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SOME INEQUALITIES FOR EIGENVALUES OF AN ELLIPTIC DIFFERENTIAL OPERATOR

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Abstract. In the present paper, we investigate the eigenvalues of an elliptic differential operator on compact Riemannian manifolds with boundary and derive a general inequality for these eigenvalues. Applying this inequality, we give universal estimates for eigenvalues on compact domains of complete submanifolds in an Euclidean space, and of complete manifolds admitting special functions. Finally, we find universal bounds on the $(k+1)$-th eigenvalue on such objects in terms of the first k eigenvalues independent of the domains.

Keywords: Universal bound, Elliptic operator, Eigenvalue, submanifolds.
2020 MSC: Primary 35P15, 53C20, 53C42, 58J 50.

1. Introduction

Let (M^n, g) be a Riemannian manifold with local coordinate system $\{x_i\}_{i=1}^n$. The most important geometric operator on M is the Laplace-Beltrami operator which is defined by

$$\Delta = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{m} \partial_i(\sqrt{\det g}\, g^{ij} \partial_j) = g^{ij}(\partial_i\partial_j - \Gamma_{ij}^k \partial_k)$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $(g^{ij}) = (g_{ij})^{-1}$ and Γ_{ij}^k are the Christoffel symbols of the Riemannian metric $g = (g_{ij})$. The Riemannian measure $d\mu$ on (M, g) is defined by $d\mu = \sqrt{\det g} dx^1 \wedge ... \wedge dx^n$. Throughout this paper, the integrations on M are always taken with respect to $d\mu$. Suppose that Ω is a bounded domain in a Riemannian manifold M. Since the equation

$$\int_\Omega v(\Delta^2 - \Delta)u = \int_\Omega u(\Delta^2 - \Delta)v,$$

holds for all functions v such that vanish on $\partial\Omega$, we infer the operator $\Delta^2 - \Delta$ is a self-adjoint and hence its eigenvalues are real and discrete. If Λ_1 is the first non-zero eigenvalue of the operator $\Delta^2 - \Delta$ and $\Lambda_i,\ i = 1, 2, \cdots$ is the i-th eigenvalue of the operator $\Delta^2 - \Delta$ then we can write

$$0 < \Lambda_1 \leq \Lambda_2 \leq \cdots .$$

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Obtaining a nice estimate of the eigenvalues of a geometric operator is very important in mathematics and physics. We will focus our attention on this problem. Payn et al. [12] established that the inequality

$$\Lambda_{k+1}-\Lambda_k\leq\frac{4}{kn}\sum_{i=1}^{k}\Lambda_i,\quad k=1,2,\cdots. \tag{1}$$

holds for the Dirichlet eigenvalues of the Laplace operator on $\Omega\subset\mathbb{R}^n$. Recently, many interesting generalizations of (1) have been obtained. For instance, in [16] Yang proved the following inequality

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_{k+1}-(1+\frac{4}{n})\Lambda_i)\leq 0,\ \text{for } k=1,2,\cdots,$$

for Dirichlet eigenvalues of the Laplace operator on $\Omega\subset\mathbb{R}^n$. Then Harrell et al. [7] extended the Yang's inequality of Dirichlet eigenvalues of the Laplacian as follows

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\leq\frac{4}{n}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_i+\frac{n^2}{4}H_0^2),$$

on bounded domain Ω in a complete Riemannian manifold M^n isometrically immersed in $\mathbb{R}^N$, where H is the mean curvature vector field of M^n and $H_0=\sup_\Omega|H|$. Also, Wang and Xia in [15] studied eigenvalues of the clamped plate problem

$$\begin{cases}\Delta^2 u=\Lambda u & in\ \Omega,\\ u=\frac{\partial u}{\partial\nu}=0 & on\ \partial\Omega,\end{cases} \tag{2}$$

and proved

$$\begin{aligned}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\quad\leq\quad&\frac{1}{n}\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(n^2H_0^2+(2n+4)\Lambda_i^{\frac{1}{2}}\right)\right\}^{\frac{1}{2}}\\&\times\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)(4\lambda_i^{\frac{1}{2}}+n^2H_0^2)\right\}^{\frac{1}{2}}.\end{aligned}$$

Here Λ is called the eigenvalue of the clamped plate problem and Λ_i, $i=1,2,\ldots$ is the i-th eigenvalue of the problem (2). When M is the unit n-sphere, Chen et al. [3] obtained the above inequality. Also, when M is an n-dimensional hypersurface in $\mathbb{R}^{n+1}$, Harrell [6] established the last inequality.

Payn et al. showed [12] that the eigenvalues of the problem (2) for $\Omega\subset\mathbb{R}^n$ satisfy

$$\Lambda_{k+1}-\Lambda_k\leq\frac{8(n+2)}{kn^2}\sum_{i=1}^{k}\Lambda_i,\quad k=1,2,\cdots. \tag{3}$$

Then, Hile and Yeh [9] obtained the following inequality as a generalization of (3),

$$\sum_{i=1}^{k} \frac{\Lambda_i^{\frac{1}{2}}}{\Lambda_{k+1}-\Lambda_i} \geq \frac{n^2 k^{\frac{3}{2}}}{8(n+2)} \left(\sum_{i=1}^{k} \Lambda_i \right)^{-\frac{1}{2}},$$

and Cheng and Yang [4] provided the estimate

$$\Lambda_{k+1} - \frac{1}{k}\sum_{i=1}^{k} \Lambda_i \leq \left(\frac{8(n+2)}{n^2} \right)^{\frac{1}{2}} \frac{1}{k} \sum_{i=1}^{k} \left(\Lambda_i(\Lambda_{k+1}-\Lambda_i) \right)^{\frac{1}{2}}.$$

Also, see [14] for the study of the spectral geometry. The goal of this article is to further investigate the relation between the spectrum of the operator $\Delta^2 - \Delta$ and the local differential geometry of submanifolds of arbitrary codimension. Suppose that Ω is a connected bounded domain with smooth boundary $\partial\Omega$ in a Riemannnian manifold M^n with $n \geq 2$ and ν is the outward unit normal vector field of $\partial\Omega$.
Motivated by the above works and [1], in our paper we investigated the Dirchelet eigenvalues of the operator $\Delta^2 - \Delta$ on Riemannian manifolds. We will use Yang's method to give a general inequality for these eigenvalues. Then we derive universal inequalities for them on compact domains of complete submanifolds in Euclidean space and of complete manifolds admitting special functions which include the Hadamard manifolds with Ricci curvature bounded below. Also, we compute universal inequalities for them on a class of warped product manifolds containing the hyperbolic space and manifolds admitting spherical eigenmaps. Our main results are as follows.

Theorem 1.1. *Suppose that M^n is a complete Riemannian manifold and Ω is a bounded domain with a smooth boundary in M. Suppose that ν is the outward unit normal of $\partial\Omega$. Let Λ_i, $i = 1, \cdots$, be the i-th eigenvalue of the problem*

$$\begin{cases} (\Delta^2 - \Delta) f = \Lambda f & \text{in } \Omega, \\ f = \frac{\partial f}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where Λ is a constant and

$$\begin{cases} (\Delta^2 - \Delta) f_i = \Lambda_i f_i & \text{in } \Omega, \\ f_i = \frac{\partial f_i}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_\Omega f_i f_j = \delta_{ij}, \quad \forall i,j = 1, \cdots, k. \end{cases} \tag{5}$$

Then for every smooth function $h : \Omega \to \mathbb{R}$, every positive integer k and any $\delta > 0$, we get

$$\sum_{i=1}^{k} (\Lambda_{k+1}-\Lambda_i)^2 \int_\Omega f_i^2 |\nabla h|^2 \leq \delta \sum_{i=1}^{k} (\Lambda_{k+1}-\Lambda_i)^2 q_i + \sum_{i=1}^{k} \frac{(\Lambda_{k+1}-\Lambda_i)}{\delta} \int_\Omega (\nabla h . \nabla f_i + \frac{f_i \Delta h}{2})^2, \tag{6}$$

where

(7)
$$q_i = \int_\Omega \left[f_i^2(\Delta h)^2 + 4(|\nabla h.\nabla f_i|^2 + u_i \Delta h \nabla h.\nabla f_i) - 2f_i|\nabla h|^2\Delta f_i + f_i^2|\nabla h|^2 \right].$$

Theorem 1.2. *Suppose that M^n is a complete Riemannian manifold and Ω is a bounded domain with a smooth boundary in M. Let Λ_i, $i = 1, \cdots$, be the i-th eigenvalue of the problem (4).*
i) Let M be isometrically immersed in $\mathbb{R}^m$ and H be its mean curvature vector. Then

$$\begin{aligned} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 &\leq \frac{1}{n} \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \left(4\Lambda_i^{\frac{1}{2}} + nH_0^2 \right) \right\}^{\frac{1}{2}} \\ &\times \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \left(nH_0^2 + (4+2n)\Lambda_i^{\frac{1}{2}} + n \right) \right\}^{\frac{1}{2}}, \end{aligned} \tag{8}$$

where $H_0 = \sup_\Omega |H|$.
ii) Let M be a minimal submanifold in $\mathbb{R}^m$, then

$$\begin{aligned} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 &\leq \frac{2}{n} \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)\Lambda_i^{\frac{1}{2}} \right\}^{\frac{1}{2}} \\ &\times \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \left((4+2n)\Lambda_i^{\frac{1}{2}} + n \right) \right\}^{\frac{1}{2}}. \end{aligned} \tag{9}$$

iii) Let there exists a function $\phi : \Omega \to \mathbb{R}$ and a constant A_0 such that

$$|\nabla \phi| = 1, \;\; |\Delta \phi| \leq A_0, \;\; \text{on } \Omega, \tag{10}$$

then

$$\begin{aligned} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 &\leq \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \left(A_0^2 + 1 + 6\Lambda_i^{\frac{1}{2}} + 4A_0\Lambda_i^{\frac{1}{4}} \right) \right\}^{\frac{1}{2}} \\ &\times \left\{ \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)(2\Lambda_i^{\frac{1}{4}} + A_0)^2 \right\}^{\frac{1}{2}}. \end{aligned} \tag{11}$$

iv) Let there exists a constant B_0 and a function $\psi : \Omega \to \mathbb{R}$ with

$$|\nabla \psi| = 1, \;\; \Delta \psi = B_0, \;\; \text{on } \Omega, \tag{12}$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(1-B_0^2+6\Lambda_i^{\frac{1}{2}}\right)\right\}^{\frac{1}{2}} \times\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)(4\Lambda_i^{\frac{1}{2}}-B_0^2)\right\}^{\frac{1}{2}}. \tag{13}$$

v) Let there exists an eigenmap $u=(u_1,u_2,\cdots,u_{m+1}):\Omega\to\mathbb{S}^m(1)$ *such that*

$$\Delta u_\alpha=\mu u_\alpha,\ \alpha=1,\cdots,m+1,\ \sum_{\alpha=1}^{m+1}u_\alpha^2=1, \tag{14}$$

for some constant μ*, then*

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(\mu+1+6\Lambda_i^{\frac{1}{2}}\right)\right\}^{\frac{1}{2}} \times\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)(4\Lambda_i^{\frac{1}{2}}+\mu)\right\}^{\frac{1}{2}}. \tag{15}$$

vi) Let there exists l *functions* $\phi_\alpha:\Omega\to\mathbb{R}$ *with*

$$\langle\nabla\phi_\alpha,\nabla\phi_\beta\rangle=\delta_{\alpha\beta},\ \ \Delta\phi_\alpha=0,\ \ \text{on } \Omega,\ \ \alpha,\beta=1,\cdots,l, \tag{16}$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\leq 2\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2(1+6\Lambda_i^{\frac{1}{2}})\right\}^{\frac{1}{2}}\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Lambda_i^{\frac{1}{2}}\right\}^{\frac{1}{2}}. \tag{17}$$

Corollary 1.3. *Suppose that* Ω *is a bounded domain with a smooth boundary in a complete Riemannian manifold* M^n*.*
i) Let M *be isometrically immersed in* $\mathbb{R}^m$ *and* H *be its mean curvature vector, then*

$$\Lambda_{k+1}\leq a_{k+1}+\sqrt{a_{k+1}^2-b_{k+1}}, \tag{18}$$

where

$$a_{k+1}=\frac{1}{k}\sum_{i=1}^{k}\Lambda_i^2+\frac{1}{kn^2}\sum_{i=1}^{k}\Lambda_i\left(4\Lambda_i^{\frac{1}{2}}+nH_0^2\right)\left(nH_0^2+(4+2n)\Lambda_i^{\frac{1}{2}}+n\right),$$

and

$$b_{k+1}=\frac{1}{k}\sum_{i=1}^{k}\Lambda_i+\frac{1}{2kn^2}\sum_{i=1}^{k}\left(4\Lambda_i^{\frac{1}{2}}+nH_0^2\right)\left(nH_0^2+(4+2n)\Lambda_i^{\frac{1}{2}}+n\right),$$

where $H_0 = \sup_\Omega |H|$.

ii) Let M *be a minimal submanifold in* $\mathbb{R}^m$*, then*

$$\Lambda_{k+1} \leq c_{k+1} + \sqrt{c_{k+1}^2 - d_{k+1}}, \tag{19}$$

where

$$c_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i^2 + \frac{4}{kn^2}\sum_{i=1}^{k}\Lambda_i^{\frac{3}{2}}\left((4+2n)\Lambda_i^{\frac{1}{2}} + n\right),$$

and

$$d_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i + \frac{2}{kn^2}\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}\left((4+2n)\Lambda_i^{\frac{1}{2}} + n\right).$$

iii) Let there exist a constant A_0 *and a function* $\phi : \Omega \to \mathbb{R}$ *with*

$$|\nabla\phi| = 1, \ \ |\Delta\phi| \leq A_0, \ \ \textit{on } \Omega, \tag{20}$$

then

$$\Lambda_{k+1} \leq t_{k+1} + \sqrt{t_{k+1}^2 - w_{k+1}}, \tag{21}$$

where

$$t_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i^2 + \frac{1}{k}\sum_{i=1}^{k}\Lambda_i(2\Lambda_i^{\frac{1}{4}} + A_0)^2\left(A_0^2 + 1 + 6\Lambda_i^{\frac{1}{2}} + 4A_0\Lambda_i^{\frac{1}{4}}\right),$$

and

$$w_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i + \frac{1}{2k}\sum_{i=1}^{k}(2\Lambda_i^{\frac{1}{4}} + A_0)^2\left(A_0^2 + 1 + 6\Lambda_i^{\frac{1}{2}} + 4A_0\Lambda_i^{\frac{1}{4}}\right).$$

iv) Let there exists a constant B_0 *and a function* $\psi : \Omega \to \mathbb{R}$ *with*

$$|\nabla\psi| = 1, \ \ \Delta\psi = B_0, \ \ \textit{on } \Omega, \tag{22}$$

then

$$\Lambda_{k+1} \leq A_{k+1} + \sqrt{A_{k+1}^2 - B_{k+1}}, \tag{23}$$

where

$$A_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i^2 + \frac{1}{k}\sum_{i=1}^{k}\Lambda_i(4\Lambda_i^{\frac{1}{2}} - B_0^2)\left(1 - B_0^2 + 6\Lambda_i^{\frac{1}{2}}\right),$$

and

$$B_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i + \frac{1}{2k}\sum_{i=1}^{k}(4\Lambda_i^{\frac{1}{2}} - B_0^2)\left(1 - B_0^2 + 6\Lambda_i^{\frac{1}{2}}\right).$$

v) Let there exist an eigenmap $u = (u_1, u_2, \cdots, u_{m+1}) : \Omega \to \mathbb{S}^m(1)$ *such that*

$$\Delta u_\alpha = \mu u_\alpha, \ \alpha = 1, \cdots, m+1, \ \sum_{\alpha=1}^{m+1} u_\alpha^2 = 1, \tag{24}$$

for some constant μ, then

$$\Lambda_{k+1} \leq C_{k+1} + \sqrt{C_{k+1}^2 - D_{k+1}}, \tag{25}$$

where

$$C_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i^2 + \frac{1}{k}\sum_{i=1}^{k}\Lambda_i(4\Lambda_i^{\frac{1}{2}} + \mu)\left(\mu + 1 + 6\Lambda_i^{\frac{1}{2}}\right),$$

and

$$D_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i + \frac{1}{2k}\sum_{i=1}^{k}(4\Lambda_i^{\frac{1}{2}} + \mu)\left(\mu + 1 + 6\Lambda_i^{\frac{1}{2}}\right).$$

vi) If there exists functions $\phi_\alpha : \Omega \to \mathbb{R}$ such that

$$\langle \nabla\phi_\alpha, \nabla\phi_\beta\rangle = \delta_{\alpha\beta}, \quad \Delta\phi_\alpha = 0, \quad \text{on } \Omega, \quad \alpha, \beta = 1, ..., l, \tag{26}$$

then

$$\Lambda_{k+1} \leq T_{k+1} + \sqrt{T_{k+1}^2 - W_{k+1}}, \tag{27}$$

where

$$T_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i^2 + \frac{4}{k}\sum_{i=1}^{k}\Lambda_i^{\frac{3}{2}}\left(1 + 6\Lambda_i^{\frac{1}{2}}\right),$$

and

$$W_{k+1} = \frac{1}{k}\sum_{i=1}^{k}\Lambda_i + \frac{2}{k}\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}\left(1 + 6\Lambda_i^{\frac{1}{2}}\right).$$

2. Proofs of the main results

By similar methods as in [1, 9, 15], we use Yang's method to give a general inequality for eigenvalues.

Proof of Theorem 1.1. For $i = 1, ..., k$, let $\phi_i : \Omega \to \mathbb{R}$ be the functions which are given by $\phi_i = hu_i - \sum_{j=1}^{k} r_{ij} f_j$ where, $r_{ij} = \int_\Omega h f_i f_j$. We have $\phi_i|_{\partial\Omega} = \frac{\partial\phi_i}{\partial\nu}|_{\partial\Omega} = 0$ and

$$\int_\Omega f_i\phi_j = 0, \quad \forall i, j = 1, \cdots, k.$$

The Rayleigh-Ritz inequality yields

$$\Lambda_{k+1} \leq \frac{\int_\Omega \phi_i(\Delta^2 - \Delta)\phi_i}{\int_\Omega \phi_i^2}. \tag{28}$$

Letting

$$p_i = \Delta h\Delta f_i + 2\nabla h.\nabla\Delta f_i + \Delta(f_i\Delta h) + 2\Delta(\nabla f_i.\nabla h) - f_i\Delta h - 2\nabla h.\nabla f_i, \tag{29}$$

we obtain

$$\begin{aligned}
\int_\Omega \phi_i(\Delta^2-\Delta)\phi_i &= \int_\Omega \phi_i\left[(\Delta^2-\Delta)(hf_i)-\sum_{j=1}^k r_{ij}\Lambda_j f_j\right]\\
&= \int_\Omega \phi_i(\Delta^2-\Delta)(hf_i)\\
&= \int_\Omega \phi_i\Big[\Delta h\Delta f_i + h\Delta^2 f_i + 2\nabla h\cdot\nabla\Delta f_i + \Delta(f_i\Delta h)\\
&\qquad +2\Delta(\nabla f_i.\nabla h) - f_i\Delta h - h\Delta f_i - 2\nabla h\cdot\nabla f_i\Big]\\
&= \int_\Omega hf_ip_i - \sum_{j=1}^k r_{ij}\int_\Omega f_jp_i + \int_\Omega h\Lambda_i f_i\phi_i\\
&= \int_\Omega hf_ip_i - \sum_{j=1}^k r_{ij}\int_\Omega f_jp_i + \Lambda_i\int_\Omega \phi_i\left[\phi_i+\sum_{j=1}^k r_{ij}f_j\right]\\
&= \int_\Omega hf_ip_i - \sum_{j=1}^k r_{ij}\int_\Omega f_jp_i + \Lambda_i\int_\Omega \phi_i^2.
\end{aligned}$$

Therefore

$$\int_\Omega \phi_i(\Delta^2-\Delta)\phi_i = \Lambda_i||\phi_i||^2 - \sum_{j=1}^k r_{ij}s_{ij} + q_i, \tag{30}$$

where

$$q_i = \int_\Omega hf_ip_i, \quad s_{ij} = \int_\Omega f_jp_i. \tag{31}$$

Multiplying both sides of $(\Delta^2-\Delta)f_i = \Lambda_i f_i$ by hf_j, we arrive at

$$hf_j(\Delta^2-\Delta)f_i = \Lambda_i hf_jf_i, \tag{32}$$

changing i and j, we conclude

$$hf_i(\Delta^2-\Delta)f_j = \Lambda_j hf_jf_i. \tag{33}$$

Using (32) and (33), we find

$$(\Lambda_j-\Lambda_i)r_{ij} = s_{ij}. \tag{34}$$

Observe that

$$\begin{aligned}
&\int_\Omega hf_i\Big[\Delta(f_i\Delta h) + 2\Delta(\nabla f_i\cdot\nabla h) + 2\nabla h\cdot\nabla\Delta f_i + \Delta h\Delta f_i\Big]\\
&= \int_\Omega \Big[f_i^2(\Delta h)^2 + 4(|\nabla h\cdot\nabla f_i|^2 + f_i\Delta h\nabla h\cdot\nabla f_i) - 2f_i|\nabla h|^2\Delta f_i\Big],
\end{aligned}$$

and

$$\int_\Omega hf_i\nabla h\cdot\nabla f_i = -\int_\Omega f_i\nabla(hf_i\nabla h)$$
$$= -\int_\Omega f_i^2|\nabla h|^2 - \int_\Omega f_ih\nabla f_i\cdot\nabla h - \int_\Omega f_i^2 h\Delta h,$$

thus

(35) $$\int_\Omega hf_i\nabla h\cdot\nabla f_i = -\frac{1}{2}\int_\Omega f_i^2|\nabla h|^2 - \frac{1}{2}\int_\Omega f_i^2 h\Delta h,$$

and

(36) $$q_i = \int_\Omega \Big[f_i^2(\Delta h)^2 + 4\big(|\nabla h\cdot\nabla f_i|^2 + f_i\Delta h\nabla h\cdot\nabla f_i\big) - 2f_i|\nabla h|^2\Delta f_i + f_i^2|\nabla h|^2\Big].$$

Hence, from (28) and (30) we get

(37) $$\Lambda_i||\phi_i||^2 - \sum_{j=1}^k r_{ij}s_{ij} + q_i \geq \Lambda_{k+1}||\phi_i||^2.$$

Plugging (34) into (37), we infer

(38) $$(\Lambda_{k+1}-\Lambda_i)||\phi_i||^2 \leq q_i + \sum_{j=1}^k(\Lambda_i-\Lambda_j)r_{ij}^2.$$

Set $t_{ij} = \int_\Omega f_j\Big(\nabla h\cdot\nabla f_i + \frac{f_i\Delta h}{2}\Big)$. We have $t_{ij} = -t_{ji}$ and

(39) $$\int_\Omega -2\phi_i\Big(\nabla h\cdot\nabla f_i + \frac{f_i\Delta h}{2}\Big) = \int_\Omega f_i^2|\nabla h|^2 + 2\sum_{j=1}^k r_{ij}t_{ij}.$$

Multiplying equation (39) by $(\Lambda_{k+1}-\Lambda_i)^2$ and applying the Schwarz inequality for any $\delta > 0$, we conclude

(40) $$(\Lambda_{k+1}-\Lambda_i)^2\left(\int_\Omega f_i^2|\nabla h|^2 + 2\sum_{j=1}^k r_{ij}t_{ij}\right)$$
$$= (\Lambda_{k+1}-\Lambda_i)^2\int_\Omega -2\phi_i\left(\nabla h\cdot\nabla f_i + \frac{f_i\Delta h}{2} - \sum_{j=1}^k t_{ij}f_j\right)$$
$$\leq \delta(\Lambda_{k+1}-\Lambda_i)^3||\phi_i||^2 + \frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\int_\Omega|\nabla h\cdot\nabla f_i + \frac{f_i\Delta h}{2} - \sum_{j=1}^k t_{ij}f_j|^2$$
$$= \delta(\Lambda_{k+1}-\Lambda_i)^3||\phi_i||^2 + \frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\left(\int_\Omega(\nabla h\cdot\nabla f_i + \frac{f_i\Delta h}{2})^2 - \sum_{j=1}^k t_{ij}^2\right).$$

Substituting (38) into (40), summing over i from 1 to k, and using $r_{ij} = r_{ji}$, $t_{ij} = -t_{ji}$, we infer

$$\begin{aligned}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\int_\Omega f_i^2|\nabla h|^2 - 2\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_i-\Lambda_j)r_{ij}t_{ij} \\ \le \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 q_i + \delta\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2(\Lambda_i-\Lambda_j)r_{ij}^2 \\ +\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\int_\Omega(\nabla h\cdot\nabla f_i+\frac{f_i\Delta h}{2})^2 - \sum_{i,j=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}t_{ij}^2.\end{aligned} \tag{41}$$

Hence,

$$\begin{aligned}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\int_\Omega f_i^2|\nabla h|^2 \le\ & \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 q_i \\ & +\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\int_\Omega(\nabla h\cdot\nabla f_i+\frac{f_i\Delta h}{2})^2 \\ & -\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Big(\sqrt{\delta}(\Lambda_i-\Lambda_j)r_{ij}-\frac{1}{\sqrt{\delta}}t_{ij}\Big)^2,\end{aligned} \tag{42}$$

which implies (6).

Proof of Theorem 1.2. Assume that $\{f_i\}_{i=1}^{\infty}$ satisfy (5).
i) We denote the standard coordinate functions of $\mathbb{R}^m$ by y_α, $\alpha = 1, \cdots, m$. Inserting $h = y_\alpha$ in (6) and summing over α, we obtain

$$\begin{aligned}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\sum_{\alpha=1}^{m}\int_\Omega f_i^2|\nabla y_\alpha|^2 \le\ & \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\sum_{\alpha=1}^{m}q_i \\ & +\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\sum_{\alpha=1}^{m}\int_\Omega\Big(\nabla y_\alpha\cdot\nabla f_i+\frac{f_i\Delta y_\alpha}{2}\Big)^2.\end{aligned} \tag{43}$$

If the manifold M is isometrically immersed in $\mathbb{R}^m$ then

$$\sum_{\alpha=1}^{m}|\nabla y_\alpha|^2 = n, \tag{44}$$

$$\sum_{\alpha=1}^{m}\int_\Omega f_i^2|\nabla y_\alpha|^2 = n, \tag{45}$$

$$\Delta(y_1, ..., y_m) = (\Delta y_1, ..., \Delta y_m) = nH, \tag{46}$$

$$\sum_{\alpha=1}^{m}(\nabla y_\alpha\cdot\nabla f_i)^2 = \sum_{\alpha=1}^{m}(\nabla f_i(y_\alpha))^2 = |\nabla f_i|^2, \tag{47}$$

and

$$\sum_{\alpha=1}^{m} \Delta y_\alpha \nabla y_\alpha \cdot \nabla f_i = \sum_{\alpha=1}^{m} \Delta y_\alpha \nabla f_i(y_\alpha) = nH.\nabla f_i = 0. \tag{48}$$

Hence,

$$\begin{aligned} \sum_{\alpha=1}^{m} q_i &= \sum_{\alpha=1}^{m} \int_\Omega \Big[f_i^2 (\Delta y_\alpha)^2 + 4(|\nabla y_\alpha \cdot \nabla f_i|^2 + f_i \Delta y_\alpha \nabla y_\alpha \cdot \nabla f_i) \\ &\quad -2u_i |\nabla y_\alpha|^2 \Delta f_i + f_i^2 |\nabla y_\alpha|^2 \Big] \\ &= \int_\Omega \Big(n u_i^2 |H|^2 + 4|\nabla f_i|^2 - 2n f_i \Delta f_i \Big) + n, \end{aligned} \tag{49}$$

and substituting (44)-(48) into (43), we conclude

$$\begin{aligned} n\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 &\leq \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \int_\Omega \Big[n u_i^2 |H|^2 + 4|\nabla f_i|^2 - 2n f_i \Delta f_i \Big] \\ &\quad + n\delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 + \sum_{i=1}^{k} \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta} \int_\Omega \left(|\nabla f_i|^2 + \frac{n f_i^2 |H|^2}{4} \right). \end{aligned} \tag{50}$$

Also,

$$\begin{aligned} \int_\Omega \Big[4|\nabla f_i|^2 - 2n f_i \Delta f_i \Big] &= -\int_\Omega \Big[4u_i \Delta f_i + 2n f_i \Delta f_i \Big] \\ &= -(4+2n) \int_\Omega f_i \Delta f_i \\ &\leq (4+2n) \left(\int_\Omega f_i^2 \right)^{\frac{1}{2}} \left(\int_\Omega (\Delta f_i)^2 \right)^{\frac{1}{2}} \\ &\leq (4+2n) \left(\int_\Omega (\Delta f_i)^2 + |\nabla f_i|^2 \right)^{\frac{1}{2}} \\ &\leq (4+2n) \Lambda_i^{\frac{1}{2}}, \end{aligned}$$

which yields

$$\begin{aligned} n\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 &\leq \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \Big(nH_0^2 + (4+2n)\Lambda_i^{\frac{1}{2}} \Big) + n\delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \\ &\quad + \sum_{i=1}^{k} \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta} \left(\Lambda_i^{\frac{1}{2}} + \frac{nH_0^2}{4} \right). \end{aligned} \tag{51}$$

In the last inequality, we have applied $|H| \leq H_0$ and

$$\int_\Omega |\nabla f_i|^2 = -\int_\Omega f_i \Delta f_i \leq \Lambda_i^{\frac{1}{2}}.$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left(\Lambda_i^{\frac{1}{2}}+\frac{nH_0^2}{4}\right)}{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(nH_0^2+(4+2n)\Lambda_i^{\frac{1}{2}}+n\right)}\right\}^{\frac{1}{2}}$$

one gets (8).
ii) Let M be a minimal submanifold of $\mathbb{R}^m$, then $H_0=0$, and (8) yields (9).
iii) Applying $h=\phi$ into (6) and using Schwarz inequality and (1.3), it follows
(52)

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\leq\delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2q_i+\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\int_\Omega\left(\nabla\phi\cdot\nabla f_i+\frac{f_i\Delta\phi}{2}\right)^2,$$

and

$$\begin{aligned}q_i &= \int_\Omega\Big[f_i^2(\Delta\phi)^2+4(|\nabla\phi\cdot\nabla f_i|^2+f_i\Delta\phi\nabla\phi.\nabla f_i)\\ &\quad -2u_i|\nabla\phi|^2\Delta f_i+f_i^2|\nabla\phi|^2\Big]\\ &\leq A_0^2+1+\int_\Omega\Big[4(|\nabla f_i|^2+A_0|f_i||\nabla f_i|)-2f_i\Delta f_i\Big].\end{aligned}\tag{53}$$

Thus,

$$\begin{aligned}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 &\leq \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(A_0^2+1+\int_\Omega\left[4(|\nabla f_i|^2+A_0|f_i||\nabla f_i|)-2f_i\Delta f_i\right]\right)\\ &\quad+\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\int_\Omega\left(|\nabla f_i|^2+A_0|f_i||\nabla f_i|+\frac{A_0^2f_i^2}{4}\right).\end{aligned}$$

We have

$$\begin{aligned}\int_\Omega|f_i||\nabla f_i| &\leq \left(\int_\Omega f_i^2\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla f_i|^2\right)^{\frac{1}{2}}=\left(\int_\Omega|\nabla f_i|^2\right)^{\frac{1}{2}}\\ &= \left(-\int_\Omega f_i\Delta f_i\right)^{\frac{1}{2}}\\ &\leq \left(\int_\Omega f_i^2\right)^{\frac{1}{4}}\left(\int_\Omega(\Delta f_i)^2\right)^{\frac{1}{4}}\\ &\leq \left(\int_\Omega(\Delta f_i)^2+|\nabla f_i|^2\right)^{\frac{1}{4}}\\ &\leq \Lambda_i^{\frac{1}{4}},\end{aligned}$$

then

$$\begin{aligned}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 &\leq \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(A_0^2+1+6\Lambda_i^{\frac{1}{2}}+4A_0\Lambda_i^{\frac{1}{4}}\right)\\ &\quad+\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}(\Lambda_i^{\frac{1}{4}}+\frac{A_0}{2})^2.\end{aligned}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_i^{\frac{1}{4}}+\frac{A_0}{2})^2}{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(A_0^2+1+6\Lambda_i^{\frac{1}{2}}+4A_0\Lambda_i^{\frac{1}{4}}\right)}\right\}^{\frac{1}{2}}$$

we have (11).

iv) Applying $h=\psi$ into (6) and using (22), we get

(54)

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\leq\delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2q_i+\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\int_\Omega\left(\nabla\psi\cdot\nabla f_i+\frac{f_iB_0}{2}\right)^2,$$

and

$$\begin{aligned}(55)\qquad q_i &= \int_\Omega\Big[f_i^2(\Delta\psi)^2+4(|\nabla\psi\cdot\nabla f_i|^2+f_iB_0\nabla\psi\cdot\nabla f_i)\\ &\qquad -2u_i|\nabla\psi|^2\Delta f_i+f_i^2|\nabla\psi|^2\Big]\\ &\leq B_0^2+1+\int_\Omega\Big[4(|\nabla f_i|^2+f_iB_0\nabla\psi\cdot\nabla f_i)-2f_i\Delta f_i\Big]\end{aligned}$$

we have $\int_\Omega f_i\nabla\psi\cdot\nabla f_i=-\frac{1}{2}\int_\Omega f_i^2\Delta\psi=-\frac{B_0}{2}$, then

$$\begin{aligned}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 &\leq \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(1-B_0^2+6\Lambda_i^{\frac{1}{2}}\right)\\ &\quad+\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}(\Lambda_i^{\frac{1}{2}}-\frac{B_0^2}{4}).\end{aligned}$$

Taking

$$\delta=\left\{\frac{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)(\Lambda_i^{\frac{1}{2}}-\frac{B_0^2}{4})}{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(1-B_0^2+6\Lambda_i^{\frac{1}{2}}\right)}\right\}^{\frac{1}{2}}$$

we obtain (13).

v) Suppose that $u=(u_1,u_2,\cdots,u_{m+1}):\Omega\to\mathbb{S}^m(1)$ satisfies in (24). Taking twice Laplacian of both sides of the equation $\sum_{\alpha=1}^{m+1}u_\alpha^2=1$ and using the condition $\Delta u_\alpha = \mu u_\alpha$, $\alpha = 1, \cdots, m+1$, we get $\sum_{\alpha=1}^{m+1} |\nabla u_\alpha|^2 = \mu$. Applying $h = u_\alpha$ into (6) and summing over α from 1 to $m+1$, we get

$$\begin{aligned} \mu \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 &\leq \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \sum_{\alpha=1}^{m+1} q_i \\ &+ \sum_{i=1}^{k} \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta} \int_\Omega (\sum_{\alpha=1}^{m+1} |\nabla u_\alpha \cdot \nabla f_i|^2 + \frac{f_i^2 \mu^2}{4}), \end{aligned} \tag{56}$$

and

$$\begin{aligned} \sum_{\alpha=1}^{m+1} q_i &= \sum_{\alpha=1}^{m+1} \int_\Omega \Big[f_i^2 (\Delta u_\alpha)^2 + 4(|\nabla u_\alpha \cdot \nabla f_i|^2 + f_i \Delta u_\alpha \nabla u_\alpha \cdot \nabla f_i) \\ &\quad -2u_i |\nabla u_\alpha|^2 \Delta f_i + f_i^2 |\nabla f_\alpha|^2 \Big]. \end{aligned} \tag{57}$$

Taking the divergence of equation $\sum_{\alpha=1}^{m+1} u_\alpha^2 = 1$ we have $\sum_{\alpha=1}^{m+1} u_\alpha \nabla u_\alpha = 0$, then $\Delta u_\alpha = \mu u_\alpha$ implies that

$$\sum_{\alpha=1}^{m+1} f_i \Delta u_\alpha \nabla u_\alpha \cdot \nabla f_i = \mu \sum_{\alpha=1}^{m+1} f_i u_\alpha \nabla u_\alpha \cdot \nabla f_i = 0,$$

also, we can write

$$\sum_{\alpha=1}^{m+1} \int_\Omega f_i^2 (\Delta u_\alpha)^2 = \sum_{\alpha=1}^{m+1} \int_\Omega f_i^2 \mu^2 u_\alpha^2 = \mu^2 \int_\Omega f_i^2 = \mu^2,$$

and

$$\sum_{\alpha=1}^{m+1} \int_\Omega |\nabla u_\alpha \cdot \nabla f_i|^2 \leq \sum_{\alpha=1}^{m+1} \int_\Omega |\nabla u_\alpha|^2 |\nabla f_i|^2 = \mu \int_\Omega |\nabla f_i|^2.$$

Since $-\int_\Omega f_i \Delta f_i \leq \Lambda_i^{\frac{1}{2}}$ and $\int_\Omega |\nabla f_i|^2 = -\int_\Omega f_i \Delta f_i \leq \Lambda_i^{\frac{1}{2}}$ we deduce

$$q_i \leq \mu^2 + 6\mu \Lambda_i^{\frac{1}{2}} + \mu.$$

Therefore

$$\begin{aligned} \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 &\leq \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left(\mu + 1 + 6\Lambda_i^{\frac{1}{2}} \right) \\ &+ \sum_{i=1}^{k} \frac{(\Lambda_{k+1} - \Lambda_i)}{\delta} (\Lambda_i^{\frac{1}{2}} + \frac{\mu}{4}). \end{aligned}$$

We get (15) by taking

$$\delta = \left\{ \frac{\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)(\Lambda_i^{\frac{1}{2}} + \frac{\mu}{4})}{\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)^2 \left(\mu + 1 + 6\Lambda_i^{\frac{1}{2}} \right)} \right\}^{\frac{1}{2}}.$$

vi) Applying $h = y_\alpha$ into (6) we conclude

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 q_i+\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\int_\Omega(\nabla\phi_\alpha\cdot\nabla f_i)^2, \tag{58}$$

and

$$q_i = \int_\Omega\left[4|\nabla\phi_\alpha\cdot\nabla f_i|^2 - 2u_i\Delta f_i + f_i^2\right] \leq 1+6\lambda_i^{\frac{1}{2}} \tag{59}$$

Hence,

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2(1+6\lambda_i^{\frac{1}{2}})+\sum_{i=1}^{k}\frac{(\Lambda_{k+1}-\Lambda_i)}{\delta}\Lambda_i^{\frac{1}{2}}.$$

We obtain (17) by considering

$$\delta = \left\{\frac{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Lambda_i^{\frac{1}{2}}}{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2(1+6\Lambda_i^{\frac{1}{2}})}\right\}^{\frac{1}{2}}.$$

From [10], we have the following lemma.

Lemma 2.1. *Suppose that $\{a_i\}_{i=1}^m$, $\{b_i\}_{i=1}^m$, and $\{c_i\}_{i=1}^m$ are decreasing, increasing, and increasing sequences of nonnegative real numbers, respectively. Then*

$$\left(\sum_{i=1}^{m}a_i^2 b_i\right)\left(\sum_{i=1}^{m}a_i c_i\right) \leq \left(\sum_{i=1}^{m}a_i^2\right)\left(\sum_{i=1}^{m}a_i b_i c_i\right). \tag{60}$$

Proof of Corollary 1.3. i) $\{(\Lambda_{k+1}-\Lambda_i)\}_{i=1}^m$ is a decreasing sequence and $\left\{\left(4\Lambda_i^{\frac{1}{2}}+nH_0^2\right)\right\}_{i=1}^m$ and $\left\{\left(nH_0^2+(4+2n)\Lambda_i^{\frac{1}{2}}+n\right)\right\}_{i=1}^m$ are increasing sequences, then from (60), we have

$$\begin{aligned}&\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left(4\lambda_i^{\frac{1}{2}}+nH_0^2\right)\right\}\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left(nH_0^2+(4+2n)\Lambda_i^{\frac{1}{2}}+n\right)\right\}\\ \leq\ &\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\right\}\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left(4\Lambda_i^{\frac{1}{2}}+nH_0^2\right)\left(nH_0^2+(4+2n)\Lambda_i^{\frac{1}{2}}+n\right)\right\}.\end{aligned}$$

From (8), it follows

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \frac{1}{n^2}\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left(4\Lambda_i^{\frac{1}{2}}+nH_0^2\right)\left(nH_0^2+(4+2n)\Lambda_i^{\frac{1}{2}}+n\right)\right\}.$$

Solving the last inequality with respect to Λ_{k+1}, we obtain (18).
ii) Let M be a minimal submanifold of $\mathbb{R}^m$, we have $H_0 = 0$. Then (18) leads to (19).
Similarly, we can prove iii)-vi).

Now, we recall from [5] some examples of complete Riemannian manifolds with functions as in Theorem 1.2.

Example 2.2. *Suppose that M is an n-dimensional Hadamard manifold with bounded Ricci curvature of below by $-(n-1)K$ for some $K \geq 0$. Let $\gamma : [0,+\infty) \to M$ be a unit speed geodesic with $d(\gamma(s), \gamma(t)) = t - s$ for any $t > s > 0$. For the Busemann function b_γ corresponding to geodesic γ given by*

$$b_\gamma(x) = \lim_{t\to+\infty}(d(x, \gamma(t)) - t)$$

we have $|\nabla b_\gamma| = 1$ ([2, 8]). Also, from [13, Theorem 3.5] we get $|\Delta b_\gamma| \leq (n-1)\sqrt{K}$ on M. Thus, any Hadamard manifold with Ricci curvature bounded below supports function satisfying (1.3).

Example 2.3. *Suppose that (N, ds_N^2) is a complete Riemannian manifold and $(M = \mathbb{R} \times N, g = dt^2 + f^2(t)ds_N^2)$, is a Riemannian manifold where f is a positive smooth function defined on $\mathbb{R}$ with $f(0) = 1$. Then (M, g) is called warped product and denoted by $M = \mathbb{R} \times_f N$ and M is a complete Riemannian manifold. Set $f = e^{-t}$ and consider the warped product $M = \mathbb{R} \times_{e^{-t}} N$. Consider $\psi : M \to \mathbb{R}$ by $\psi(t, x) = t$. From [5], we have $|\nabla\psi| = 1$ and $\Delta\psi = n - 1$, then ψ satisfies the conditions (22).*
Let $\mathbb{H}^n$ be the n-dimensional hyperbolic space with constant curvature -1. Using the upper half-space model, $\mathbb{H}^n$ is defined by $\mathbb{R}^n_+ = \{(x_1, \cdots, x_n)|x_n > 0\}$ with metric

$$ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

The map $\phi : \mathbb{R} \times_{e^{-t}} \mathbb{R}^{n-1} \to \mathbb{H}^n$ given by $\phi(t, x) = (x, e^t)$ is an isometry. Therefore $\mathbb{H}^n$ admits a warped product model, $\mathbb{H}^n = \mathbb{R} \times_{e^{-t}} \mathbb{R}^{n-1}$.

Example 2.4. *Any compact homogeneous Riemannian manifold admits eigenmaps to some unit sphere with the first positive eigenvalue of the Laplacian ([11]).*

Example 2.5. *Suppose thta N is a complete Riemannian manifold and $M = \mathbb{R}^l \times N = \{(x_1, \cdots, x_l, z)|(x_1, \cdots, x_l) \in \mathbb{R}^l,\ z \in N\}$ is the product of $\mathbb{R}^l$ and N endowed with the product metric. Consider functions $\phi_\alpha : M \to \mathbb{R}$, $\alpha = 1, 2, \cdots, l$, defined by $\phi_\alpha(x_1, \cdots, x_l, z) = x_\alpha$. The functions $\{\phi_\alpha\}_{\alpha=1}^l$ satisfy (26).*

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All authors have read and agreed to the published version of the manuscript.

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