Journal of Mahani Mathematical Research
Shahid Bahonar University of Kerman
Print ISSN: 2251-7952 Online ISSN: 2645-4505

BEST STATES FOR WOMEN TO WORK: ANALYSIS USING MATHEMATICS OF UNCERTAINLY

J. N. Mordeson, S. Mathew ✉, and D. S. Malik

Article type: Research Article

(Received: 25 July 2024, Received in revised form 14 August 2024)

(Accepted: 18 August 2024, Published Online: 07 September 2024)

Abstract. In this paper, we determine the fuzzy similarity measure between the ranking of states with respect to best the states for work for women and work in general. We break the states into regions and determine the same fuzzy similarity measures for each region. We find that the fuzzy similarity measures range from high to very high. We develop new fuzzy similarity measures to be used in rankings.

Keywords: Women, work, state rankings, fuzzy similarity measures, distance functions.

2020 MSC: Primary 03B52, 03E72.

1. Introduction

In [7], states are ranked with respect to the best states to work. In [2], states are ranked with respect to the best state to work in general. We determine the fuzzy similarity measure of these to rankings. We find the similarity to be high. We then break the United States into regions and determine the fuzzy similarity measure of these two rankings for each region. Similarity plays a role in many fields. There exists many special definitions of similarity which have been used in different areas. We choose to use fuzzy similarity measures which seem appropriate in rankings. In fact, we develop some new measures. The paper is written as two parts; the first part focuses on the theoretical development of similarity measures and the second about its application in finding the best states for women to live and work. There are several possible fields like human trafficking and illegal immigration where we can use similarity measures effectively.

Let X be a set with n elements. We let $\mathcal{FP}(X)$ denote the fuzzy power set of X. We let $\wedge$ denote minimum and $\vee$ maximum. For two fuzzy subsets μ, ν of X, we write $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \epsilon X$. If μ is a fuzzy subset of μ, we let μ^c denote the complement of μ, i.e., $\mu^c(x) = 1 - \mu(x)$ for all $x \epsilon X$

✉ sm@nitc.ac.in, ORCID: 0000-0001-8478-9245
https://doi.org/10.22103/jmmr.2024.23791.1681
Publisher: Shahid Bahonar University of Kerman
How to cite: J. N. Mordeson, S. Mathew, D. S. Malik, *Best states for women to work: Analysis using mathematics of uncertainly*, J. Mahani Math. Res. 2025; 14(1): 219-231.

2. Fuzzy Similarity Measures and Distance Functions

Definition 2.1. [8] Let S be a function of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ into $[0,1]$. Then S is called a **fuzzy similarity measure** on $\mathcal{FP}(X)$ if the following properties hold: $\forall \mu, \nu, \rho \epsilon \mathcal{FP}(X)$,

(1) $S(\mu,\nu) = S(\nu,\mu)$;

(2) $S(\mu,\nu) = 1$ if and only if $\mu = \nu$;

(3) If $\mu \subseteq \nu \subseteq \rho$, then $S(\mu,\rho) \leq S(\mu,\nu) \wedge S(\nu,\rho)$;

(4) If $S(\mu,\nu) = 0$, then $\forall x \epsilon X, \mu(x) \wedge \nu(x) = 0$.

In [5], condition (4) of Definition 2.1 is replaced by: $S(\mu,\mu^c) = 0$ if and only if μ is crisp.

Example 2.2. *[8] Let μ and ν be fuzzy subsets of a finite set X. Let S and M be functions of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ into $[0,1]$. Then S and M are fuzzy similarity measures of X, where*

(1)$S(\mu,\nu) = 1 - \frac{\sum_{x\epsilon X} |\mu(x)-\nu(x)|}{\sum_{x\epsilon X}(\mu(x)+\nu(x))}$;

(2)$M(\mu,\nu) = \frac{\sum_{x\epsilon X}(\mu(x)\wedge\nu(x))}{\sum_{x\epsilon X}(\mu(x)\vee\nu(x))}$.

Definition 2.3. Let d be a function of $\mathcal{FP}(X) \times \mathcal{FP}(X)$ into $[0,1]$. Then d is called a **distance function** on $\mathcal{FP}(X)$ if the following properties hold: $\forall \mu, \nu, \rho \epsilon \mathcal{FP}(X)$,

(1) $d(\mu,\nu) = d(\nu,\mu)$;

(2) $d(\mu,\nu) = 0$ if and only if $\mu = \nu$;

(3) If $\mu \subseteq \nu \subseteq \rho$, then $d(\mu,\nu) \leq d(\mu,\rho)$ and $d(\nu,\rho) \leq d(\mu,\rho)$;

(4) If $d(\mu,\nu) = 1$, then $\forall x \epsilon X, \mu^c(x) \vee \nu^c(x) = 1$.

In [5], condition (4) of Definition 2.3 is replaced by $d(\mu,\mu^c) = 1$ if and only if μ is crisp.

Let A be a one-to-function of X onto $\{1,2,...,n\}$. Then A is called a **ranking** of X. Define the fuzzy subset μ_A of X by for all $x\epsilon X, \mu_A(x) = \frac{A(x)}{n}$. Then μ_A is called the **fuzzy subset associated** with A. For A a ranking of X, we have $\sum_{x\epsilon X} A(x) = \frac{n(n+1)}{2}$ and $\sum_{x\epsilon X} \mu_A(x) = \frac{n+1}{2}$ since $\sum_{x\epsilon X} A(x) = 1+2+...+n$.

Define $A^* : X \to [0,1]$ by $\forall x\epsilon X, A^*(x) = n+1-A(x)$. Then A^* is called the **reverse ranking** of A. It should be noted that A^* yields the smallest fuzzy similarity measure that A can have with any ranking of X, [4]. Note also that $\mu_{A^*}(x) = \mu_{A^c}(x) + \frac{1}{n}$, where μ_{A^c} is the complement of μ_A.

$\sum_{x\epsilon X} |\mu(x) - \nu(x)| = \sum_{x\epsilon X}(\mu(x) \vee \nu(x) - (\mu(x) \wedge \nu(x))$ and $\sum_{x\epsilon X}(\mu(x) + \nu(x)) = \sum_{x\epsilon X}(\mu(x) \vee \nu(x) + (\mu(x) \wedge \nu(x))$.

Definition 2.4. [1] (1) Define $d_H : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0,1]$ by for all $(\mu,\nu)\epsilon\mathcal{FP}(X) \times \mathcal{FP}(X), d_H(\mu,\nu) = \frac{1}{n}\sum_{i=1}^{n}|\mu(x_i) - \nu(x_i)|$. Then d_H is called the **normalized Hamming distance**.

(2) Define $S_H : \mathcal{FP}(X) \times \mathcal{FP}(X) \to [0,1]$ by for all $(\mu,\nu)\epsilon\mathcal{FP}(X) \times \mathcal{FP}(X), S_H(\mu,\nu) = \frac{1}{1+d_H(\mu,\nu)}$.

Theorem 2.5. *S_H is a fuzzy similarity measure.*

Proof. We have $S_H(\mu,\nu) = \frac{1}{1+\frac{1}{n}\sum_{i=1}^{n}|\mu(x_i)-\nu(x_i)|} = \frac{n}{n+\sum_{i=1}^{n}|\mu(x_i)-\nu(x_i)|}$. $S_H(\mu,\nu) = 1 \Leftrightarrow 1 = \frac{1}{1+d_H(\mu,\nu)} \Leftrightarrow d_H(\mu,\nu) = 0 \Leftrightarrow \mu = \nu$.Let $\mu,\nu,\nu,\rho\epsilon\mathcal{FP}(X)$ be such that $\mu \subseteq \nu \subseteq \rho$. Then

$$\begin{aligned}
S_H(\mu,\rho) &\leq S_H(\mu,\nu) \\
&\Leftrightarrow \frac{n}{n+\sum_{i=1}^{n}|\mu(x_i)-\rho(x_i)|} \leq \frac{n}{n+\sum_{i=1}^{n}|\mu(x_i)-\nu(x_i)|} \\
&\Leftrightarrow n+\sum_{i=1}^{n}|\mu(x_i)-\rho(x_i)| \geq n+\sum_{i=1}^{n}|\mu(x_i)-\nu(x_i)| \\
&\Leftrightarrow \sum_{i=1}^{n}(\mu(x_i)\vee\rho(x_i) - \mu(x_i)\wedge\rho(x_i)) \geq \sum_{i=1}^{n}(\mu(x_i)\vee\nu(x_i) - \mu(x_i)\wedge\nu(x_i)) \\
&\Leftrightarrow \sum_{i=1}^{n}(\rho(x_i)-\mu(x_i)) \geq \sum_{i=1}^{n}(\nu(x_i)-\mu(x_i)).
\end{aligned}$$

Since the latter statement is true, we have $S_H(\mu,\rho) \leq S_H(\mu,\nu)$.
A similar argument shows that $S_H(\mu,\rho) \leq S_H(\nu,\rho)$. □

Definition 2.6. Define $M_H : \mathcal{FP}(X)\times\mathcal{FP}(X) \to [0,1]$ by for all $(\mu,\nu)\epsilon\mathcal{FP}(X)\times \mathcal{FP}(X), M_H(\mu,\nu) = \frac{1}{1+2d_H(\mu,\nu)}$.

Theorem 2.7. *M_H is a fuzzy similarity measure.*

Proof. $M(\mu,\nu) = 1 \Leftrightarrow \frac{1}{1+2d_H(\mu,\nu)} = 1 \Leftrightarrow d_H(\mu,\nu) = 0 \Leftrightarrow \mu = \nu$. Let $\mu,\nu,\rho\epsilon\mathcal{FP}(X)$ be such that $\mu \subseteq \nu \subseteq \rho$.Then

$$\begin{aligned}
M_H(\mu,\rho) &\leq M_H(\mu,\nu) \\
&\Leftrightarrow \frac{1}{1+2d(\mu,\rho)} \leq \frac{1}{1+2d(\mu,\nu)} \\
&\Leftrightarrow \frac{2}{n}\sum_{i=1}^{n}|\mu(x_i)-\nu(x_i)| \geq \frac{2}{n}\sum_{i=1}^{n}|\mu(x_i)-\nu(x_i)|.
\end{aligned}$$

That the latter statement is true follows as in the previous theorem.
A similar argument shows that $M_H(\mu,\rho) \leq M_H(\nu,\rho)$. □

Theorem 2.8. $M_H = \frac{S_H}{2-S_H}$.

Proof. Let $\mu, \nu \epsilon \mathcal{FP}(X)$. Then

$$\frac{S_H(\mu,\nu)}{2-S_H(\mu,\nu)} = \frac{\frac{1}{1+d_H(\mu,\nu)}}{2-\frac{1}{1+d_H(\mu,\nu)}} = \frac{\frac{1}{1+d_H(\mu,\nu)}}{\frac{2+2d_H(\mu,\nu)-1}{1+d_H(\mu,\nu)}} = \frac{1}{1+2d_H(\mu,\nu)} = M_H(\mu,\nu). \quad \square$$

Define $d_M : \mathcal{FP}(X)\times\mathcal{FP}(X) \to [0,1]$ by for all $(\mu,\nu)\epsilon\mathcal{FP}(X)\times\mathcal{FP}(X)$, $d_M(\mu,\nu) = -1 + \frac{n}{n-\sum_{i=1}^n |\mu(x_i)-\nu(x_i)|}$. □

Theorem 2.9. *d_M is a distance function.*

Proof. $d_H(\mu,\nu) = 0 \Leftrightarrow -1 + \frac{n}{n-\sum_{i=1}^n |\mu(x_i)-\nu(x_i)|} = 0 \Leftrightarrow \frac{n}{n-\sum_{i=1}^n |\mu(x_i)-\nu(x_i)|} = 1 \Leftrightarrow n = n - \sum_{i=1}^n |\mu(x_i)-\nu(x_i)| \Leftrightarrow \sum_{i=1}^n |\mu(x_i)-\nu(x_i)| = 0 \Leftrightarrow \mu = \nu$. We have

$$\begin{aligned}
d_M(\mu,\rho) &\geq d_M(\mu,\nu) \\
&\Leftrightarrow -1 + \frac{n}{n-\sum_{i=1}^n |\mu(x_i)-\rho(x_i)|} \geq -1 + \frac{n}{n-\sum_{i=1}^n |\mu(x_i)-\nu(x_i)|} \\
&\Leftrightarrow \frac{1}{n-\sum_{i=1}^n |\mu(x_i)-\rho(x_i)|} \geq \frac{1}{n-\sum_{i=1}^n |\mu(x_i)-\nu(x_i)|} \\
&\Leftrightarrow n - \sum_{i=1}^n |\mu(x_i)-\rho(x_i)| \leq n - \sum_{i=1}^n |\mu(x_i)-\nu(x_i)| \\
&\Leftrightarrow \sum_{i=1}^n |\mu(x_i)-\rho(x_i)| \geq \sum_{i=1}^n |\mu(x_i)-\nu(x_i)|.
\end{aligned}$$

The latter condition is true so we have the result.

A similar argument shows that $d_M(\mu,\rho) \geq d_M(\nu,\rho)$ □

Let A and B be rankings of X.Then the following properties hold: $\forall x \epsilon X$, either (1) or (2) or (3) or (4) holds, where

(1) $\mu_A(x) \leq \mu_B(x) \leq \mu_{A^*}(x)$,

(2) $\mu_A(x) \geq \mu_B(x) \geq \mu_{A^*}(x)$,

(3) $\mu_B(x) < \mu_A(x) \wedge \mu_{A^*}(x)$,

(4) $\mu_B(x) > \mu_A(x) \vee \mu_{A^*}(x)$.

Let $E = \{x\epsilon X | A(x) = B(x) = A^*(x)\}$ and $X_i = \{x\epsilon X | x \notin E$ and (i) holds$\}$ $i = 1,2$. Let $X_i = \{x\epsilon X|\ (i)$ holds$\}, i = 3,4$. Let $n_i = |X_i|, i = 1,2,3,4$.

We see that $\{X_1, X_2, X_3, X_4, E\}$ is a partition of X.

Recall that $d_H(\mu_A, \nu_B) = \frac{1}{n}\sum_{x\epsilon X} |\mu_A(x) - \mu_B|$ and $d_H(\mu_{A^*}, \nu_B) = \frac{1}{n}\sum_{x\epsilon X} |\mu_A^*(x) - \mu_B|$.

The proof of the following result lies in the proof of Theorem of 4.1, [4].

Lemma 2.10. *[4]* $\sum_{x\epsilon X}|\mu_A(x)-\mu_B|-\sum_{x\epsilon X}|\mu_{A^*}(x)-\mu_B|=$

$$-n_1-\frac{n_1}{n}+\sum_{x\epsilon X_1}2\mu_B(x)+n_2+\frac{n_2}{n}+\sum_{x\epsilon X_2}(-2\mu_B(x))$$
$$-n_3-\frac{n_3}{n}+\sum_{x\epsilon X_3}2\mu_A(x)+n_4+\frac{n_4}{n}+\sum_{x\epsilon X_4}(-2\mu_A(x)).$$

Theorem 2.11. $d_H(\mu_A,\mu_B)-d_H(\mu_{A^*},\mu_B)=$

$$\frac{1}{n}(-n_1-\frac{n_1}{n}+\sum_{x\epsilon X_1}2\mu_B(x)+n_2+\frac{n_2}{n}+\sum_{x\epsilon X_2}(-2\mu_B(x))$$
$$-n_3-\frac{n_3}{n}+\sum_{x\epsilon X_3}2\mu_A(x)+n_4+\frac{n_4}{n}+\sum_{x\epsilon X_4}(-2\mu_A(x)))$$

Proof. $d_H(\mu_A,\mu_B)-d_H(\mu_{A^*},\mu_B)=\frac{1}{n}(\ \sum_{x\epsilon X}|\mu_A(x)-\mu_B|-\sum_{x\epsilon X}|\mu_{A^*}(x)-\mu_B|)=$

$$\frac{1}{n}(-n_1-\frac{n_1}{n}+\sum_{x\epsilon X_1}2\mu_B(x)+n_2+\frac{n_2}{n}+\sum_{x\epsilon X_2}(-2\mu_B(x))$$
$$-n_3-\frac{n_3}{n}+\sum_{x\epsilon X_3}2\mu_A(x)+n_4+\frac{n_4}{n}+\sum_{x\epsilon X_4}(-2\mu_A(x))).$$

□

Let $f(A,B,A^*)=$

$$-n_1-\frac{n_1}{n}+\sum_{x\epsilon X_1}2\mu_B(x)+n_2+\frac{n_2}{n}+\sum_{x\epsilon X_2}(-2\mu_B(x))$$
$$-n_3-\frac{n_3}{n}+\sum_{x\epsilon X_3}2\mu_A(x)+n_4+\frac{n_4}{n}+\sum_{x\epsilon X_4}(-2\mu_A(x)).$$

From [[4], Theorem 4.1, we have that $S(\mu_A,\mu_B)-S(\mu_{A^*},\mu_B)=-\frac{1}{n+1}f(A,B,A^*)$ and from Lemma 2.10 , $d_H(\mu_A,\mu_B)-d_H(\mu_{A^*},\mu_B)=\frac{1}{n}f(A,B,A^*)$. Thus the following results holds.

Theorem 2.12. *Let S be the similarity measure defined in Example 2.2. Then* $n(d_H(\mu_A,\mu_B)-d_H(\mu_{A^*},\mu_B))=-(n+1)(S(\mu_A,\mu_B)-S(\mu_{A^*},\mu_B))$.

3. Complementary Fuzzy Similarity Measures and Distance Functions

Theorem 3.1 ([3], pp. 12-14)**.** *Suppose* $\{a_1,a_2,...,a_n\}=\{1,2,...,n\}=\{b_1,b_2,...,b_n\}$.

(1) If n is even, then the largest $\sum_{i=1}^{n}|a_i-b_i)|$ *can be is* $\frac{n^2}{2}$.

(2) If n is odd, then the largest $\sum_{i=1}^{n}|a_i-b_i|$ *can be is* $\frac{n^2-1}{2}$.

Corollary 3.2. *Let A and B be rankings of X.*

(1) If n is even, then the largest $\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|$ can be is $\frac{n}{2}$.

(1) If n is odd, then the largest $\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|$ can be is $\frac{n^2-1}{2n}$.

Theorem 3.3. *Let d and S map $\mathcal{FP}(X)\times\mathcal{FP}(X)$ into $[0,1]$. Suppose $\forall(\mu,\nu)\epsilon\mathcal{FP}(X)\times\mathcal{FP}(X)$ that $d(\mu,\nu) = 1 - S(\mu,\nu)$. Then d is a distance function if and only if S is a fuzzy similarity measure.*

Proof. Let $(\mu,\nu)\epsilon\mathcal{FP}(X) \times \mathcal{FP}(X)$. Then $d(\mu,\nu) = d(\nu,\mu)$ if and only if $S(\mu,\nu) = S(\nu,\mu)$. Also, $d(\mu,\nu) = 0$ if and only if $S(\mu,\nu) = 1$. Let $\mu,\nu,\rho\epsilon\mathcal{FP}(X)$ be such that $\mu \subseteq \nu \subseteq \rho$. Then $d(\mu,\nu) \leq d(\mu,\rho) \Leftrightarrow S(\mu,\nu) \geq S(\mu,\rho)$ and $d(\nu,\rho) \leq d(\mu,\rho) \Leftrightarrow S(\nu,\rho) \geq S(\mu,\rho)$. Condition (4) for a fuzzy similarity measure states that $S(\mu,\mu^c) = 0 \Leftrightarrow \mu$ is crisp and that condition (4) for a distance function states that $d(\mu,\mu^c) = 1 \Leftrightarrow \mu$ is crisp. Now $S(\mu,\nu) = 0 \Leftrightarrow d(\mu,\nu) = 1$. Thus the desired result holds. Suppose that conditions (4) are $S(\mu,\nu) = 0 \Rightarrow \forall x\epsilon X, \mu(x) \wedge \nu(x) = 0$ and $d(\mu,\nu) = 1 \Rightarrow \forall x\epsilon X, \mu^c(x) \vee \nu^c(x) = 1$. Then for rankings, $S(\mu,\nu) = 0$ never holds and $d(\mu,\nu) = 1$ never holds. □

Define $S_C(\mu_A,\mu_B) = 1 - \frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_B(x)|}{n}$. Then $S_C(\mu_A,\mu_B)$ is called a **complementary fuzzy similarity measure** with respect to the Hamming distance.

The smallest $S(\mu_A,\mu_B) = 1 - \frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_{B(x)}|}{n+1}$ (S of Example 2.2) and $S_C(\mu_A,\mu_B) = 1 - \frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_B(x)|}{n}$ can be is when $\sum_{x\epsilon X}|\mu_A(x) - \mu_B(x)|$ is the largest it can be.

Now $S_C(\mu_A,\mu_B) = S(\mu_A,\mu_B) - \frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_{B(x)}|}{n(n+1)}$ for the largest $\sum_{x\epsilon X}|\mu_A(x) - \mu_B(x)| : S = 1 - \frac{1}{n+1}\sum_{x\epsilon X}|\mu_A(x) - \mu_B(x)|$ and $S_C = 1 - \frac{1}{n}\sum_{x\epsilon X}|\mu_A(x) - \mu_B(x)|$.Thus $(S(\mu_A,\mu_B)-1)(n+1) = (S_C(\mu_A,\mu_B)-1)n$. Hence $S(\mu_A,\mu_B)n + S(\mu_A,\mu_B) - 1 = S_C(\mu_A,\mu_B)n$ and so $S_C(\mu_A,\mu_B) = S(\mu_A,\mu_B) + \frac{1}{n}S(\mu_A,\mu_B) - \frac{1}{n}$. Therefore, $S_C(\mu_A,\mu_B) = S(\mu_A,\mu_B) + \frac{1}{n}(1 - \frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_B(x)|}{n+1}) - \frac{1}{n} = S(\mu_A,\mu_B) - \frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_B(x)|}{n+1}$.

Theorem 3.4. *Let A and B be rankings of X.*

(1) Suppose that n is even. Then $S_C(\mu_A,\mu_B) = S(\mu_A,\mu_B) - \frac{1}{2(n+1)}$

(2) Suppose that n is odd. Then $S_C(\mu_A,\mu_B) = S(\mu_A,\mu_B) - \frac{1}{2n} + \frac{1}{2n^2}$

Proof. $S_C(\mu_A,\mu_B) = S(\mu_A,\mu_B) - \frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_{B(x)}|}{n(n+1)}$ for the largest $\sum_{x\epsilon X}|\mu_A(x) - \mu_B(x)|$

(1) $S_C(\mu_A,\mu_B) = S(\mu_A,\mu_B) - \frac{\frac{1}{n}\frac{n^2}{2}}{n(n+1)} = S(\mu_A,\mu_B) - \frac{n}{2n(n+1)} = S(\mu_A,\mu_B) - \frac{1}{2(n+1)}$.

(2) $S_C(\mu_A, \mu_B) = S(\mu_A, \mu_B) - \frac{\frac{1}{n}\frac{(n-1)^2}{2}}{n(n+1)} = S(\mu_A, \mu_B) - \frac{n^2-1}{2n}\frac{1}{n(n+1)} = S(\mu_A, \mu_B) - \frac{1}{2n} + \frac{1}{2n^2}$. □

Theorem 3.5. *Let A and B be rankings of X.*

(1) Suppose that n is even. Then the smallest $S_C(\mu_A, \mu_B)$ *can be is* $\frac{1}{2}$.

(2) Suppose that n is odd. Then the smallest $S_C(\mu_A, \mu_B)$ *can be is* $\frac{1}{2} + \frac{1}{2n^2}$.

Proof. The smallest $S_C(\mu_A, \mu_B)$ can be is the smallest $S(\mu_A, \mu_B)$ minus the largest the largest $\frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_{B(x)}|}{n(n+1)}$. The smallest S and largest $\frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_{B(x)}|}{n(n+1)}$ occur at the same time.

(1) By [[3], p. 13] and Theorem 3.4, the smallest $S_C(\mu_A, \mu_B)$ can be is $\frac{n/2+1}{n+1} - \frac{1}{2(n+1)}\frac{1}{n} = \frac{n+2-1}{2(n+1)} = \frac{1}{2}$.

(2) By [[3], p. 14] and Theorem 3.4, the smallest $S_C(\mu_A, \mu_B)$ can be is $\frac{1}{2} + \frac{1}{2n} - \frac{n^2-1}{2n}\frac{1}{n+1}\frac{1}{n} = \frac{1}{2} + \frac{1}{2n} - \frac{n-1}{2n^2} = \frac{1}{2} + \frac{1}{2n^2}$. □ □

Another way to prove Theorem 3.5 is as follows: $S_C = 1 - \frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_B(x)|}{n}$. Thus the smallest S_C can be is 1 minus the largest $\frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_B(x)|}{n}$ can be. We have by Corollary 3.2 that for n even, the largest $\frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_B(x)|}{n}$ is $\frac{n/2}{n} = \frac{1}{2}$ and for n odd, the largest $\frac{\sum_{x\epsilon X}|\mu_A(x)-\mu_B(x)|}{n}$ can be is $\frac{n^2-1}{2n}\frac{1}{n} = \frac{1}{2} - \frac{1}{2n^2}$.

We next consider M. We have by Theorem 2.8 that $M_H = \frac{S_H}{2-S_H}$. Let s_n be the smallest S_H can be. Now $\frac{s_H}{2-s_H} \leq s_n$ since $\frac{1}{2-s_H} \leq 1$. We show that there does not exist s with $M_H(\mu_A, \mu_B)(x) = s$ for some $x\epsilon X$ such that $\frac{s}{2-s} < \frac{s_n}{2-s_n}$. If such an s exists, then $2s - ss_n < 2s_n - ss_n$ and so $s < s_n$ which is impossible since s_n is the smallest value S_H can be. Thus we have the following result.

Theorem 3.6. *The smallest value* M_H *can be is* $\frac{s_n}{2-s_n}$ *, where* s_n *is the smallest value* S_H *can be.*

4. Best States to Work

Suppose s denotes the smallest value for S. Define

$$\widehat{S}(\mu_A, \mu_B) = \frac{S(\mu_A, \mu_B) - s}{1 - s}.$$

Then $\widehat{S}(\mu_A, \mu_B)$ varies between 0 and 1. For values of $\widehat{S}(\mu_A, \mu_B)$ between 0 and 0.2, we say that the fuzzy similarity is very low, between 0.2 and 0.4 low, between 0.4 and 0.6 medium, between 0.6 and 0.8 high, between 0.8 and 1 very high. A similar approach can be taken for M and S_C.

It is stated in [2] that states have had to step up for workers and their families in the past few decades, as Congress has stalled on taking action. For example, while the federal minimum wage has been stuck at $7.25 an hour for

14 years, most states have mandated higher wages. In [2], The Best States to Work Index provides how the states rank overall and by policy area.

In [7], it is stated that since women make up the majority of the workforce-and-many are supporting families-this dimension considers how far the tipped minium wage goes to cover the cost of living for a family of three (one wage earner and two children). In [7], The Best States for Working Women Index provides how the states rank overall and by policy area.

In both rankings, the District of Columbia and Puerto Rica are included. In Table 1, the ranking of the best place for women to work is provided first and then the overall best place to work ranking is given.

TABLE 1. United States

State	Women	Overall	State	Women	Overall
Oregon	1	2	Florida	27	30
California	2	1	Michigan	28	26
New York	3	4	Missouri	29	31
Washington	4	5	South Dakota	30	27
Connecticut	5	7	Indiana	31	37
Massachusetts	6	6	Ohio	32	22
New Jersey	7	9	Iowa	33	36
Nevada	8	20	Idaho	34	41
Colorado	9	8	Pennsylvania	35	32
Hawaii	10	15	Kentucky	36	38
Puerto Rico	11	19	Oklahoma	37	44
Illinois	12	10	Wisconsin	38	34
District of Columbia	13	3	North Dakota	39	40
Vermont	14	11	Kansas	40	43
Maine	15	12	Arizona	41	18
Rhode Island	16	14	Louisiana	42	39
New Mexico	17	16	Arkansas	43	42
Minnesota	18	17	West Virginia	44	33
Maryland	19	13	Utah	45	46
Virginia	20	28	Wyoming	46	35
Delaware	21	21	South Carolina	47	49
Alaska	22	23	Texas	48	47
Nebraska	23	25	Mississippi	49	51
Montana	24	24	Alabama	50	48
Tennessee	25	45	Georgia	51	50
New Hampshire	26	29	North Carolina	52	52

Here $n = 52$. $S = 1 - \frac{212}{2756} = 0.9231$. The smallest S can be is $\frac{n/2+1}{n+1} = \frac{27}{53} = 0.5094$. Thus $\widehat{S} = \frac{0.9231-0.5094}{1-0.5094} = 0.8433$. The fuzzy similarity measure is high.

Now $M = \frac{S}{2-S} = \frac{0.9231}{1.8572} = 0.8572$. The smallest M can be is $\frac{0.5094}{2-0.5094} = \frac{0.5094}{1.4906} = 0.3417$. Thus $\widehat{M} = \frac{0.8472-0.3417}{1-0.3417} = \frac{0.5055}{0.6583} = 0.7679$.

By Theorem 3.4, $S_C = S - \frac{1}{2(n+1)} = 0.9231 - \frac{1}{106} = 0.9231 - 0.0094 = 0.9137$. The smallest S_C can be is 0.5000. Thus $\widehat{S_C} = \frac{0.9137-0.5000}{1-0.5000} = \frac{0.4137}{0.5000} = 0.8274$.

5. **Regions**

Table 2. West

State	Women Region rank	Overall Region Rank
Oregon	1	2
California	2	1
Montana	3	8
Washington	4	3
Nevada	5	6
Colorado	6	4
Hawaii	7	5
Alaska	8	7
Idaho	9	10
Utah	10	11
Wyoming	11	9

Here $n = 11$. $S = 1 - \frac{18}{32} = 1 - 0.1364 = 0.8656$. The smallest S can be is $\frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \frac{1}{2n} = 0.5 + \frac{1}{22} = 0.5455$. Thus $\widehat{S} = \frac{0.8656-0.5455}{1-0.5455} = \frac{0.3201}{0.4545} = 0.7043$. The fuzzy similarity measure is high.

Now $M = \frac{S}{2-S} = \frac{0.8656}{2-0.8656} = \frac{0.8656}{1.1344} = 0.7630$. The smallest M can be is $\frac{0.5455}{2-0.5455} = \frac{0.5455}{1.4545} = 0.3750$. Thus $\widehat{M} = \frac{0.7630-0.3750}{1-0.3750} = \frac{0.3890}{0.6250} = 0.6224$.

By Theorem 3.4, $S_C = S - \frac{1}{2n} + \frac{1}{2n^2} = 0.8656 - \frac{1}{22} + \frac{1}{242} = 0.8242$. The smallest S_C can be is $\frac{1}{2} + \frac{1}{2n^2} = 0.5 + 0.0041 = 0.5041$. Thus $\widehat{S_C} = \frac{0.8242-0.5041}{1-0.5041} = \frac{0.3201}{0.4959} = 0.6455$.

Table 3. South West

State	Women Region rank	Overall Region Rank
New Mexico	1	1
Oklahoma	2	3
Arizona	3	2
Texas	4	4

Here $n = 4.S = 1 - \frac{2}{20} = 0.9$.The smallest S can be is $\frac{n/2+1}{n+1} = \frac{3}{5} = 0.6$. Thus $\widehat{S} = \frac{0.9-0.6}{1-0.6} = \frac{0.3}{0.4} = 0.75$. The fuzzy similarity measure is high.

Now $M = \frac{S}{2-S} = \frac{0.9}{2-0.9} = \frac{0.9}{1.1} = 0.8182$. The smallest M can be is $\frac{0.6}{2-0.6} = \frac{0.6}{1.4} = 0.4288$. Thus $\widehat{M} = \frac{0.8182-0.4288}{1-0.4288} = \frac{0.3894}{0.5712} = 0.6817$.

By Theorem 3.4, $S_C = S - \frac{1}{2(n-1)} = 0.9 - \frac{1}{6} = 0.7333$. The smallest $S_C(\mu_A, \mu_B)$ can be is $\frac{1}{2}$. Thus $\widehat{S_C} = \frac{0.7333-0.5000}{1-0.5000} = \frac{0.2333}{0.5} = 0.4666$.

Table 4. Mid West

State	Women Region rank	Overall Region Rank
Illinois	1	1
Minnesota	2	2
Nebraska	3	4
Michigan	4	5
Missouri	5	7
South Dakota	6	6
Indiana	7	10
Ohio	8	3
Iowa	9	9
Wisconsin	10	8
North Dakota	11	11
Kansas	12	12

Here $n = 12$. $S = 1 - \frac{14}{156} = 1 - 0.0897 = 0.9103$. The smallest S can be is $\frac{n/2n+1}{n+1} = \frac{7}{13} = 0.5385$. Thus $\widehat{S} = \frac{0.9103-0.5385}{1-0.5385} = \frac{0.3718}{0.4615} = 0.8056$.. The fuzzy similarity measure is very high.

Now $M = \frac{S}{2-S} = \frac{0.9103}{2-0.9103} = \frac{0.9103}{1.0897} = 0.8453$. The smallest M can be is $\frac{0.5385}{2-0.5385} = \frac{0.5385}{1.4615} = 0.3685$. Thus $\widehat{M} = \frac{0.8354-0.3685}{1-0.3685} = \frac{0.4669}{0.6315} = 0.7393$.

By Theorem 3.4, $S_C = S - \frac{1}{2(n-1)} = 0.9103 - \frac{1}{22} = 0.8648$. The smallest S_C can be is $\frac{1}{2}$. Thus $\widehat{S_C} = \frac{0.8648-0.5000}{1-0.5000} = \frac{0.3648}{0.5000} = 0.7296$.

Here $n = 14$. $S = 1 - \frac{18}{210} = 1 - 0.0857 = 0.9143$. The smallest S can be is $\frac{1/2n+1}{n+1} = \frac{8}{15} = 0.5333$. Thus $\widehat{S} = \frac{0.9143-0.5333}{1-0.5333} = \frac{0.3810}{0.4667} = 0.8164$.. The fuzzy similarity measure is very high.

Now $M = \frac{S}{2-S} = \frac{0.9143}{2-0.9143} = \frac{0.9143}{1.0857} = 0.8427$. The smallest M can be is $\frac{0.5333}{2-0.5333} = \frac{0.5333}{1.4667} = 0.3636$. Thus $\widehat{M} = \frac{0.8427-0.3636}{1-0.3636} = \frac{0.4791}{0.6364} = 0.7528$.

By Theorem 3.4, $S_C = S - \frac{1}{2(n-1)} = 0.9143 - \frac{1}{26} = 0.8758$. The smallest S_C can be is $\frac{1}{2}$.Thus $\widehat{S_C} = \frac{0.8758-0.5000}{1-0.5000} = \frac{0.3758}{0.5000} = 0.7516$.

Here $n = 11$. $S = 1 - \frac{4}{132} = 1 - 0.0303 = 0.9697$. The smallest S can be is $\frac{1}{2} + \frac{2}{2n} = 0.5 + \frac{1}{22} = 0.5455$. Thus $\widehat{S} = \frac{0.9697-5455}{1-0.5455} = \frac{0.4242}{0.4545} = 0.9335$. Thus the fuzzy similarity measure is very high. Now $M = \frac{S}{2-S} = \frac{0.9697}{1.0303} = 0.9412$. The

Table 5. South East

State	Women Region rank	Overall Region Rank
Puerto Rico	1	2
Washington D. C.	2	1
Virginia	3	3
Tennessee	4	9
Florida	5	4
Kentucky	6	6
Louisiana	7	7
Arkansas	8	8
West Virginia	9	5
South Carolina	10	11
Mississippi	11	13
Alabama	12	10
Georgia	13	12
North Carolina	14	14

Table 6. North East

State	Women Region rank	Overall Region Rank
New York	1	1
Connecticut	2	3
Massachusetts	3	2
New Jersey	4	4
Vermont	5	5
Maine	6	6
Rhode Island	7	8
Maryland	8	7
Delaware	9	9
New Hampshire	10	10
Pennsylvania	11	11

smallest M can be is $\frac{0.5455}{2-0.5455} = \frac{0.5455}{1.454.4} = 0.3750$. Hence $\widehat{M} = \frac{0.9412-0.3750}{1-0.3750} = \frac{0.5662}{0.6250} = 0.9059$.

By Theorem 3.4, $S_C = S - \frac{1}{2n} + \frac{1}{2n^2} = 0.9697 - \frac{1}{22} + \frac{1}{242} = 0.9243$. The smallest S_C can be is $\frac{1}{2} + \frac{1}{2n^2} = 0.5 + 0.0041 = 0.5041$. Thus $\widehat{S_C} = \frac{0.9697-0.5041}{1-0.5041} = \frac{0.4656}{0.4959} = 0.9389$.

6. Conclusion

We determined the fuzzy similarity between the ranking of states with respect to best the states for work for women and overall. We also broke the states into regions and used the same fuzzy similarity measures for each region. We find that the fuzzy similarity measures range from high to very high. We developed new fuzzy similarity measures to be used in rankings. A new research possibility is to use the state rankings of the best place to work with the rankings in [6] with the respect to the best and worst states to be a woman. Another research possibility is to find the fuzzy similarity measures of the best states for women to work and those rankings with respect to violence off women, [4]. These ideas can be extended to different countries as a future work. Also fuzzy similarity measures can be used in comparing the effectiveness of controlling human trafficking and illegal migration.

7. Author Contributions

Coceptualization, methodology J N Mordeson and D S Malik resources, J N Mordeson and D S Malik; writing—original draft preparation, validation, J N Mordeson, S Mathew; writing—review and editing,S. Mathew; Conceptualization, S Mathew.

8. Data Availability Statement

Data used in the work is from references [2], [4] and [7].

9. Funding

No funding was received for the work.

10. Conflict of interest

The authors declare no conflict of interest.

References

[1] Beg, I., and Ashraf, S., Similarity measures for fuzzy sets, Applied and Computational Mathematics, vol. 8. no. 2,(2009) 196-202.
[2] Best and Worst states to Work in America 2023, https://www.oxfamamerica.org/stateaction2023.org., OXFAM.
[3] Mordeson, J. N., & Mathew, S. (2021). Mathematics of Uncertainty for Coping with World Challenges, Climate Change, World Hunger, Modern Slavery, Coronavirous, Human Trafficking, Springer, Studies in Systems, Decision and Control 353, 2021.
[4] D. S. Malik, J. N. Mordeson, and S. Mathew, Domestic violence and homicide: fuzzy similarity measures, Journal of Mahani Mathematical Research, 2024; 13(3): 1 - 23.
[5] Peng, X., New similarity measure and distance measure for Pythagorean fuzzy sets, Complex and Intelligent Systems (2019) 5: 101-111.
[6] The Best and Worst States to Be a Women. Introducing the U. S. Women, Peace, and Security Index 2020. GIWPS Georgetown Institute for Women, Peace and Security.
[7] The Three Policy Areas: How the States Rank for Working Women, State Scorecards, https://www.oxfamamerica.org/statesreport 2022).
[8] Wang, X., De Baets, B., and Kerre, E., A comparative study of similarity measure, Fuzzy Sets and Systems, 73 (1995) 259-268.

John N. Mordeson
Orcid number: 0000-0001-5808-0793
Department of Mathematics
Creighton University
Omaha, USA
Email address: mordes@creighton.edu

Sunil Mathew
Orcid number: 0000-0001-8478-9245
Department of Mathematics
National Institute of Technology Calicut
Calicut, India
Email address: sm@nitc.ac.in

Davender S. Malik
Orcid number: 0009-0005-2290-0960
Department of Mathematics
Creighton University
Omaha, USA
Email address: malik@creighton.edu