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FUZZY MODELING USING THE SIMILARITY-BASED APPROXIMATE REASONING SYSTEM

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Abstract. Just as we humans use many different types of inferential procedures to help us understand things or to make decisions, there are many different fuzzy logic inferential procedures, including similarity-based approaches. Similarity measures can be seen not only as a general notion but also as a particular family of fuzzy relations which play crucial roles for the motivation and the whole design of similarity-based reasoning. In the context of similarity-based reasoning, several issues merit concern. One is the representation of implication relation and two is the composition of a fuzzy implication relation with an observed system fact. The others are continuity and robustness of these systems which are the soul that must be inherited in the newly setup frameworks. Therefore, the purpose of this study is to introduce a new similarity-based approximate reasoning system which is based on introducing a new class of similarity measure on the space of LR-fuzzy numbers. Therefore, first, a new class of similarity measures is introduced between fuzzy sets. The similarity measure is needed in order to activate rules which are in terms of linguistic variables. Second, it is proved that the proposed measures satisfy the properties of the axiomatic definition as well as the other properties by a theorem. Next, we validate the effectiveness of the proposed similarity measure in a bidirectional approximate reasoning system in order to provide a non-linear mapping of fuzzy input data into fuzzy output data. Finally, using existing experimental data from Uniaxial Compressive Strength (UCS) testing, the fuzzy inference system constitutive model is produced to describe the influence of joint geometry (joint location, trace length and orientation) on the UCS of rock. The numerical results will show that the proposed model based on similarity-based approximate reasoning systems has better performance compared with the Mamdani fuzzy inference systems and the multivariate regression.

Keywords: Approximate reasoning system, Fuzzy modeling, Similarity measure, Statistical learning.
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1. Motivation and Introduction

Statistical learning theory provides the theoretical basis for many of today's machine learning algorithms and is arguably one of the most developed

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branches of artificial intelligence in general. Providing the basis of new learning algorithms, however, was not the only motivation for developing statistical learning theory but drawing valid conclusions/patterns from empirical data. Therefore, in recent years, the problem of modeling and prediction from observed data has been one of the most commonly encountered research topics in data analysis. A simple way to describe a system is the regression analysis which is used in determining the best fitted model for describing and forecasting the relationship between input-output variables [1].

Conventional linear regression models would be usual when both independent and dependent variables are treated as real numbers. However, in real situations, regression variables may be given as non-numerical entities such as linguistic variables. Real-world problems usually encounter uncertainty that would be due to

- impreciseness,
- incompleteness,
- vagueness,
- judgments,
- ambiguousness associated with the data,
- not knowledge-based data, and/or
- stochastic nature of an event's results.

Except the last one, all the first six uncertainties are based on fuzzy logic [30]. Therefore, due to vague essence and uncertainties in many real life problems, it would be better to use a fuzzy modeling approach under uncertainty. Fuzzy regression analysis studies the relationship between a response variable and a set of explanatory variables in complex systems involving imprecise data and/or imprecise relationships. Most of the researches that have been done on the topic of fuzzy regression analysis can be classified in the following categories:

- The class of possibilistic methods [38, 41, 42, 52].
- The class of distance methods [10, 14, 15].
- The class of heuristic methods [3, 4, 11].

Remark 1.1. It should be noted that, in contrast to what happens in the real case for the above items, the numerical fitting problem and the statistical estimation problem for the linear regression are different in the fuzzy case. The reason is that the lack of linearity of the space of fuzzy data makes that considering or not the data generation process lead to different restrictions in the minimization problem [3, 4, 7]. Therefore, in many real-world situations, where the complexity of the physical system calls for the development of a more general viewpoint, the estimation of the model is concerned with some new techniques. Therefore, some novel soft modeling techniques in fuzzy environment needs to be proposed by combining soft concepts (such as fuzzy logic, fuzzy inference systems, artificial neural networks), the possibilistic concepts, least-squares and least-absolutes estimation methods together [10]. So, during recent years, these methods have provided quantitative recommendations for design and it is shown that in some cases these combined techniques provide results with greater accuracy than the conventional methods. These methods yet provide quantitative recommendations for design and their results perform slightly better than the conventional method [8, 28].

Statistical learning is involved in a wide range of basic and higher-order cognitive functions and is taken to be an important building block of virtually all current theories of information processing [27, 39]. In the last two decades, a large and continuously growing research community has therefore focused on the ability to extract embedded patterns of regularity in time and space [20]. For instance, Berk [5] investigated statistical learning from a regression perspective. Xu and Zeevi [48] introduced a principled framework dubbed "uniform localized convergence" and characterized sharp problem-dependent rates for central statistical learning problems. Imam et al. [25] investigated air quality monitoring using statistical learning models for a sustainable environment. [43] classified quantum measurements using statistical learning. Cox et al. [13] revisited the effects of selective attention and animacy on visual statistical learning. Using kernel-based statistical learning theory, Fiedler et al. [19] considered the situation when the number of input variables goes to infinity. MacDonell [32] considered the impact of sampling and rule set size on generated fuzzy inference system predictive accuracy. Atanasove et al. [2] presented a succinct derivation of the training and generalization performance of a variety of high-dimensional ridge regression models using the basic tools of random matrix theory and free probability. Feng et al. [18] studied the nonparametric modal regression problem systematically from a statistical learning viewpoint. Zhang [53] investigated the mathematical analysis of machine learning algorithms. Wagner et al. [45] provided a similarity-based inference engine for non-singleton fuzzy logic systems. Dvořák et al. [17] studied similarity-based reasoning fuzzy inference systems from the point of view of extensionality. Mazandarani and Li [33] by proposing fractional fuzzy inference system, considered a new generation of fuzzy inference systems. Hothorn [24] provided CRAN task view of machine learning and statistical learning (see also [27]). Sharifani and Amini [40] provided a review of the methods and applications of machine learning and deep learning, including their strengths and weaknesses, as well as their potential future directions.

Finally, this paper concerns a new soft estimation problem in fuzzy modeling. In recent years, analytical approaches using fuzzy logic have become increasingly popular due to their ability to provide some impressive advantages over more conventional analytical approaches. Since the inception of fuzzy logic techniques, their application to problems in various fields (including decision making [16], control theory [36], pattern recognition and artificial intelligence) has become popular due to the ability of fuzzy logic techniques to handle semi-quantitative datasets or datasets with significant uncertainty [31]. Fuzzy logic is a theory developed to relate classes of objects without sharply defined boundaries in which membership of an object to a class is a matter of degree [33]. Fuzzy logic system approaches allow greater understanding of the origin of dependence between variables, allow modification of sensitivity in analysis, provide greater flexibility in analysis and are better able to tolerate imprecise or chaotic data [44]. Additionally, they can be used in combination with conventional analytical techniques and are based on simplistic and understandable linguistic variables. The mapping of the output of the fuzzy modeling from the given inputs is formulated by the basic fuzzy operator, known as the fuzzy inference system (FIS) [30]. Several different types of fuzzy models exist, with some of the more popular including the Mamdani-, Sugeno-, Tsukamoto- and Singleton-type models [37]. Despite its success in various systems, researchers have indicated certain drawbacks in the mechanism of Compositional Rule of Inference (CRI) in the forward approximate reasoning systems [6]. In the context of CRI, several issues merit concern. One is the representation of implication relation and two is the composition of a fuzzy implication relation with an observed system fact. The others are continuity and robustness of these systems which are the soul that must be inherited in the newly setup frameworks. By continuity and robustness, we mean that changes in the output variable must be in balance with the changes in the input variable. By robustness, we also mean that the model performs consistently on target and is relatively insensitive to uncontrollable noise factors. It is usually assumed that noise factors are uncontrollable in the field, but can be controlled during model development for purposes of a designed experiment.

This motivated us to use the similarity-based approximate reasoning method in modeling fuzzy input-output variables. Notice, in similarity-based reasoning schemes, from a given fact, the desired conclusion is derived using only a measure of similarity between A (the antecedent of the rules) and A^* (the given fact). This makes the similarity-based approximate reasoning satisfies reversibility property and provides forward and backward similarity-based approximate reasoning systems. Therefore, the aim of the present paper is to study the forward and backward similarity-based approximate reasoning systems to construct a model between input-output linguistic variables. In this regard, first a notion of similarity measure between fuzzy numbers will be recalled. In general, since fuzzy logic system is a nonlinear mapping of input data into outputs, therefore, in this paper, we use this technique to provide a model between fuzzy input-output data. We shall use similarity-based approximate reasoning systems because we want the input value of the rules to be linguistic, i.e. the output linguistic variable is predicted by the input linguistic variable (not with a crisp value like in the Mamdani fuzzy inference systems). The results of a comparative applied numerical example will show that the proposed model based on similarity-based approximate reasoning systems has ability to closely approximate a wide variety of processes, from the Mamdani fuzzy inference systems and the traditional methods such as multivariate regression.

This paper is organized as follows. In Section 2, a new definition and relevant properties with respect to similarity measures between fuzzy sets will be proposed and discussed. In Section 3, based on the similarity measures, a new approach to develop a similarity-based approximate reasoning model is proposed to describe the relationship between input-output linguistic variables. Also, two illustrative examples will be presented here. A comparative study is provided in Section 4 showing the fact that the proposed modeling technique performs better than the Mamdani fuzzy inference systems and the multivariate regression. In Section 5, some final remarks conclude the paper.

2. A similarity measures between fuzzy sets

Similarity-based reasoning systems consider the notion of similarity as crucial for their motivation and the whole design. However, similarity can be seen not only as a general notion but also as a particular family of binary fuzzy relations modeling a fuzzy equality [9,12,22,23]. Therefore, this section deals with the well-known notion of similarity measures between fuzzy sets and proposes a variety of similarity measures for fuzzy sets. Moreover, it proves that the proposed measures satisfy the properties of the axiomatic definition for similarity measures. For practical reasons, we then combine the proposed similarity measures with Yang and Shih's [49] algorithm for clustering fuzzy data.

2.1. **Preliminary concepts.** A fuzzy set $\widetilde{N}$ of the universal set $\mathbb{X}$ is defined by its membership function $\widetilde{N}(x) : \mathbb{X} \to [0,1]$ [50]. In this paper, we consider $\mathbb{R}$ (the real line) as the universal set. We denote by $\widetilde{N}_\alpha = \{x \in \mathbb{R} : \widetilde{N}(x) \geq \alpha\}$ the α-level set (α-cut) of the fuzzy set $\widetilde{N}$, for every $\alpha \in (0,1]$, and for $\alpha = 0$, $\widetilde{N}_0$ is the closure of the set $\{x \in \mathbb{R} : \widetilde{N}(x) > 0\}$.

Definition 2.1. The LR-fuzzy number $\widetilde{N} = (a, b; l, r)_{LR}$ with central values of the interval $[a, b] \subset \mathbb{R}$, left and right spreads $l \in \mathbb{R}^+$, $r \in \mathbb{R}^+$, decreasing left and right shape functions $L : \mathbb{R}^+ \to [0,1]$, $R : \mathbb{R}^+ \to [0,1]$, with $L(0) = R(0) = 1$, has the following membership function and α-cut

$$\widetilde{N}(x) = \begin{cases} L(\frac{a-x}{l}) & \text{if} \quad x \leq a, \\ 1 & \text{if} \quad a < x \leq b, \\ R(\frac{x-b}{r}) & \text{if} \quad x \geq b. \end{cases}$$

$$\widetilde{N}_\alpha = [a - L^{-1}(\alpha)l, b + R^{-1}(\alpha)r], \quad \alpha \in [0,1].$$

Remark 2.2. In practice, it is usually preferred to use simple shapes for functions L and R such as $L(x) = R(x) = \max\{1 - x, 0\}$. We denote by $\mathcal{F}(\mathbb{R})$, the set of all LR-fuzzy numbers of $\mathbb{R}$.

Definition 2.3 ([30]). The following operators on $\widetilde{B}$, $\widetilde{C} \in \mathcal{F}(\mathbb{R})$ will be used in the sequel.

(1) Equality: $\widetilde{A} = \widetilde{B} \Leftrightarrow \widetilde{A}(x) = \widetilde{B}(x)$, $\forall x \in \mathbb{R}$.
(2) Inclusion: $\widetilde{A} \subseteq \widetilde{B} \Leftrightarrow \widetilde{A}(x) \leq \widetilde{B}(x)$, $\forall x \in \mathbb{R}$,

(3) Inclusion: $\widetilde{A} \subseteq \widetilde{B} \Leftrightarrow \widetilde{A}_\alpha \subseteq \widetilde{B}_\alpha$, $\forall \alpha \in [0,1]$.
(4) Complement: $\widetilde{A}^c(x) = 1 - \widetilde{A}(x)$, $\forall x \in \mathbb{R}$.
(5) Union: $(\widetilde{A} \cup \widetilde{B})(x) = \max\{\widetilde{A}(x), \widetilde{B}(x)\}$, $\forall x \in \mathbb{R}$.
(6) Intersection: $(\widetilde{A} \cap \widetilde{B})(x) = \min\{\widetilde{A}(x), \widetilde{B}(x)\}$, $\forall x \in \mathbb{R}$.

Definition 2.4. Let $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C} \in \mathcal{F}(\mathbb{R})$, then $S : \mathcal{F}(\mathbb{R}) \otimes \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ is a similarity measure between fuzzy sets if satisfies the following:

(1) $S(\widetilde{A}, \widetilde{B}) \in [0,1]$.
(2) $S(\widetilde{A}, \widetilde{B}) = S(\widetilde{B}, \widetilde{A})$.
(3) $S(\widetilde{A}, \widetilde{B}) = 1$ if and only if $\widetilde{A} = \widetilde{B}$.
(4) If $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$, then $S(\widetilde{A}, \widetilde{C}) \leq \min\{S(\widetilde{A}, \widetilde{B}), S(\widetilde{B}, \widetilde{C})\}$.

2.2. **Similarity measure.** Let $\widetilde{A}$, $\widetilde{B} \in \mathcal{F}(\mathbb{R})$. The similarity measure $S : \mathcal{F}(\mathbb{R}) \otimes \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ is defined as follows [23]

$$S(\widetilde{A}, \widetilde{B}) = f\left(\int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| d\alpha\right),$$

where

(1) $\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha = A_\alpha \cup B_\alpha - A_\alpha \cap B_\alpha$,
(2) $|\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha|$ is the length of the interval $\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha \in \mathbb{R}$,
(3) $f : \mathbb{R}^+ \to [0,1]$ with the condition $f(0) = 1$ is a decreasing function (for instance, $f(x) = \frac{1-x}{1+x}$, $f(x) = \frac{1}{1+x^p}$ or $f(x) = e^{-x^p}$, $p > 0$).

Theorem 2.5. *Let $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C} \in \mathcal{F}(\mathbb{R})$, then $S : \mathcal{F}(\mathbb{R}) \otimes \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ satisfies the following:*

(1) $S(\widetilde{A}, \widetilde{B}) \in [0,1]$.
(2) $S(\widetilde{A}, \widetilde{B}) = S(\widetilde{B}, \widetilde{A})$.
(3) $S(\widetilde{A}, \widetilde{B}) = 1$ *if and only if* $\widetilde{A} = \widetilde{B}$.
(4) *If* $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$, *then* $S(\widetilde{A}, \widetilde{C}) \leq \min\{S(\widetilde{A}, \widetilde{B}), S(\widetilde{B}, \widetilde{C})\}$.
(5) $S(\widetilde{A} \cap \widetilde{B}, \widetilde{A} \cup \widetilde{B}) = S(\widetilde{A}, \widetilde{B})$.
(6) $S(\widetilde{A}^c, \widetilde{B}^c) = S(\widetilde{A}, \widetilde{B})$.
(7) $S(\widetilde{A} \cap \widetilde{C}, \widetilde{B} \cap \widetilde{C}) \geq S(\widetilde{A}, \widetilde{B})$.
(8) $S(\widetilde{A} \cup \widetilde{C}, \widetilde{B} \cup \widetilde{C}) \geq S(\widetilde{A}, \widetilde{B})$.

Proof. (1) It is immediately derived from the properties of the function f.
(2) The result can easily be checked because $|\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| = |\widetilde{B}_\alpha \triangle \widetilde{A}_\alpha|$, for every $\alpha \in [0,1]$.
(3) If $\widetilde{A} = \widetilde{B}$, then, $S(\widetilde{A}, \widetilde{B}) = f(\int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| d\alpha) = f(0) = 1$. Conversely, assume that $S(\widetilde{A}, \widetilde{B}) = 1$. It is clear that $\int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| d\alpha = f^{-1}(1) = 0$. Therefore, $|\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| = 0$, for each $\alpha \in [0,1]$, which concludes that $\widetilde{A}_\alpha = \widetilde{B}_\alpha$ for each $\alpha \in [0,1]$, or equivalently $\widetilde{A} = \widetilde{B}$.

(4) Let $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{C}$ be three arbitrary nested fuzzy numbers, then $\widetilde{A}_\alpha \subseteq \widetilde{B}_\alpha \subseteq \widetilde{C}_\alpha$, for every $\alpha \in [0,1]$. We can prove that for every $\alpha \in [0,1]$, $\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha \subseteq \widetilde{A}_\alpha \triangle \widetilde{C}_\alpha$, and $\widetilde{B}_\alpha \triangle \widetilde{C}_\alpha \subseteq \widetilde{A}_\alpha \triangle \widetilde{C}_\alpha$. Therefore, we obtain the inequalities $|\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| \leq |\widetilde{A}_\alpha \triangle \widetilde{C}_\alpha|$, and $|\widetilde{B}_\alpha \triangle \widetilde{C}_\alpha| \leq |\widetilde{A}_\alpha \triangle \widetilde{C}_\alpha|$. The proof can be easily checked because f is a strictly decreasing function.

(5) For every $\alpha \in [0,1]$, because $(\widetilde{A}_\alpha \cup \widetilde{B}_\alpha) \triangle (\widetilde{A}_\alpha \cap \widetilde{B}_\alpha) = \widetilde{A}_\alpha \triangle \widetilde{B}_\alpha$, therefore, we can get the result.

(6) It can be proved that $(\widetilde{A}_\alpha)^c \triangle (\widetilde{B}_\alpha)^c = \widetilde{A}_\alpha \triangle \widetilde{B}_\alpha$, for every $\alpha \in [0,1]$.

(7) For given fuzzy numbers $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, note that

$$(\widetilde{A}_\alpha \cap \widetilde{C}_\alpha) \triangle (\widetilde{B}_\alpha \cap \widetilde{C}_\alpha) \subseteq \widetilde{A}_\alpha \triangle \widetilde{B}_\alpha.$$

Therefore, we obtain the following inequality

$$\int_0^1 |(\widetilde{A}_\alpha \cap \widetilde{C}_\alpha) \triangle (\widetilde{B}_\alpha \cap \widetilde{C}_\alpha)| \, d\alpha \leq \int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| \, d\alpha.$$

The proof can be easily completed, because f is a strictly decreasing function.

(8) For every $\alpha \in [0,1]$, we can obtain that

$$(\widetilde{A}_\alpha \cup \widetilde{C}_\alpha) \triangle (\widetilde{B}_\alpha \cup \widetilde{C}_\alpha) \subseteq \widetilde{A}_\alpha \triangle \widetilde{B}_\alpha.$$

Now, similar to item 7, the proof can be completed.

□

2.3. **Clustering illustrative example.** Fuzzy clustering, which is one of the major techniques in pattern recognition, is a method for decomposing a given data set into groups or clusters of similar individuals with uncertain boundaries. Different algorithms have been developed in fuzzy cluster analysis, which can be roughly divided into two main categories [34]: the first category is based on objective functions, while the other one is based on a relation matrix such as similarity relation, correlation relation, fuzzy equivalence relations, and the like (for more on this topic, see e.g. [26, 27]). The investigation considered in the following example focuses on the second category of fuzzy clustering methods based on fuzzy relations that can be made in the beginning with a similarity matrix [23].

Assume that there are nine patterns denoted as follows:

$$\begin{aligned}
&\widetilde{A}_1 = (13.0, 0.27, 1.00)_T, \quad \widetilde{A}_2 = (14.0, 1.95, 0.93)_T, \quad \widetilde{A}_3 = (14.4, 0.56, 1.17)_T,\\
&\widetilde{A}_4 = (14.7, 0.89, 0.88)_T, \quad \widetilde{A}_5 = (14.9, 0.12, 1.21)_T, \quad \widetilde{A}_6 = (15.3, 1.19, 0.41)_T,\\
&\widetilde{A}_7 = (15.1, 1.82, 0.90)_T, \quad \widetilde{A}_8 = (15.6, 0.38, 1.38)_T, \quad \widetilde{A}_9 = (16.0, 1.97, 0.12)_T.
\end{aligned}$$

The similarity measures between these patterns are shown in Figure 1 based on the following similarity measure function

$$S(\widetilde{A}, \widetilde{B}) = \frac{1}{1 + \left(\int_0^1 |\widetilde{A}_\alpha \triangle \widetilde{B}_\alpha| d\alpha\right)^2},$$

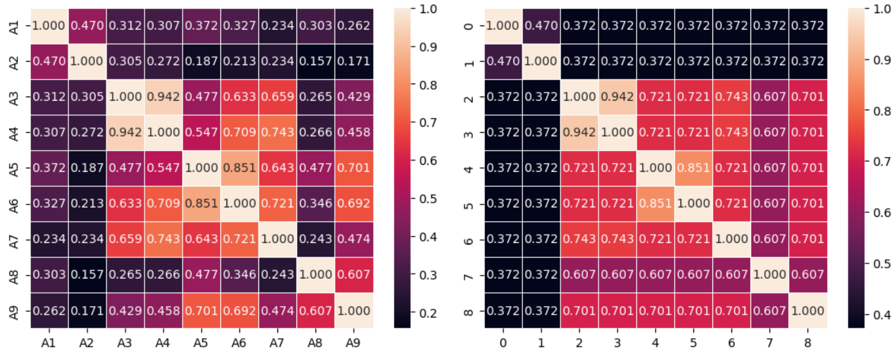

FIGURE 1. Left: Similarity measure matrix $S^{(0)}$ between nine patterns $\widetilde{A}_1 - \widetilde{A}_9$. Rigth: $\max - \min$ decomposed similarity measure matrix $S^{(3)}$

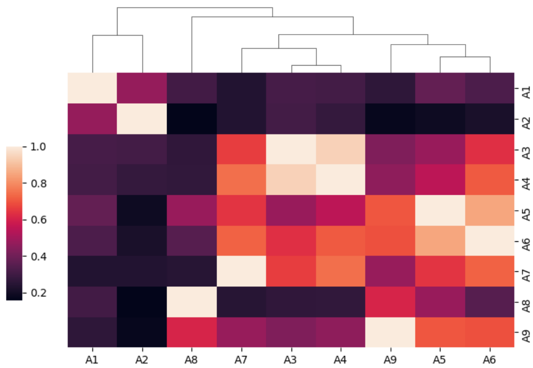

FIGURE 2. Dendrogram for cluster analysis based on $\max - \min$ decomposed similarity measure matrix $S^{(3)}$

To obtain a fuzzy cluster for these patterns, we are using Yang and Shih's [49] algorithm, which creates a clustering algorithm for the $\max - \min$ similarity relation matrix. Here, by $\max - \min$ composition, first, we have to obtain $S^{(0)} < S^{(1)} < \ldots < S^{(n)} = S^{(n+1)} = \ldots$ which is denoted as

$$\forall x, y \in \{1, \ldots, 9\} \quad S^{(n)}(x, y) = \max_{z \in \{1,\ldots,9\}} \{\min(S^{(n-1)}(x, z), S^{(n-1)}(z, x))\}.$$

Therefore, by beginning with the initial similarity matrix $S^{(0)}$ given in Figure 1, the $\max - \min$ compositions are obtained as $S^{(0)} < S^{(1)} < S^{(2)} < S^{(3)} = S^{(4)} = \ldots$. Then, by beginning with the matrix $S^{(3)}$ shown in Figure 1 and applying the clustering algorithm, the hierarchical clustering are obtained and presented in Figure 2 and Table 1. The results illustrate the partitions that

Table 1. Clustering results according to different levels

δ	Clustering results
$0.000 < \delta \leq 0.371$	$\{\widetilde{A}_1, \widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_8, \widetilde{A}_9\}$
$0.371 < \delta \leq 0.470$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_8, \widetilde{A}_9\}$
$0.470 < \delta \leq 0.606$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_9\}, \{\widetilde{A}_8\}$
$0.606 < \delta \leq 0.700$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7\}, \{\widetilde{A}_8\}, \{\widetilde{A}_9\}$
$0.700 < \delta \leq 0.720$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_7\}, \{\widetilde{A}_5, \widetilde{A}_6\}, \{\widetilde{A}_8\}, \{\widetilde{A}_9\}$
$0.720 < \delta \leq 0.743$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4\}, \{\widetilde{A}_5, \widetilde{A}_6\}, \{\widetilde{A}_7\}, \{\widetilde{A}_8\}, \{\widetilde{A}_9\}$
$0.743 < \delta \leq 0.851$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3, \widetilde{A}_4\}, \{\widetilde{A}_5\}, \{\widetilde{A}_6\}, \{\widetilde{A}_7\}, \{\widetilde{A}_8\}, \{\widetilde{A}_9\}$
$0.851 < \delta \leq 0.941$	$\{\widetilde{A}_1\}, \{\widetilde{A}_2\}, \{\widetilde{A}_3\}, \{\widetilde{A}_4\}, \{\widetilde{A}_5\}, \{\widetilde{A}_6\}, \{\widetilde{A}_7\}, \{\widetilde{A}_8\}, \{\widetilde{A}_9\}$

Table 2. Similarity measures between nine patterns $\widetilde{A}_1 - \widetilde{A}_9$

Clusters		$S(\widetilde{A}, \widetilde{A}_i)$	$S(\widetilde{A}, \mathcal{C}_j) = \max\{S(\widetilde{A}, \widetilde{A}_i) \vert \widetilde{A}_i \in \mathcal{C}_j\}$
$\mathcal{C}_1$	$\widetilde{A}_1$	0.689	0.689
$\mathcal{C}_2$	$\widetilde{A}_2$	0.759	0.759
	$\widetilde{A}_3$	0.309	
	$\widetilde{A}_4$	0.293	
	$\widetilde{A}_5$	0.277	
$\mathcal{C}_3$	$\widetilde{A}_6$	0.263	0.309
	$\widetilde{A}_7$	0.248	
	$\widetilde{A}_9$	0.214	
$\mathcal{C}_4$	$\widetilde{A}_8$	0.230	0.230

have been made at levels of δ. For instance, when $\delta \in (0.470, 0.606]$, the following distinct clusters are obtained

$$\mathcal{C}_1 = \{\widetilde{A}_1\}, \quad \mathcal{C}_2 = \{\widetilde{A}_2\}, \quad \mathcal{C}_3 = \{\widetilde{A}_3, \widetilde{A}_4, \widetilde{A}_5, \widetilde{A}_6, \widetilde{A}_7, \widetilde{A}_9\}, \quad \mathcal{C}_4 = \{\widetilde{A}_8\}.$$

Suppose the new pattern $\widetilde{A} = (13.7, 1.1, 0.8)_T$ is recorded and we have to decide to which cluster it belongs to. To do so, considering the calculations provided in Table 2 and using the principle of largest similarity between the new pattern $\widetilde{A}$ and the clusters $\mathcal{C}_j$, $j = 1, \ldots, 4$, it is concluded that

$$\arg\max_{j \in \{1,\ldots,4\}} S(\widetilde{A}, \mathcal{C}_j) = \mathcal{C}_2,$$

Therefore, using the principle of maximum degree of similarity, it indicates that the new pattern $\widetilde{A}$ belongs to the cluster $\mathcal{C}_2$, because $\widetilde{A}$ has the most similarity degree with cluster $\mathcal{C}_2$, i.e. $S(\widetilde{A}, \mathcal{C}_2) = 0.759$.

Remark 2.6. To study calculations with more details on various applied numerical examples of such similarity measures in the contexts of pattern recognition, decision making, clustering, and approximate reasoning, we refer the reader to to Hesamian and Chachi [23].

3. A Similarity-based fuzzy inference system model

In the following, we use fuzzy logic systems to provide a fuzzy model between fuzzy input-output data. The fuzzy logic modeling technique contains "if" part (or the antecedent) and "then" part (or the consequent) that each fuzzy implication is a parameterized equation of the model's input-output variables. In this structure of a typical fuzzy if-then rule we use the "and" fuzzy operator. From every rule equation, one fuzzy output obtains. Therefore, we should find a way to aggregate all of fuzzy outputs and generate the final fuzzy output. Based on this approach we can also predict for various values of the input linguistic variables.

The basic structure of the fuzzy inference system consists of three components: a rule base, a database, and an inference procedure [30]. The rule base contains the selection of fuzzy if-then rules activated by a certain value of interest, the database defines the membership functions adopted in the fuzzy if-then rules, then the inference procedure provides a fuzzy reasoning based on information aggregation from the activated fuzzy rules. The detailed description of fuzzy inference systems can be found in [6,32,44]. The structure of a typical fuzzy if-then rule that uses the "and" fuzzy operator is demonstrated in the example statement below [37]:

$$\text{rule}_i \quad : \quad \text{if } \widetilde{X}_1 \text{ is } \widetilde{B}_{1i} \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_{2i}, \ldots \ \& \ \widetilde{X}_p \text{ is } \widetilde{B}_{pi} \implies \widetilde{Y} \text{ is } \widetilde{C}_i.$$

In this scheme, rule_i $(i = 1, \ldots, n)$ is the ith production rule, n is the number of rules, $\widetilde{X}_j$ $(j = 1, \ldots, p)$ is the fuzzy input (antecedent) variables, $\widetilde{Y}$ is the fuzzy output (consequent) variable, $\widetilde{B}_{ji}$'s are fuzzy sets of the universe of discourse $\mathbb{X}_j$ for the antecedent variables, and $\widetilde{C}_i$'s are fuzzy sets of the universe of discourse $\mathbb{Y}$ for the consequent variable. Gorgin et al. [21] described a hardware realization framework for fuzzy inference system optimization. Now, we introduce the way of estimating a fuzzy model, based on the fuzzy inference system. Below, the steps taken in the production of the fuzzy inference system constitutive model for this work are systematically described.

Let us first consider the forward approximate reasoning scheme based on fuzzy sets. Suppose that the antecedent statement is demonstrated as follows

$$\textit{Antecedent} \quad : \quad \widetilde{X}_1 \text{ is } \widetilde{B}^*_1 \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}^*_2, \ldots \ \& \ \widetilde{X}_p \text{ is } \widetilde{B}^*_p,$$

where $\widetilde{B}^*_j$'s $(j = 1, \ldots, p)$ are fuzzy sets of the universe of discourse $\mathbb{X}_j$ for the antecedent variables. We need to determine the consequence of the approximate reasoning scheme, which is

$$\textit{Consequence} \quad : \quad \widetilde{Y} \text{ is } \widetilde{C}^*,$$

where $\widetilde{C}^*$ is a fuzzy set of the universe of discourse $\mathbb{Y}$ for the consequent variable. Now, we propose the following algorithm for ith rule:

(1) Compute $s_{ji} = S(\widetilde{B}_{ji}, \widetilde{B}_j^*)$, the similarity measurement between fuzzy sets $\widetilde{B}_{ji}$ and $\widetilde{B}_j^*$, $j = 1, \ldots, p$.
(2) Let $s_i = \max_{1 \le j \le p} s_{ji}$, and $\widetilde{C}_i^* = s_i \otimes \widetilde{C}_i$.
(3) The deduced consequence of rule$_i$ is "$\widetilde{Y}$ is $\widetilde{C}_i^*$".

Thus, the deduced consequence of the approximate reasoning scheme is "$\widetilde{Y}$ is $\widetilde{C}^*$", where

$$\widetilde{C}^* = \widetilde{C}_1^* \cup \widetilde{C}_2^* \cup \ldots \cup \widetilde{C}_n^*.$$

Conversely, let us consider the backward approximate reasoning scheme based on fuzzy sets. Suppose that the consequence statement is demonstrated as follows

$$\textit{Consequence} \quad : \quad \widetilde{Y} \text{ is } \widetilde{C}^*.$$

We need to determine the antecedent of the approximate reasoning scheme, which is

$$\textit{Antecedent} \quad : \quad \widetilde{X}_1 \text{ is } \widetilde{B}_1^* \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_2^*, \ldots \ \& \ \widetilde{X}_p \text{ is } \widetilde{B}_p^*.$$

Similarly, we can derive the following results for ith rule:

(1) Compute $s_i = S(\widetilde{C}_i, \widetilde{C}^*)$, the similarity measurement between fuzzy sets $\widetilde{C}_i$ and $\widetilde{C}^*$.
(2) Let $\widetilde{B}_{ji}^* = s_i \otimes \widetilde{B}_{ji}$, $j = 1, \ldots, p$.
(3) The deduced antecedent of rule rule$_i$ is

$$\widetilde{X}_1 \text{ is } \widetilde{B}_{1i}^* \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_{2i}^*, \ldots \ \& \ \widetilde{X}_p \text{ is } \widetilde{B}_{pi}^*.$$

By aggregating the above results, the deduced antecedent of the approximate reasoning scheme is

$$\textit{Antecedent} \quad : \quad \widetilde{X}_1 \text{ is } \widetilde{B}_1^* \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_2^*, \ldots \ \& \ \widetilde{X}_p \text{ is } \widetilde{B}_p^*,$$

where

$$\widetilde{B}_j^* = \widetilde{B}_{j1}^* \cup \widetilde{B}_{j2}^* \cup \ldots \cup \widetilde{B}_{jn}^*, \quad j = 1, \ldots, p.$$

Forward approximate reasoning: An illustrative example. Consider the forward approximate reasoning scheme is as follows

$$\begin{aligned}
\text{rule}_1 &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{11} \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_{21} \ \& \ X_3 \text{ is } \widetilde{B}_{31} \implies \widetilde{Y} \text{ is } \widetilde{C}_1,\\
\text{rule}_2 &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{12} \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_{22} \ \& \ X_3 \text{ is } \widetilde{B}_{32} \implies \widetilde{Y} \text{ is } \widetilde{C}_2,\\
\text{rule}_3 &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{13} \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_{23} \ \& \ X_3 \text{ is } \widetilde{B}_{33} \implies \widetilde{Y} \text{ is } \widetilde{C}_3,\\
\text{rule}_4 &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{14} \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_{24} \ \& \ X_3 \text{ is } \widetilde{B}_{34} \implies \widetilde{Y} \text{ is } \widetilde{C}_4,\\
\text{rule}_5 &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{15} \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_{25} \ \& \ X_3 \text{ is } \widetilde{B}_{35} \implies \widetilde{Y} \text{ is } \widetilde{C}_5,\\
\text{rule}_6 &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{16} \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}_{26} \ \& \ X_3 \text{ is } \widetilde{B}_{36} \implies \widetilde{Y} \text{ is } \widetilde{C}_6.
\end{aligned}$$

Table 3. The triangular fuzzy sets in the forward approximate reasoning

i	$\widetilde{B}_{1i}$	$\widetilde{B}_{2i}$	$\widetilde{B}_{3i}$	$\widetilde{C}_i$
1	$(0.40, 0.20, 0.15)$	$(0.80, 0.40, 0.50)$	$(25, 15, 15)$	$(26, 6, 6)$
2	$(0.40, 0.20, 0.15)$	$(0.80, 0.40, 0.50)$	$(45, 15, 15)$	$(26, 6, 6)$
3	$(0.60, 0.25, 0.10)$	$(1.20, 0.40, 0.40)$	$(45, 15, 15)$	$(18, 6, 4)$
4	$(0.60, 0.25, 0.10)$	$(0.80, 0.40, 0.50)$	$(25, 15, 15)$	$(18, 6, 4)$
5	$(0.55, 0.30, 0.05)$	$(1.20, 0.40, 0.40)$	$(25, 15, 15)$	$(10, 8, 5)$
6	$(0.55, 0.30, 0.05)$	$(1.20, 0.40, 0.40)$	$(45, 15, 15)$	$(10, 8, 5)$

Table 4. The values of $s_{ji} = S(\widetilde{B}_{ji}, \widetilde{B}^*_i)$ $(j = 1, 2, 3,\ i = 1, \ldots, 6)$ between fuzzy sets in the forward approximate reasoning

i	s_{1i}	s_{2i}	s_{3i}	$s_i = \max\{s_{1i}, s_{2i}, s_{3i}\}$	$\widetilde{C}^*_i = s_i \otimes \widetilde{C}_i$
1	0.865	0.424	0.043	0.865	$(22.49, 5.19, 5.19)$
2	0.865	0.424	0.090	0.865	$(22.49, 5.19, 5.19)$
3	0.769	0.404	0.090	0.769	$(13.84, 4.61, 3.08)$
4	0.769	0.424	0.043	0.769	$(13.84, 4.61, 3.08)$
5	0.771	0.404	0.043	0.771	$(7.71, 6.17, 3.85)$
6	0.771	0.404	0.090	0.771	$(7.71, 6.17, 3.85)$

The triangular fuzzy numbers $\widetilde{B}_{1i}$, $\widetilde{B}_{2i}$, $\widetilde{B}_{3i}$, and $\widetilde{C}_i$ used in this forward approximate reasoning are provided in Table 3. Now, let the antecedent statement be as follows

$$\textit{Antecedent} \quad : \quad \widetilde{X}_1 \text{ is } \widetilde{B}^*_1 \ \& \ \widetilde{X}_2 \text{ is } \widetilde{B}^*_2 \ \& \ \widetilde{X}_3 \text{ is } \widetilde{B}^*_3,$$

where

$$\widetilde{B}^*_1 = (0.45, 0.45, 0.20)_T, \quad \widetilde{B}^*_2 = (1, 0.80, 0.80)_T, \quad \widetilde{B}^*_3 = (40, 15, 15)_T.$$

We have to obtain the consequence of the forward approximate reasoning scheme, which is

$$\textit{Consequence} \quad : \quad \widetilde{Y} \text{ is } \widetilde{C}^*.$$

Applying

$$S(\widetilde{B}_{ji}, \widetilde{B}^*_i) = \frac{1}{1 + \int_0^1 |\widetilde{B}_{ji\alpha} \triangle \widetilde{B}^*_{i\alpha}| d\alpha}, \qquad j = 1, 2, 3,\ i = 1, \ldots, 6$$

on the fuzzy sets given in Table 4, we obtain the similarities shown in Table 4. Thus, according to the procedure proposed in the previous section, $\widetilde{C}^*$ is obtained as follows

$$\widetilde{C}^* = \cup_{i=1}^6 \widetilde{C}^*_i,$$

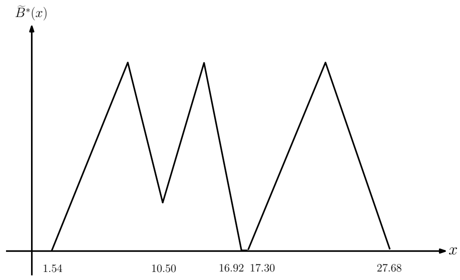

FIGURE 3. The membership function of fuzzy set $\widetilde{C}^*$ in Eq. 1

where the fuzzy sets $\widetilde{C}_i^*$, $i = 1, \ldots, 6$, are given in Table 4. The closed formula of $\widetilde{C}^*$ is written as follows which is also depicted in Figure 3

$$\widetilde{C}^*(x) = \begin{cases} \frac{x-1.54}{6.17} & 1.54 < x \leq 7.71, \\ \frac{11.56-x}{3.85} & 7.71 < x \leq 10.50, \\ \frac{x-9.23}{4.61} & 10.50 < x \leq 13.84, \\ \frac{16.92-x}{3.08} & 13.84 < x \leq 16.92, \\ \frac{x-17.3}{5.19} & 17.30 < x \leq 22.49, \\ \frac{27.68-x}{5.19} & 22.49 < x \leq 27.68. \end{cases} \tag{1}$$

Backward approximate reasoning: An illustrative example. Conversely, let us consider the above forward approximate reasoning scheme as a backward approximate reasoning scheme. Now, suppose that the consequence statement of the approximate reasoning scheme is demonstrated as follows

$$Consequence \quad : \quad \widetilde{Y} \text{ is } \widetilde{C}^* = (18, 6, 6)_T.$$

We need to determine the antecedent statement of the approximate reasoning scheme, which is

$$Antecedent \quad : \quad \widetilde{X}_1 \text{ is } \widetilde{B}_1^* \;\&\; \widetilde{X}_2 \text{ is } \widetilde{B}_2^* \;\&\; \widetilde{X}_3 \text{ is } \widetilde{B}_3^*.$$

the results are summarized in Table 5. Thus, we obtain

$$\widetilde{B}_j^* = \cup_{i=1}^{6} \widetilde{B}_{ji}^*, \quad j = 1, 2, 3,$$

The functions $\widetilde{B}_1^*(\cdot)$, $\widetilde{B}_2^*(\cdot)$, and $\widetilde{B}_3^*(\cdot)$ are plotted in Figures 4.

4. Comparison study: practical example

In this section, using the real world dataset from Uniaxial Compressive Strength (UCS) testing [46], the similarity based fuzzy inference system constitutive model is produced to describe the relationship between three input variables,

- joint location (X_1),
- joint trace length ratio (X_2),

Table 5. Similarity measures in backward approximate reasoning

i	$s_i = S(\widetilde{C}_i, \widetilde{C}^*)$	$\widetilde{B}^*_{1i} = s_i \otimes \widetilde{B}_{1i}$	$\widetilde{B}^*_{2i} = s_i \otimes \widetilde{B}_{2i}$	$\widetilde{B}^*_{3i} = s_i \otimes \widetilde{B}_{3i}$
1	0.086	$(0.034, 0.017, 0.013)_T$	$(0.069, 0.034, 0.043)_T$	$(2.15, 1.29, 1.29)_T$
2	0.086	$(0.034, 0.017, 0.013)_T$	$(0.069, 0.034, 0.043)_T$	$(3.87, 1.29, 1.29)_T$
3	0.500	$(0.300, 0.125, 0.050)_T$	$(0.600, 0.200, 0.200)_T$	$(22.50, 7.50, 7.50)_T$
4	0.500	$(0.300, 0.125, 0.050)_T$	$(0.400, 0.200, 0.250)_T$	$(12.50, 7.50, 7.50)_T$
5	0.079	$(0.043, 0.024, 0.004)_T$	$(0.095, 0.032, 0.032)_T$	$(1.98, 1.19, 1.19)_T$
6	0.079	$(0.043, 0.024, 0.004)_T$	$(0.095, 0.032, 0.032)_T$	$(3.56, 1.19, 1.19)_T$

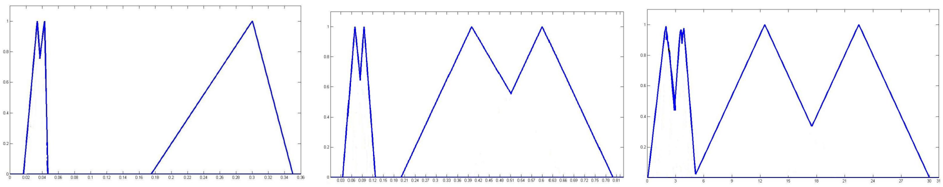

Figure 4. Membership functions of $\widetilde{B}^*_1$ (left), $\widetilde{B}^*_2$ (center) and $\widetilde{B}^*_3$ (rigth)

Table 6. The dataset.

$No.$	X_1	X_2	X_3	y
1	0.24	0.40	1.05	30.80
2	0.24	0.60	1.05	27.50
3	0.24	0.80	1.05	25.05
4	0.24	1.00	1.05	19.70
5	0.24	1.50	1.05	5.86
6	0.18	0.71	1.31	25.61
7	0.18	0.71	1.05	22.55
8	0.18	0.71	0.79	21.46
9	0.18	0.71	0.52	27.96
10	0.18	0.71	0.26	34.81
11	0.17	1.20	1.05	9.61
12	0.22	1.20	1.05	8.94
13	0.27	1.20	1.05	10.97
14	0.34	1.20	1.05	12.09
15	0.41	1.20	1.05	12.90

- joint orientation (X_3),

and determine

- UCS (as the output variable y).

Table 7. The fuzzy sets of the levels of input variables $\widetilde{X}_j$, $j = 1, 2, 3$, are denoted by $\widetilde{B}_{ji}$, $i = 1, \ldots, 5$, in the rules of the similarity based forward approximate reasoning given in Appendix. Also the fuzzy sets of the levels of output variable y are denoted by $\widetilde{C}_i$, $i = 1, \ldots, 5$.

i	$\widetilde{B}_{1i}$	$\widetilde{B}_{2i}$	$\widetilde{B}_{3i}$	$\widetilde{C}_i$
1	$(0.00, 0.00, 0.45)_T$	$(0.00, 0.00, 0.40)_T$	$(10, 0, 20)_T$	$(0, 0, 10)_T$
2	$(0.80, 0.15, 0.15)_T$	$(0.40, 0.20, 0.20)_T$	$(35, 15, 15)_T$	$(10, 6, 5)_T$
3	$(0.45, 0.05, 0.10)_T$	$(0.80, 0.30, 0.40)_T$	$(50, 15, 15)_T$	$(18, 4, 4)_T$
4	$(0.60, 0.10, 0.10)_T$	$(1.40, 0.40, 0.40)_T$	$(80, 20, 0)_T$	$(25, 5, 8)_T$
5	$(0.90, 0.10, 0.00)_T$	$(2.00, 0.60, 0.00)_T$		$(35, 10, 0)_T$

Table 6 shows the dataset and the required fuzzy sets for such the shame are given in Table 7 (for more details see [46]). The structure of the fuzzy if-then rules are demonstrated in the Appendix. Now, suppose in the forward approximate reasoning scheme based on fuzzy sets, the antecedent statement is demonstrated as follows

$$\textit{Antecedent} \quad : \quad \widetilde{X}_1 \text{ is } \widetilde{B}_1^* \;\&\; \widetilde{X}_2 \text{ is } \widetilde{B}_2^* \;\&\; \widetilde{X}_3 \text{ is } \widetilde{B}_3^*,$$

where,

$$\widetilde{B}_1^* = (0.3, 0.2, 0.1)_T, \quad \widetilde{B}_2^* = (1, 0.4, 0.3)_T, \quad \widetilde{B}_3^* = (40, 10, 15)_T.$$

Applying the similarity measure S used in the two previous examples, we obtain the results summarized in Table 9 given in the Appendix. Thus,

$$\widetilde{C}^* = \cup_{i=1}^{98} \widetilde{C}_i^*,$$

where the fuzzy sets $\widetilde{C}_i^*$, $i = 1, \ldots, 98$, are given in Table 9. The function $\widetilde{C}^*(\cdot)$ is plotted in Figure 5. Using the center of gravity defuzzification method thorough following formula

$$\frac{\int x\widetilde{C}^*(x)\, dx}{\int \widetilde{C}^*(x)\, dx}$$

the crisp value of 16.87 obtained as the crisp value from the similarity based fuzzy inference system.

Once the fuzzy model is formed (see the Appendix), it can be used to predict the value of an output variable for various values of the input variables. By applying the proposed model to the data set given in the first part of Table 6, the predicted crisp values of output variable y in the second part of Table 6 are obtained. In order to determine the ability of the proposed model to predict the output variable y using input variables X_1, X_2 and X_3, the ordinary regression model [1] and the fuzzy inference system model (FIS model) [46,47] are also used for modeling this data set. We also compare the proposed method

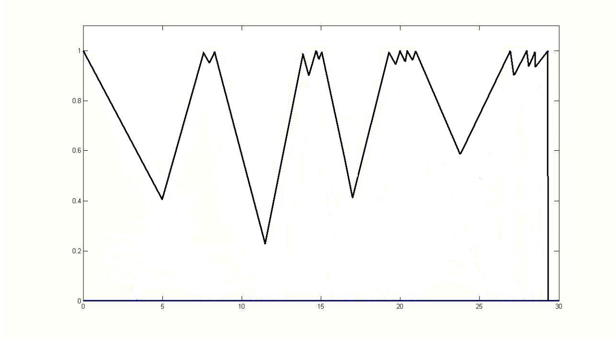

FIGURE 5. Membership function of $\widetilde{C}^* = \cup_{i=1}^{98} \widetilde{C}_i^*$

with the method proposed by Wagner et al. [45] that is a similarity-based inference engine for fuzzy logic systems. For $\widetilde{X}$ and $\widetilde{B}$ representing the input and antecedent FISs in the single rule

$$\text{rule} \quad : \quad \text{if } \widetilde{X} \text{ is } \widetilde{B} \implies \widetilde{Y} \text{ is } \widetilde{C}.$$

they employed the value of the following similarity measure

$$S(\widetilde{X}, \widetilde{B}) = \frac{\int \min\{\widetilde{X}(x), \widetilde{B}(x)\}\, dx}{\int \max\{\widetilde{X}(x), \widetilde{B}(x)\}\, dx},$$

as a degree of firing for a given pair of input and antecedent FISs. Thus, if this is the only rule of the rule-base, the output fuzzy set will be

$$\widetilde{C}^*(y) = \min\{S(\widetilde{X}, \widetilde{B}), \widetilde{C}(y)\}$$

For input-output mapping within the rule, Wagner et al. [45] employed minimum as the standard t-norm [29]. For multiple-input and multiple-rule system, see [35, 45].

The observed and predicted crisp values of output variable y based on the ordinary regression model, Wagner et al. [45] and the FIS model are shown in Table 8. The following performance indices were used to measure the performance of the three models in terms of accuracy of prediction

$$\begin{aligned} \mathbf{E} &= \sqrt{\frac{1}{n} \sum (y - \widehat{y})^2}, \\ \mathbf{v} &= \left(1 - \frac{var(y - \widehat{y})}{var(y)}\right) 100\%, \end{aligned}$$

where y is the output recorded value, $\widehat{y}$ is the corresponding predicted value and n is the sample size of instances. Generally, lower values (minimum 0) of $\mathbf{E}$ and higher values (maximum 100%) of $\mathbf{v}$ are indicative of better performance for a predictive model. Calculated values for $\mathbf{E}$ and $\mathbf{v}$ for the four constitutive models are given in Table 8, in favor of the proposed model.

TABLE 8. Calculated **E** and **v**% values as well as the corresponding predicted values of output variable y obtained from four competitive models.

	Observed	Predicted values of y			
No.	y	Our model	FIS [46]	Reg.	Wagner et al. [45]
1	30.80	31.1	32.0	28.1	29.9
2	27.50	26.3	26.9	27.6	29.3
3	25.05	21.4	21.8	26.3	19.2
4	19.70	16.6	16.7	21.2	10.8
5	5.86	4.5	3.9	10.7	8.9
6	25.61	26.6	25.4	26.3	24.3
7	22.55	23.6	23.3	26.3	21.9
8	21.46	22.9	24.0	26.3	18.5
9	27.96	27.6	27.6	26.3	20.1
10	34.81	34.5	34.0	29.2	33.0
11	9.61	10.6	10.6	10.3	8.2
12	8.94	9.8	11.3	10.7	7.7
13	10.97	11.8	12.0	10.7	11.7
14	12.09	11.8	12.9	11.7	15.2
15	12.90	12.8	13.8	14.5	12.9
E	—	1.48	1.68	2.27	3.78
v%	—	97.18	96.29	90.75	84.51

Remark 4.1. It should be noted that the superiority of the methods stated in Table 8 is based on the numerical results of a practical example. Since such the models are data dependent, therefore the comprehensive superiority of one method over another should be:

(1) based on diverse and extensive data sets; and/or
(2) based on relationships and pure formulas.

The last item above is an important topic which can be one of the research guideline in the future.

5. Conclusions

In this paper, we validated the applicability and effectiveness of functional dependence modeling between fuzzy input-output data based on the similarity-based approximate reasoning system. The models based on fuzzy logic have ability to closely approximate a wide variety of processes, from the traditional methods such as multivariate regression. Unlike many traditional modeling methods that typically represent an input-output relationship with a single equation that applies globally, the fuzzy logic modeling technique represents the correlation with numerous local equations (rules) that are combined to globally represent the process. An efficient structure identification algorithm ensures that the fuzzy model contains the appropriate fuzzy rules so that the predictive capability of the model is maximized. The experiments ascertain the fact that the proposed modeling technique based on the similarity-based approximate reasoning system performs better than the Mamdani fuzzy inference systems and the multivariate regression. The ability to incorporate physical understanding and prior knowledge is an important aspect of this model.

In our future work, we will pursue further this approach and study the interplay between the matching functions proposed in this paper and the remaining components of similarity-based reasoning systems, namely, robustness, continuity, modification, monotonicity, interpolativity and aggregation functions.

Author Contributions

Conceptualization, methodology, software, Jalal Chachi; formal analysis, investigation, writing—original draft preparation, Jalal Chachi and Mehdi Jalalvand; writing—review and editing, Mehdi Jalalvand. Both authors have read and agreed to the published version of the manuscript.

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Conflict of interest

The authors declare no conflict of interest.

Appendix: Rules and Tables

$$
\begin{aligned}
\text{rule}_1 &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{11} \;\&\; \widetilde{X}_2 \text{ is } \widetilde{B}_{25} \;\&\; \widetilde{X}_3 \text{ is } \widetilde{B}_{33} \implies \widetilde{Y} \text{ is } \widetilde{C}_1,\\
\text{rule}_2 &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{11} \;\&\; \widetilde{X}_2 \text{ is } \widetilde{B}_{25} \;\&\; \widetilde{X}_3 \text{ is } \widetilde{B}_{32} \implies \widetilde{Y} \text{ is } \widetilde{C}_1,\\
\text{rule}_3 &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{11} \;\&\; \widetilde{X}_2 \text{ is } \widetilde{B}_{25} \;\&\; \widetilde{X}_3 \text{ is } \widetilde{B}_{34} \implies \widetilde{Y} \text{ is } \widetilde{C}_2,\\
\vdots &\;\;\vdots\\
\text{rule}_{96} &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{15} \;\&\; \widetilde{X}_2 \text{ is } \widetilde{B}_{21} \;\&\; \widetilde{X}_3 \text{ is } \widetilde{B}_{32} \implies \widetilde{Y} \text{ is } \widetilde{C}_1,\\
\text{rule}_{97} &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{15} \;\&\; \widetilde{X}_2 \text{ is } \widetilde{B}_{21} \;\&\; \widetilde{X}_3 \text{ is } \widetilde{B}_{34} \implies \widetilde{Y} \text{ is } \widetilde{C}_1,\\
\text{rule}_{98} &: \text{ if } \widetilde{X}_1 \text{ is } \widetilde{B}_{15} \;\&\; \widetilde{X}_2 \text{ is } \widetilde{B}_{21} \;\&\; \widetilde{X}_3 \text{ is } \widetilde{B}_{31} \implies \widetilde{Y} \text{ is } \widetilde{C}_1.
\end{aligned}
$$

TABLE 9. Similarity measures $s_{ji} = S(\widetilde{B}_{ji}, \widetilde{B}_i^*)$ $(j = 1, 2, 3, i = 1, \ldots, 98)$ between fuzzy sets in the forward approximate reasoning in comparison study

i	s_{1i}	s_{2i}	s_{3i}	$s_i = \max_{j\in\{1,2,3\}} s_{ji}$	$\widetilde{C}_i^* = s_i \otimes \widetilde{C}_i$
1	0.837	0.606	0.066	0.837	$(0.00, 0.00, 8.37)_T$
2	0.837	0.606	0.080	0.837	$(0.00, 0.00, 8.37)_T$
3	0.837	0.606	0.043	0.837	$(8.37, 5.02, 4.19)_T$
4	0.837	0.617	0.066	0.837	$(8.37, 5.02, 4.19)_T$
5	0.837	0.617	0.080	0.837	$(8.37, 5.02, 4.19)_T$
6	0.837	0.617	0.043	0.837	$(8.37, 5.02, 4.19)_T$
7	0.837	0.617	0.030	0.837	$(8.37, 5.02, 4.19)_T$
8	0.837	0.800	0.066	0.837	$(20.93, 4.19, 6.70)_T$
9	0.837	0.800	0.080	0.837	$(20.93, 4.19, 6.70)_T$
10	0.837	0.800	0.043	0.837	$(20.93, 4.19, 6.70)_T$
11	0.837	0.800	0.030	0.837	$(20.93, 4.19, 6.70)_T$
12	0.837	0.645	0.066	0.837	$(20.93, 4.19, 6.70)_T$
13	0.837	0.645	0.080	0.837	$(20.93, 4.19, 6.70)_T$
14	0.837	0.645	0.043	0.837	$(20.93, 4.19, 6.70)_T$
15	0.837	0.645	0.030	0.837	$(20.93, 4.19, 6.70)_T$
16	0.837	0.645	0.066	0.837	$(29.30, 8.37, 0.00)_T$
17	0.837	0.645	0.080	0.837	$(29.30, 8.37, 0.00)_T$
18	0.837	0.645	0.043	0.837	$(29.30, 8.37, 0.00)_T$
19	0.837	0.645	0.030	0.837	$(29.30, 8.37, 0.00)_T$
20	0.769	0.606	0.066	0.769	$(0.00, 0.00, 7.69)_T$
21	0.769	0.606	0.080	0.769	$(0.00, 0.00, 7.69)_T$
22	0.769	0.606	0.043	0.769	$(7.69, 4.61, 3.85)_T$
23	0.769	0.606	0.030	0.769	$(7.69, 4.61, 3.85)_T$
24	0.769	0.617	0.066	0.769	$(13.84, 3.08, 3.08)_T$
25	0.769	0.617	0.080	0.769	$(13.84, 3.08, 3.08)_T$
26	0.769	0.617	0.043	0.769	$(19.23, 3.85, 6.15)_T$
27	0.769	0.617	0.030	0.769	$(19.23, 3.85, 6.15)_T$
28	0.769	0.800	0.066	0.800	$(20.00, 4.00, 6.40)_T$
29	0.769	0.800	0.080	0.800	$(20.00, 4.00, 6.40)_T$
30	0.769	0.800	0.043	0.800	$(20.00, 4.00, 6.40)_T$
31	0.769	0.800	0.030	0.800	$(20.00, 4.00, 6.40)_T$
32	0.769	0.645	0.066	0.769	$(26.92, 7.69, 0.00)_T$
33	0.769	0.645	0.080	0.769	$(26.92, 7.69, 0.00)_T$
34	0.769	0.645	0.043	0.769	$(26.92, 7.69, 0.00)_T$
35	0.769	0.645	0.030	0.769	$(26.92, 7.69, 0.00)_T$
36	0.769	0.645	0.066	0.769	$(26.92, 7.69, 0.00)_T$
37	0.769	0.645	0.080	0.769	$(26.92, 7.69, 0.00)_T$
38	0.769	0.645	0.043	0.769	$(26.92, 7.69, 0.00)_T$
39	0.769	0.645	0.030	0.769	$(26.92, 7.69, 0.00)_T$
40	0.816	0.606	0.066	0.816	$(0.00, 0.00, 8.16)_T$
41	0.816	0.606	0.080	0.816	$(0.00, 0.00, 8.16)_T$
42	0.816	0.606	0.043	0.816	$(8.16, 4.90, 4.08)_T$
43	0.816	0.606	0.030	0.816	$(8.16, 4.90, 4.08)_T$
44	0.816	0.617	0.066	0.816	$(14.69, 3.26, 3.26)_T$
45	0.816	0.617	0.043	0.816	$(20.40, 4.08, 6.53)_T$
46	0.816	0.617	0.030	0.816	$(20.40, 4.08, 6.53)_T$
47	0.816	0.800	0.043	0.816	$(20.40, 4.08, 6.53)_T$
48	0.816	0.800	0.080	0.816	$(20.40, 4.08, 6.53)_T$
49	0.816	0.800	0.043	0.816	$(28.56, 8.16, 0.00)_T$
50	0.816	0.800	0.030	0.816	$(28.56, 8.16, 0.00)_T$
51	0.816	0.645	0.066	0.816	$(28.56, 8.16, 0.00)_T$
52	0.816	0.645	0.030	0.816	$(28.56, 8.16, 0.00)_T$
53	0.816	0.645	0.043	0.816	$(28.56, 8.16, 0.00)_T$
54	0.816	0.645	0.030	0.816	$(28.56, 8.16, 0.00)_T$
55	0.816	0.645	0.066	0.816	$(28.56, 8.16, 0.00)_T$
56	0.816	0.645	0.080	0.816	$(28.56, 8.16, 0.00)_T$
57	0.816	0.645	0.043	0.816	$(28.56, 8.16, 0.00)_T$
58	0.816	0.645	0.030	0.816	$(28.56, 8.16, 0.00)_T$
59	0.833	0.606	0.043	0.833	$(8.33, 5.00, 4.17)_T$
60	0.833	0.606	0.080	0.833	$(8.33, 5.00, 4.17)_T$
61	0.833	0.606	0.043	0.833	$(14.99, 3.33, 3.33)_T$
62	0.833	0.606	0.030	0.833	$(14.99, 3.33, 3.33)_T$
63	0.833	0.617	0.066	0.833	$(20.83, 4.17, 6.66)_T$

Table 9: Continued

i	s_{1i}	s_{2i}	s_{3i}	$s_i = \max_{j\in\{1,2,3\}} s_{ji}$	$\widetilde{C}_i^* = s_i \otimes \widetilde{C}_i$
64	0.833	0.617	0.080	0.833	$(20.83, 4.17, 6.66)_T$
65	0.833	0.617	0.043	0.833	$(29.16, 8.33, 0.00)_T$
66	0.833	0.617	0.030	0.833	$(29.16, 8.33, 0.00)_T$
67	0.833	0.800	0.066	0.833	$(20.83, 4.17, 6.66)_T$
68	0.833	0.800	0.080	0.833	$(20.83, 4.17, 6.66)_T$
69	0.833	0.800	0.043	0.833	$(29.16, 8.33, 0.00)_T$
70	0.833	0.800	0.030	0.833	$(29.16, 8.33, 0.00)_T$
71	0.833	0.645	0.066	0.833	$(29.16, 8.33, 0.00)_T$
72	0.833	0.645	0.080	0.833	$(29.16, 8.33, 0.00)_T$
73	0.833	0.645	0.043	0.833	$(29.16, 8.33, 0.00)_T$
74	0.833	0.645	0.030	0.833	$(29.16, 8.33, 0.00)_T$
75	0.833	0.645	0.066	0.833	$(29.16, 8.33, 0.00)_T$
76	0.833	0.645	0.080	0.833	$(29.16, 8.33, 0.00)_T$
77	0.833	0.645	0.043	0.833	$(29.16, 8.33, 0.00)_T$
78	0.833	0.645	0.030	0.833	$(29.16, 8.33, 0.00)_T$
79	0.769	0.606	0.066	0.769	$(13.84, 3.08, 3.08)_T$
80	0.769	0.606	0.080	0.769	$(13.84, 3.08, 3.08)_T$
81	0.769	0.606	0.043	0.769	$(19.23, 3.85, 6.15)_T$
82	0.769	0.606	0.030	0.769	$(19.23, 3.85, 6.15)_T$
83	0.769	0.617	0.066	0.769	$(26.92, 7.69, 0.00)_T$
84	0.769	0.617	0.080	0.769	$(26.92, 7.69, 0.00)_T$
85	0.769	0.617	0.043	0.769	$(26.92, 7.69, 0.00)_T$
86	0.769	0.617	0.030	0.769	$(26.92, 7.69, 0.00)_T$
87	0.769	0.800	0.066	0.800	$(28.00, 8.00, 0.00)_T$
88	0.769	0.800	0.080	0.800	$(28.00, 8.00, 0.00)_T$
89	0.769	0.800	0.043	0.800	$(28.00, 8.00, 0.00)_T$
90	0.769	0.800	0.030	0.800	$(28.00, 8.00, 0.00)_T$
91	0.769	0.645	0.066	0.769	$(26.92, 7.69, 0.00)_T$
92	0.769	0.645	0.080	0.769	$(26.92, 7.69, 0.00)_T$
93	0.769	0.645	0.043	0.769	$(26.92, 7.69, 0.00)_T$
94	0.769	0.645	0.030	0.769	$(26.92, 7.69, 0.00)_T$
95	0.769	0.645	0.066	0.769	$(0.00, 0.00, 7.69)_T$
96	0.769	0.645	0.080	0.769	$(0.00, 0.00, 7.69)_T$
97	0.769	0.645	0.043	0.769	$(0.00, 0.00, 7.69)_T$
98	0.769	0.645	0.030	0.769	$(0.00, 0.00, 7.69)_T$

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