[8] Gilbert, JP. (1962), Random censorship. Ph.D. Thesis, University of Chicago.

[9] Golosnoy, V., & Liesenfeld, R. (2011). Interval shrinkage estimators, J. Appl. Stat., 38, 465–477.

[10] Hussein Ali AL-Hakeema, Jayanthi Arasanb, & Mohd Shafie Bin Mustafab. (2023). Parameter estimation for the generalized exponential distribution in the presence of interval censored data and covariate. Int. J. Nonlinear Anal. Appl., 14(1) 739–751.

[11] Krishna, H., & Goel, N. (2017). Classical and Bayesian Inference in Two Parameter Exponential Distribution with Randomly Censored Data. Computational Statistics, 33, 249–275.

[12] Kourouklis, S. (1994). Estimation in The Two-Parameter Exponential Distribution with Prior Information. IEEE Transactions on reliability, 43, 446–450.

[13] Koziol JA., & Green, SB. (1976). A Cramer-von Mises statistic for random censored data. Biometrika, 63, 465–474.

[14] Lam, BK., Sinha, & Z, Wu. (1994). Estimation of Parameters in a Two-Parameter Exponential Distribution using Ranked Set Sample. Ann. Inst. Statist. Math., 46(4), 723–736.

[15] Nasiri, P. (2022). Interval shrinkage estimation of the parameter of exponential distribution in the presence of outliers under loss functions. Statistics in Transition new series, 23(3), 65–78.

[16] Sohrabi, E., & Jabbari Nooghabi, M. (2024). Estimation of the parameters of Weibull distribution and net premium against outliers. Iranian Journal of Insurance Research, 13(1), 43–60.

[17] Thompson, JR. (1968). Some shrinkage techniques for estimating the mean. Journal of American Statistical Association, 63, 113–122.

[18] Upadhyay, SK., & Singh, U. (2007). Bayes Estimator for Two-Parameter Exponential Distribution. Commun. Statist. Theor. Meth., 24(1), 227–240.

Ali Soori
Orcid number: 0000-0003-2675-8728
Department of Statistics
Ahvaz Islamic Azad University
Ahvaz, Iran
Email address: ali.sori50@gmail.com

Parviz Nasiri
Orcid number: 0000-0002-0827-4853
Department of Statistics
Payam Noor University, 4697-19395
Tehran, Iran
Email address: pnasiri@pnu.ac.ir

Mehdi Jabbari Nooghabi
Orcid number: 0000-0002-5636-2209
Department of Statistics
Ferdowsi University of Mashhad
Mashhad, Iran
Email address: jabbarinm@um.ac.ir

Farshin Hormozinejad
Orcid number: 0000-0001-6064-1596
Department of Statistics
Ahvaz Islamic Azad University
Ahvaz, Iran
Email address: hormozi-nejad@iauahvaz.ac.ir

Mohammadreza Ghalani
Orcid number: 0009-0007-2856-7999
Department of Statistics
Ahvaz Islamic Azad University
Ahvaz, Iran
Email address: m_ghalani@yahoo.com