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EXTENSION OF STABILIZERS ON SUBTRACTION ALGEBRAS

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Special issue dedicated to Professor Esfandiar Eslami

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Abstract. This paper explores the intersection between the class of bounded subtraction algebras and the class of Boolean algebras, demonstrating their equivalence. It introduces the concepts of stabilizers for subsets and the stabilizers of one subset with respect to another within subtraction algebras. The study reveals that both the stabilizer of a subset and the stabilizer of an ideal with respect to another ideal are, in fact, ideals themselves. Investigating the impact of stabilizers on product and quotient subtraction algebras is a focal point. Additionally, a novel concept termed the "stabilizer operation" is defined, and it is proven that the collection of ideals endowed with a binary stabilizer operator forms a bounded Hilbert algebra.

Keywords: (Bounded) Subtraction algebra, Stabilizer, Hilbert algebra.
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1. **Introduction**

The notion of subtractions on algebraic structures has been a subject of extensive investigation, initially with a focus on Boolean algebras. B.M. Schein [12] explored a system of $(\Phi, \circ, \setminus)$ where Φ denotes a set of functions closed under function composition "$\circ$", i.e. $(\Phi, \circ)$ is a function semigroup, and set theoretic subtraction "$\setminus$", i.e. $(\Phi, \setminus)$ is a subtraction algebra following prior definitions [1]. He showed that every subtraction semigroup is isomorphic to a differente semigroup of invertible functions. Algebraic structures such as subtraction algebras are among the most fundamental and crucial in logical algebras. Schein's findings laid the foundation for subsequent investigations into the properties and structures of these systems. B. Zelinka [15] delved into a problem proposed by Schein regarding the structure of multiplication within subtraction semigroups, achieving resolution specifically for atomic subtraction algebras. Building upon this, Y.B. Jun, H. S. Kim, and E.H. Roh [7] introduced the concept of ideals within subtraction algebras, focusing on their characterization.

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Further research expanded the scope of inquiry. Y.B. Jun and H. S. Kim [8] explored the notion of ideals generated by sets within subtraction algebras, while Y.B. Jun, Y.H. Kim, and K.A. Oh [9] introduced the concept of complicated subtraction algebras and scrutinized their properties. The study by K.J. Lee, Y.B. Jun, and Y.H. Kim [11] introduced weak subtraction algebras, accompanied by a method for constructing them from quasi-ordered sets. Y. Çeven and M.A. Öztürk [3] made notable contributions by introducing essential notions such as subalgebras, bounded subtraction algebras, and unions of subtraction algebras.

In parallel, the notion of stabilizers has been explored in various algebraic contexts such as BL-algebras [5], Residuated Lattices [2], and triangle algebras [14]. Haveshki and Mohamadhasani [5] initiated the study of stabilizers in BL-algebras, while Saeid and Mohtashamnia [2] introduced novel types of stabilizers in residuated lattices, exploring their relationships with filters. Zahiri, Saeid, and Eslami [14] extended the investigation to triangle algebras, introducing and studying stabilizers in detail, particularly focusing on their relations with special IVRL-filters.

Motivated by these foundational works, this article aims to present a novel perspective on subtraction algebras, specifically focusing on the introduction and development of stabilizer theory within this context. The main objectives include establishing the equivalence between bounded subtraction algebras and Boolean algebras, introducing the notions of stabilizer of a subset and stabilizer of a subset with respect to another subset, investigating their properties, and exploring the relationships between stabilizers and ideals in subtraction algebras. Since the ideals play significant role in considering these algebras, we prove that the stabilizer of a subset and the stabilizer of a subset with respect to another subset are ideals. In addition, some theorems that determine relationships among these ideals have been determined. One of our aims is to introduce specific sets in subtraction algebras and consider them in details. Finally, it is demonstrated that the set of all ideals endowed with a binary stabilizer operator forms a bounded Hilbert algebra, contributing to the broader understanding of these algebraic structures.

2. **Preliminaries**

This section examines some key terms and findings that will be crucial in the upcoming sequel.

Definition 2.1. [9] An algebra $\langle A, -\rangle$ with a single binary operation $-$ is called a *subtraction algebra* if for all $x, y, z \in A$ the following equations hold:

(S1) $x - (y - x) = x$,
(S2) $x - (x - y) = y - (y - x)$,
(S3) $(x - y) - z = (x - z) - y$.

Lemma 2.1. [9]If $\langle A, -\rangle$ is a subtraction algebra, then for every $x, y \in A$

(1) $(x - y) - y = x - y$
(2) $x - x = y - y$

Corollary 2.2. For any $a \in A$, $0 := a - a$ is the minimum element of A and for every $x \in A$

$$x - 0 = x \text{ and } 0 - x = 0.$$

Proposition 2.3. [9] The subtraction determines a partial order relation on A by

$$a \leq b \Longleftrightarrow a - b = 0 \tag{1}$$

where $0 = a - a$ is an element that by Lemma 2.12 does not depend on the choice of $a \in A$.

Proof. The reflexivity property is clear. For antisymmetry property, if $a \leq b$ and $b \leq a$, then we have $a - b = 0$ and $b - a = 0$. So by (S_2)

$$a = a - (a - b) = b - (b - a) = b.$$

For transitivity property we have, if $a \leq b$ and $b \leq c$, then $a - b = 0$ and $b - c = 0$. Thus by (S_2) and (S_3) we have

$$\begin{aligned} a - c &= (a - c) - 0 \\ &= (a - c) - (a - b) \\ &= (a - (a - b)) - c \\ &= (b - (b - a)) - c \\ &= (b - c) - (b - a) \\ &= 0 - (b - a) = 0. \end{aligned}$$

□

Proposition 2.4. [9] In a subtraction algebra $\langle A, -, 0\rangle$ the followings are true for all $x, y, z \in A$

(1) $x - 0 = x$ and $0 - x = 0$,
(2) $(x - y) - x = 0$,
(3) $x - (x - y) \leq y$,
(4) $(x - y) - (y - x) = x - y$,
(5) $x - (x - (x - y)) = x - y$,
(6) $(x - y) - (z - y) \leq x - z$,
(7) $x \leq y$ iff $x = y - w$ for some $w \in A$,
(8) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$,

note that, the infimum of two elements of a subtraction algebra always exists. However, not only might the supremum of them not exist, but they also might not have an upper bound at all.

Proposition 2.5. The infimum of two elements of a subtraction algebra always exists and it is $a \wedge b = a - (a - b)$.

Proof. By (S_3), we have

$$\begin{aligned}(a-(a-b))-a &= (a-a)-(a-b)\\ &= 0-(a-b)\\ &= 0.\end{aligned}$$

Thus $a-(a-b)\leq a$, and by (S_2) we have $a-(a-b)=b-(b-a)\leq b$.

If $x\leq a,b$, then $x-a=0$ and $x-b=0$. Therefore by (S_3), (S_2) and Proposition 2.6

$$\begin{aligned}((x-(x-a))-(x-b))-(a-(a-b)) &= ((x-(x-a))-(a-(a-b))-(x-b)\\ &= (a-(a-x))-(a-(a-b)))-(x-b)\\ &= ((a-(a-(a-b)))-(a-x))-(x-b)\\ &= ((a-b)-(a-x))-(x-b)\\ &= ((a-(a-x))-b)-(x-b)\\ &\leq (a-(a-x))-x\\ &= (a-x)-(a-x)=0.\end{aligned}$$

So $((x-(x-a))-(x-b))\leq(a-(a-b))$, by Proposition 2.6 we have $x-(a-(a-b))\leq x-((x-(x-a))-(x-b))$. Thus

$$\begin{aligned}x-(a-(a-b)) &\leq x-((x-0)-0)\\ &= x-(x-0)\\ &= x-x\\ &= 0.\end{aligned}$$

Therefore $x\leq a-(a-b)$, and $a-(a-b)$ is the infimum of a and b. □

Proposition 2.6. [9] In a subtraction algebra $\langle A,-,0\rangle$ the followings are true for all $x,y,z\in A$

(1) $x,y\leq z$ implies $x-y=x\wedge(z-y)$,
(2) $(x\wedge y)-(x\wedge z)\leq x\wedge(y-z)$,
(3) $(x-y)-z=(x-z)-(y-z)$.

Proposition 2.7. [7] Let $\langle A,-,0\rangle$ be a subtraction algebra and let $x,y\in A$. If $w\in A$ is an upper bound for x and y, then the least upper bound of x and y exists and

$$x\vee y=w-\big((w-y)-x\big).$$

Definition 2.2. [3] Let $\langle A,-,0\rangle$ be a subtraction algebra. If there is an element $1\in A$ satisfying $x\leq 1$ for all $x\in A$, then A is called a *bounded subtraction algebra* denoted by $\langle A,-,0,1\rangle$.

In a bounded subtraction algebra A, we denote $1-x$ by x'.

Definition 2.3. [9] Let $\langle A, -, 0\rangle$ be a subtraction algebra. For any $a, b \in A$, let

$$G(a, b) = \{x \in A : x - a \leq b\}$$

then A is said to be complicated if for any $a, b \in A$ the set $G(a, b)$ has a greatest element.

Definition 2.4. [7] Let $\langle A, -, 0\rangle$ be a subtraction algebra. A non-empty subset $I \subseteq A$ is called an *ideal* of A if for every $x, y \in A$,

(1) $0 \in I$,
(2) $y \in I$, $(x - y) \in I$ implies $x \in I$.

Definition 2.5. [10]

(i) Let $\langle A, -, 0\rangle$ be a subtraction algebra. A *prime ideal* of A is defined to be an ideal P of A such that $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

(ii)Let $(A, \wedge, \vee)$ be a lattice. A *lattice ideal* of A is a non-empty subset of A such that:

- I is a sublattice of A,
- for any $a \in I$ and $b \in A$, $a \wedge b \in I$.

Lemma 2.8. [7] An ideal I of a subtraction algebra A has the following property, for all $x, y \in A$:

If $y \in I, x \leq y$ *then* $x \in I$.

Definition 2.6. [4] Let $\langle X, -, 0\rangle$ and $\langle Y, \ominus, \mathit{0}\rangle$ be any two subtraction algebras. If the mapping $\varphi : X \to Y$ for all $x, y \in X$, satisfies

$$\varphi(x - y) = \varphi(x) \ominus \varphi(y),$$

then φ is called a *homomorphism*.

Definition 2.7. [6] A *Hilbert algebra* is a triple $\langle H, \rightarrow, 1\rangle$, where " $\rightarrow$ " is a binary operation on H, $1 \in H$ and for all $x, y, z \in H$ the following conditions are satisfied:

(H1) $x \to (y \to x) = 1$,
(H2) $\big(x \to (y \to z)\big) \to \big((x \to y) \to (x \to z)\big) = 1$,
(H3) $x \to y = 1$ and $y \to x = 1$ implies $x = y$.

A Hilbert algebra A is *bounded* if there exists an element $0 \in A$ such that $0 \leq x$ for all $x \in A$, where " $\leq$ " is defined by: $a \leq b$ iff $a \to b = 1$.

3. Bounded subtraction Algebras

This section explores the intersection between the class of bounded subtraction algebras and the class of Boolean algebras, demonstrating their equivalence.

Proposition 3.1. In a subtraction algebra $\langle A, -, 0\rangle$ the following is satisfied for all $x, a, b \in A$,

$$(x - a) \vee (x - b) = x - (a \wedge b)$$

Proof. Since x is an upper bound for $x-a$ and $x-b$ by Proposition 2.62 and (1), according to Proposition 2.7 their least upper bound exists. By Proposition 2.68 we have $x - a \leq x - (a \wedge b)$ and $x - b \leq x - (a \wedge b)$. Consequently

$$(x-a) \vee (x-b) \leq x - (a \wedge b).$$

Conversely by (S3) and Proposition 2.66 we have

$$\begin{aligned}\Big(\big(x-(x-b)\big)-(x-a)\Big)-a &= \big((x \wedge b)-a\big)-(x-a)\\ &\leq (x \wedge b) - x = 0.\end{aligned}$$

Thus $\Big(\big(x-(x-b)\big)-(x-a)\Big) \leq a$, and by (S3) with changing the role of a and b we have $\Big(\big(x-(x-b)\big)-(x-a)\Big) \leq b$. Thus $\Big(\big(x-(x-b)\big)-(x-a)\Big) \leq a \wedge b$. Now from Proposition 2.68 we have

$$x - a \wedge b \leq x - \Big(\big(x-(x-b)\big)-(x-a)\Big)$$

which by Proposition 2.7 means

$$x - a \wedge b \leq (x-a) \vee (x-b). \qquad \square$$

Proposition 3.2. In every bounded subtraction algebra $\langle A, -, 0, 1\rangle$, the following properties hold for all $x, y, z \in A$:

(1) $(x')' = x$,
(2) $x' - y' = y - x$,
(3) $x \wedge y' - y = x \wedge y'$,
(4) $x \wedge y = x - y'$,
(5) $(x \wedge y)' = (x' \vee y')'$,
(6) $(x \vee y)' = (x' \wedge y')'$,
(7) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Proof.

(1) $(x')' = 1 - (1 - x) = x - (x - 1) = x - 0 = x$.
(2) From (S3) and part 1. we have

$$x' - y' = (1-x) - (1-y) = (1-(1-y)) - x = (y')' - x = y - x.$$

(3)

$$\begin{aligned}x \wedge y' - y &= \big(y' - (y' - x)\big) - y\\ &= (y' - y) - (y' - x)\\ &= y' - (y' - x) = y' \wedge x.\end{aligned}$$

(4)

$$\begin{aligned} x \wedge y - (x - y') &= \big(x - (x - y)\big) - (x - y') \\ &= \big(x - (x - y')\big) - \big((x - y) - (x - y')\big) \\ &= x \wedge y' - \Big(\big(x - (x - y')\big) - y\Big) \\ &= x \wedge y' - \big((x \wedge y') - y\big) \\ &= x \wedge y' - x \wedge y' = 0, \end{aligned}$$

thus $x \wedge y \leq x - y'$. Conversely $x \leq 1$ by use of Proposition 2.6 8 implies $(x - (x - y)) \geq (x - (1 - y))$, i.e. $x \wedge y \geq x - y'$.

(5) Let $x = 1$ in Proposition 3.1.

(6) It is clear by parts 1 and 5.

(7) By using of parts 1, 4, 5 and Proposition 3.1

$$\begin{aligned} x \wedge (y \vee z) &= x \wedge (y' \wedge z')' \\ &= x - (y' \wedge z') \\ &= (x - y') \vee (x - z') \\ &= (x \wedge y) \vee (x \wedge z). \quad \square \end{aligned}$$

The following theorem is a result of the above discussions:

Theorem 3.3. If $\langle A, -, 0, 1\rangle$ is a bounded subtraction algebra, then $(A, \wedge, \vee, {}', 0, 1)$ is a Boolean algebra, where

$$\begin{aligned} x \wedge y &:= x - (x - y), \\ x \vee y &:= 1 - ((1 - y) - x), \\ x' &:= 1 - x. \end{aligned}$$

Conversely every Boolean algebra with $a - b := a \wedge b'$ can be interpreted as a subtraction algebra. Thus the class of bounded subtraction algebras which have been introduced by Y. Çeven, and M.A. Öztürk [3] and the class of Boolean algebras coincide. In [3], they introduced bounded subtraction algebras and defined the set $S(A) = \{x \in A : (x')' = x\}$ and showed that $S(A)$ is a bounded subalgebra of A. They proved that if A is a bounded complicated subtraction algebra, then $S(A)$ is a complicated subalgebra, which all are degenerated, because $S(A) = A$.

4. Stabilizers on subtraction algebras

In this section, we will introduce the concepts of stabilizers for subsets and the stabilizers of one subset with respect to another within subtraction algebras. Then we consider them in details.

For the subsequent sections, $\langle A, -, 0\rangle$ or simply A is a subtraction algebra.

Definition 4.1. If X, Y are non-empty subsets of A, then

$$X^\star = \left\{a \in A \;\middle|\; x - a = x \;\; \textit{for all} \;\; x \in X\right\}$$

is called the *stabilizer* of X in A, and

$$(X, Y)^\star = \left\{a \in A \;\middle|\; x - (x - a) \in Y \;\; \textit{for all} \;\; x \in X\right\}$$

is called the stabilizer of X with respect to Y in A.

We see that in a special case where $Y = \{0\}$, the concept of $(X, Y)^\star$ coincides with the concept of $X^\star$.

Example 4.1. Let $\langle A, -, 0\rangle$ be a subtraction algebra with $A = \{0, a, b, c, d\}$ and the following table:

-	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	a
b	b	b	0	b	b
c	c	c	c	0	c
d	d	d	d	d	0

and let $X = \{c\}$ and $Y = A$. Then

$$\begin{aligned} X^\star &= \left\{t \in A \mid x - t = x, \textit{ for all } x \in X\right\} \\ &= \left\{t \in A \mid c - t = c\right\} \\ &= \left\{0, a, b, d\right\} \end{aligned}$$

$$(X, Y)^\star = (X, A)^\star = \left\{t \in A : x - (x - t) \in A, \textit{ for all } x \in X\right\} = A.$$

In the following example, we show that for $X, Y \subseteq A$, $(Y, X)^\star \neq (X, Y)^\star$.

Example 4.2. In Example 4.1, we have $(X, Y)^\star = A \neq (Y, X)^\star = \emptyset$.

Proposition 4.3. If X is a non-empty subset of A then $X^\star$ is an ideal.

Proof. Since $(\forall x \in X)(x - 0 = x)$, $0 \in X^\star$.
Now suppose $a - b \in X^\star$, $b \in X^\star$ and let $x \in X$. Then by Proposition 2.66 we have

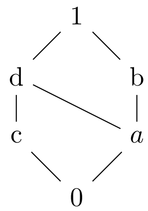
$$x = x - (a - b) = (x - b) - (a - b) \leq x - a \leq x.$$

Thus $(\forall x \in X)(x - a = x)$, that is $a \in X^\star$. Hence $X^\star$ is an ideal. □

A. Borumand Saeid and N. Mohtashamnia in [2] claim that for every x in the residuated lattice L the right stabilizer of the set $\{x\}$ is a prime filter of L. However, this assertion is incorrect. The following provides counterexample:

Example 4.4. [13] Let $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a residuated lattice with Hasse diagram and Cayley tables as follows:

Then $\{a\}^*_R = \{1\}$ is not a prime filter because $b \vee d \in \{a\}^*_R$, but $b \notin \{a\}^*_R$ and $d \notin \{a\}^*_R$.



$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
1	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Similarly, this claim does not hold in subtraction algebras. The following example shows that $\{x\}^\star$ the stabilizer of $\{x\}$ is not necessarily a prime ideal.

Example 4.5. Let $\langle A, -, 0\rangle$ be a subtraction algebra, with $A = \{0, a, b, c\}$ and the following table:

-	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Then $\{c\}^\star = \{0\}$ which is not prime, because $a \wedge b = a - (a - b) = 0 \in \{c\}^\star$, but neither $a \in \{c\}^\star$ nor $b \in \{c\}^\star$.

Theorem 4.6. Let A be a linearly ordered subtraction algebra. Then for every $\emptyset \neq X \subseteq A$, $X^\star$ is a prime ideal of A.

Proof. Let $a, b \in A$ such that $a \wedge b \in X^\star$. Then $x - (a \wedge b) = x$, for all $x \in X$. Since A is linearly ordered, $a \leq b$ or $b \leq a$, so $x = x - (a \wedge b) = x - a$, for all $x \in X$ or $x = x - (b \wedge a) = x - b$, for all $x \in X$. Hence $a \in X^*$ or $b \in X^*$. So X^* is a prime ideal. □

Theorem 4.7. If X, Y are ideals of A, then $(X, Y)^\star$ is an ideal of A.

Proof. Since Y is an ideal, $(x - (x - 0)) = x - x = 0 \in Y$. Thus $0 \in (X, Y)^\star$. Now suppose $b, a - b \in (X, Y)^\star$ then

$$\big(x - (x - b)\big) \in Y \qquad \textit{for all } x \in X,$$

(2) $$\Big(x - \big(x - (a - b)\big)\Big) \in Y \qquad \textit{for all } x \in X.$$

Now let $x \in X$ by Lemma 2.8 we have $x - b \in X$. So by (2)

$$(x - b) - \Big((x - b) - (a - b)\Big) \in Y \tag{3}$$

Since $(x - (x - b)) \in Y$, by Lemma 2.8, we have:

$$\Big(\big(x - (x - b)\big) - (x - a)\Big) \in Y.$$

Hence by Proposition 2.63

$$\Big(\big(x - (x - a)\big) - \big((x - b) - (x - a)\big)\Big) \in Y. \tag{4}$$

On the other hand by Proposition 2.66

$$(x - b) - (a - b) \leq x - a$$

and Proposition 2.68 implies

$$(x - b) - \big((x - b) - (a - b)\big) \geq (x - b) - (x - a).$$

Again by Proposition 2.68

$$\begin{aligned} &\big(x - (x - a)\big) - \Big((x - b) - \big((x - b) - (a - b)\big)\Big) \\ &\qquad\qquad \leq \big(x - (x - a)\big) - \big((x - b) - (x - a)\big). \end{aligned} \tag{5}$$

Since Y is an ideal, Lemma 2.8 and equations (4), (5) imply that

$$\big(x - (x - a)\big) - \Big((x - b) - \big((x - b) - (a - b)\big)\Big) \in Y. \tag{6}$$

Since Y is an ideal, by equations (3), (6) we have

$$\big(x - (x - a)\big) \in Y$$

Thus $a \in (X, Y)^\star$, and consequently $(X, Y)^\star$ is an ideal. □

By the following example, we show that if X, Y are not ideal of A, then $(X, Y)^\star$ is not ideal of A:

Example 4.8. In Example 4.5, let $X = \{c\}$ and $Y = \{a\}$. Then we have

$$(X, Y)^\star = \{k \in A \mid c - (c - k) = a\} = \{b\}.$$

That $\{b\}$ is not an ideal of A.

Proposition 4.9. Let X, Y are non-empty subsets of A. Then

(1) $X^\star = \bigcap \big\{\{x\}^\star : x \in X\big\}$,
(2) $X \subseteq Y$ implies $Y^\star \subseteq X^\star$,
(3) $A^\star = \{0\}$, $\{0\}^\star = A$,
(4) Let $h : A \to A$ be a homomorphism and $x \in A$. Then $h\left(\{x\}^\star\right) \subseteq \{h(x)\}^\star$.

Proof.

(1)

$$\begin{aligned} a \in X^\star &\iff & x - a = x &\quad \textit{for all } x \in X \\ &\iff & a \in \{x\}^\star &\quad \textit{for all } x \in X \\ &\iff & a \in \bigcap\{\{x\}^\star : x \in X\} & \end{aligned}$$

(2) By part 1 is clear.
(3) Since $(\forall x \in X)(x - 0 = x)$, $0 \in A^\star$, and if $t \in A^\star$, then $t \in A$ and $x - t = x$ for all $x \in A$. If $x = t$, then $t - t = 0$ and $t = 0$. Consequently $A^\star = \{0\}$. Since $(\forall x \in X)(0 - x = 0)$, thus $\{0\}^\star = A$.
(4) If $t \in h(\{x\}^\star)$, then $t = h(a)$ for some $a \in \{x\}^\star$, thus $x - a = x$. Since h is a homomorphism we have

$$h(x) = h(x - a) = h(x) - h(a) = h(x) - t \tag{7}$$

Hence $t \in \{h(x)\}^\star$. □

In the following, we give an example in which $\{h(x)\}^\star \not\subseteq h(\{x\}^\star)$.

Example 4.10. Let $A = \{0, a, b, c, d\}$ be the subtraction algebra of Example 4.1 and $h : A \to A$ a homomorphism with the following table:

x	0	a	b	c	d
h(x)	0	a	b	c	0

then $\{h(d)\}^\star = \{0\}^\star = A$, but $h(\{d\}^\star) = h(\{0, a, b, c\}) = \{0, a, b, c\}$.

Theorem 4.11. Let I be an ideal of A and X, Y, X_i, Y_i subsets of A, for all $i \in J$ such that $\bigcap_{i\in J} X_i \neq \emptyset$ and $\bigcap_{i\in J} Y_i \neq \emptyset$. Then

(1) $(X, Y)^\star = A$ implies $X \subseteq Y$,
(2) $(I, Y)^\star = A$ iff $I \subseteq Y$,
(3) $(I, I)^\star = A$,
(4) If $X_i \subseteq X_j$ and $Y_i \subseteq Y_j$, then $(X_j, Y_i)^\star \subseteq (X_i, Y_j)^\star$,
(5) $(X, \{0\})^\star = X^\star$,
(6) $X^\star \subseteq (X, I)^\star$,
(7) $(\{0\}, I)^\star = A$,
(8) $(X, \bigcap_{i\in J} Y_i)^\star \subseteq \bigcap_{i\in J}(X, Y_i)^\star$,
(9) $\bigcap_{i\in J}(X_i, Y)^\star = (\bigcup_{i\in J} X_i, Y)^\star$,
(10) $\bigcap_{i\in J}(X, Y_i)^\star = (X, \bigcap_{i\in J} Y_i)^\star$,
(11) $\bigcap_{i\in J} Y_i^\star = (\bigcup_{i\in J} Y_i)^\star$,
(12) $(\bigcup_{i\in J} X_i, \bigcap_{i\in J} Y_i)^\star \subseteq \bigcap_{i\in J}(X_i, Y_i)^\star \subseteq (\bigcap_{i\in J} X_i, \bigcap_{i\in J} Y_i)^\star$,
(13) $\bigcup_{i\in J}(X_i, Y_i)^\star \subseteq (\bigcap_{i\in J} X_i, \bigcup_{i\in J} Y_i)^\star$,
(14) $\bigcup_{a\in I}\{a\}^\star = A$.

Proof.

(1) Let $x \in X$, since $(X, Y)^\star = A$ we have $1 \in (X, Y)^\star$, thus $x = x - (x - 1) \in Y$, Hence $X \subseteq Y$.

(2) Assume $a \in A$, $x \in I$ and $I \subseteq Y$, since $x - (x - a) \leq x$ we have $x - (x - a) \in I$, thus $x - (x - a) \in Y$, Hence $a \in (I, Y)^\star$. The converse is clear by part1.
(3) It is clear by part2.
(4) Let $x \in (X_j, Y_i)^\star$, then $a - (a - x) \in Y_i$ for all $a \in X_j$, and so $a - (a - x) \in Y_j$ for all $a \in X_i$, i.e. $x \in (X_i, Y_j)^\star$.
(5) Since $x - (x - a) = 0$ iff $x - a = x$, thus

$$(X, \{0\})^\star = \{a : x - (x - a) = 0\} = \{a : x - a = x\} = X^\star.$$

(6) It is clear by parts 4, 5, and that $0 \in I$.
(7) It is clear by part 2, since $\{0\}$ is an ideal.
(8) It is clear by part 4.
(9) $x \in \bigcap_{i\in J}(X_i, Y)^\star$ iff $(\forall i \in J)(x \in (X_i, Y)^\star)$ iff $(\forall i \in J)(\forall y \in X_i)(y - (y - x) \in Y)$ iff $(\forall y \in \bigcup_{i\in J} X_i)(y - (y - x) \in Y)$ iff $x \in (\bigcup_{i\in J} X_i, Y)^\star$.
(10) $x \in \bigcap_{i\in J}(X, Y_i)^\star$ iff $(\forall i \in J)(x \in (X, Y_i)^\star)$ iff $(\forall i \in J)(\forall y \in X)(y - (y - x) \in Y_i)$ iff $(\forall y \in X)(\forall i \in J)(y - (y - x) \in Y_i)$ iff $x \in (X, \bigcap_{i\in J} Y_i)^\star$.
(11) It is derived by parts 5, 9.
(12) It is clear by part 4.
(13) It is clear by part 4.
(14) By Proposition 4.9, $\{0\}^\star = A$ and since $0 \in I$ we have $\bigcup_{a\in I}\{a\}^\star = A$. □

In the following example, we show that the converse of part (i) is not true:

Example 4.12. In Example 4.1, if $X = \{a\} \subseteq Y = \{a, b\}$, then $(X, Y)^\star \neq A$.

Proposition 4.13. If A, B are two subtraction algebras, $X, F \subseteq A$ and $Y, G \subseteq B$, then $(X \times Y, F \times G)^\star = (X, F)^\star \times (Y, G)^\star$.

Proof. For every $a \in A, b \in B$, we have $(a, b) \in (X \times Y, F \times G)^\star$

$$\text{iff } (x, y) - ((x, y) - (a, b)) \in F \times G, \text{ for all } x \in X, y \in Y$$
$$\text{iff } (x - (x - a), y - (y - b)) \in F \times G, \text{ for all } x \in X, y \in Y$$
$$\text{iff } (a, b) \in (X, F)^\star \times (Y, G)^\star.$$

□

Corollary 4.14. Let $\{F_i\}_{i\in I}$ be a family of ideals of A and $\{X_i\}_{i\in I}$ be a family of subsets of A. Then $(\prod_{i\in I} X_i, \prod_{i\in I} F_i)^\star = \prod_{i\in I}(X_i, F_i)^\star$.

Theorem 4.15. Let A be a subtraction algebra which is also a lattice, and X be a subset of A
(i) If J is a (lattice) ideal of A, then $(X, J)^\star$ is a (lattice) ideal of A.
(ii) If J is a prime ideal of A, then $(X, J)^\star$ is a prime ideal of A.

Example 4.16. In Example 4.5, If $X = \{a\}, Y = \{c, b\}$, then we have $(X, Y)^* = \{0, a\}$ that is a prime ideal but Y is not a prime ideal.

Theorem 4.17. Let A be a linearly subtraction algebra and X, Y be subsets of A such that $X \nsubseteq Y$. Then $(X, Y)^\star \subseteq Y$.

Proof. Let $a \in (X, Y)^\star$. Then $a \wedge x \in Y$, for all $x \in X$. Since A is lineary, $a \in Y$ or $x \in Y$, for all $x \in X$. Also since $X \nsubseteq Y$, we have $a \in Y$, so $(X, Y)^\star \subseteq Y$. $\square$

Proposition 4.18. If F and G are ideals of A and $X \subseteq A$, then $(X, F/G)^\star = (X, F)^\star/G$.

Proof.

$$\begin{aligned}(X, F/G)^\star &= \{[a] \in A/G \mid x - (x - [a]) \in F/G, \text{ for all } x \in X\}\\ &= \{[a] \in A/G \mid x - (x - a) \in F, \text{ for all } x \in X\}\\ &= \{[a] \in A/G \mid a \in (X, F)^\star\}\\ &= (X, F)^\star/G.\end{aligned}$$

$\square$

Proposition 4.19. Let $f : A \longrightarrow B$ be a subtraction algebra monomorphism and $X \subset A$. Then $(X, \ker f)^\star = f^{-1}(f(X)^\star)$.

Proof. It is clear that $\ker(f)$ is an ideal of A.

$$\begin{aligned}a \in (X, \ker(f))^\star &\Leftrightarrow x - (x - a) \in \ker(f), \text{ for all } x\\ &\Leftrightarrow f(x - (x - a)) = 0, \text{ for all } x\\ &\Leftrightarrow f(x) - (f(x) - f(a))) = 0, \text{ for all } x\\ &\Leftrightarrow f(x) - f(a) = f(x), \text{ for all } x\\ &\Leftrightarrow a \in f^{-1}(f(X)^\star).\end{aligned}$$

$\square$

Let us to denote the set of all ideals of the subtraction algebra A by $\mathcal{I}(A)$. We define the operation " $\Rightarrow$ " on $\mathcal{I}(A)$ by $I \Rightarrow J := (I, J)^\star$. Let us call it an "Stabilizer Operator".

Lemma 4.20. For all $I, J, K \in \mathcal{I}(A)$, the following is satisfied.

$$I \cap J \subseteq K \quad \textit{iff} \quad I \subseteq J \Rightarrow K$$

Proof. Suppose $I \cap J \subseteq K$, fix $x \in I$, and let y be an arbitrary element of J. Since I, J are ideals we have $x - (x - y) \in I$ and $y - (y - x) \in J$. On the other hand $x - (x - y) = y - (y - x)$. Thus $y - (y - x) \in I \cap J$. Hence $y - (y - x) \in K$, for all $y \in J$, i.e. $x \in J \Rightarrow K$.
Conversely suppose $I \subseteq J \Rightarrow K$, if $x \in I \cap J$, then $x \in J$ and $x \in J \Rightarrow K$, thus $y - (y - x) \in K$, for all $y \in J$, now by letting $y = x$, we have $x = x - (x - x) \in K$. $\square$

Corollary 4.21. For every subtraction algebra $\langle A, -, 0\rangle$ and for all $I, J \in \mathcal{I}(A)$ the following holds:

$$I \cap (I \Rightarrow J) \subseteq J$$

Theorem 4.22. $\langle \mathcal{I}(A), \Rightarrow, A \rangle$ is a bounded Hilbert algebra.

Proof. According to Theorem 4.7, " $\Rightarrow$ " is a binary operation on $\mathcal{I}(A)$, and clearly $A \in \mathcal{I}(A)$.
Since $I \cap J \subseteq I$, by Lemma 4.20 $I \subseteq J \Rightarrow I$. Now by Theorem 4.112 and new notation we have

$$I \Rightarrow (J \Rightarrow I) = A \tag{8}$$

Let $x \in (I \Rightarrow J) \cap \big(I \Rightarrow (J \Rightarrow K)\big)$. Thus for all $y \in I$ we have $y - (y - x) \in J$ and $y - (y - x) \in (J \Rightarrow K)$, and by Corollary 4.21 we have $y - (y - x) \in K$ for all $y \in I$. Hence $x \in I \Rightarrow K$. Thus $(I \Rightarrow J) \cap \big(I \Rightarrow (J \Rightarrow K)\big) \subseteq I \Rightarrow K$, which by corollary 4.21 implies

$$\big(I \Rightarrow (J \Rightarrow K)\big) \subseteq (I \Rightarrow J) \Rightarrow (I \Rightarrow K)$$

By Theorem 4.112 we have

$$\big(I \Rightarrow (J \Rightarrow K)\big) \Rightarrow \big((I \Rightarrow J) \Rightarrow (I \Rightarrow K)\big) = A \tag{9}$$

Again by Theorem 4.112 we have

$$\Big((I \Rightarrow J) = A \textit{ and } (J \Rightarrow I) = A\Big) \quad \textit{implies} \quad I = J \tag{10}$$

Since $\{0\} \subseteq I$ for all $I \in \mathcal{I}(A)$, by (8), (9) and (10) the algebra $\langle \mathcal{I}(A), \Rightarrow, A \rangle$ is a bounded Hilbert algebra. □

5. Conclusion

The notion of subtraction algebras is defined by B. M. Schein [1]. Subtraction algebras have been investigated by different people and different results have been presented so far. In this paper, we show that the class of bounded subtraction algebras and the class of Boolean algebras coincide. Then, we introduced the notions of the stabilizer of a subset X and the stabilizer of X with respect to Y in a subtraction algebra A. Moreover, we show that $X^\star$ and $(G, F)^\star$ are ideals of A if $X \subseteq A$ and F, G are ideals of A. We investigated many important properties of the stabilizers. Finally, we proved that in every subtraction algebra A, $\langle \mathcal{I}(A), \Rightarrow, A \rangle$ is a bounded Hilbert algebra. The investigation of other such special sets can be an interesting topic for future research.

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