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INVESTIGATING MODULES WITH PARTIAL ENDOMORPHISMS HAVING μ-SMALL KERNELS

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Abstract. In this paper, we introduce and study the concept of generalized monoform modules ($G-M$ modules, for short) which is a proper generalization of that of monoform modules. We present some of their examples, properties and characterizations. It is shown that over a commutative ring R, the properties monoform, small monoform, $G-M$, compressible, uniform and weakly co-Hopfian are all equivalent. Moreover, we demonstrate that a ring R is an injective semisimple ring iff any R-module is $G-M$. Further, we prove a similar theorem to Hilbert's basis theorem for monoform, small monoform and $G-M$ modules.

Keywords: Monoform modules, Small monoform modules, $G-M$ modules.
2020 MSC: Primary 16D40, 16D10, 16P40.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary R-modules. Let L be an R-module, for submodules A and B of L, $A \leq B$ denotes that A is a submodule of B, $A \leq^{\oplus} L$ denotes that A is a direct summand of L, $E(L)$ denotes the injective hull of L and $\text{End}_R(L)$ denotes the ring of endomorphisms of L. The study of modules by properties of their endomorphisms is a classical research subject. In [18], Zelmanowitz introduced the concept of monoform modules. Recall that a partial endomorphism of a module M is a homomorphism from a submodule of M into M. A module is monoform if any of its non-zero partial endomorphism is monomorphism. A monoform module is uniform (i.e., any two non-zero submodules have non-zero intersection). A submodule N of L is called a small submodule of L if whenever $N + K = L$ for some submodule K of L, we have $L = K$, and in this case we write $N \ll L$. A module L is called small if it is a small submodule of some module. In [11], Inaam Hadi and Hassan Marhun introduced and studied the notion of small monoform modules. An R-module L is called a small monoform module, if any non-zero partial endomorphism of L has a small kernel.

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Recently, some works have been done concerning some variations of Hopfian, co-Hopfian and also monoform modules see [2–4, 7].

Motivated by the above-mentioned works, we are interested in introducing a new generalization of monoform modules namely $G-M$ modules. We call a module L $G-M$ if for every non-zero submodule N of L and for each $f \in Hom_R(N, L)$, $f \neq 0$ implies $Kerf \ll_\mu N$. The concept of $G-M$ modules forms a proper generalization of monoform modules (Example 2.2). It is obvious that any small monoform module is $G-M$. Example 2.18 demonstrates that the converse is false, in general.

Our paper is structured as follows:

In Section 1, we give some known results which we will cite or use throughout this paper.

In Section 2, we present some equivalent properties and characterizations of $G-M$ modules. A non-zero R-module L is called compressible provided for each non-zero submodule N of L there exists a monomorphism $f : L \to N$. We have the obvious implications: compressible $\Rightarrow$ monoform $\Rightarrow$ small monoform $\Rightarrow$ $G-M$. We will see later that under certain conditions the properties monoform, uniform, compressible, small monoform and $G-M$ are coincide (Corollary 2.9). The dual of an R-module L is $Hom_R(L, R)$, this will be denoted by L^*. If the natural map $L \to (L^*)^*$ is bijective, L will be called reflexive. We prove that for a quasi-Frobenius principal ring R, if L is a $G-M$ cosingular R-module, then L is reflexive and $E(L^*)$ is finitely generated (Proposition 2.11). In proposition 2.15, we obtain that if L is a fully retractable R-module such that for every $0 \neq N \leq L$, the kernel of any non-zero endomorphism of N is μ-small, then L is $G-M$. In [5], we investigated and introduced the concept of μ-Hopfian modules. An R-module L is called μ-Hopfian if every surjective endomorphism of L has a μ-small kernel. We show that for a semisimple quasi-injective R-module L, the properties μ-Hopfian and $G-M$ are equivalent (Proposition 2.16).

In Section 3, we consider the monoform, small monoform and $G-M$ properties of the $R[x]/(x^{n+1})$-modules $L[x]/(x^{n+1})$.

For a ring R and a right R-module L, let $Z^*(L) = \{x \in L\colon\ xR \text{ is small}\}$. If $Z^*(L) = 0$ (resp., $Z^*(L) = L$), then L is called a noncosingular (resp., cosingular) module (see [13]).

A submodule K of L is said to be μ-small in L ($K \ll_\mu L$), if $L = K + H$ with L/H is cosingular, then $L = H$, see [17]. It is clear that if $B \ll L$, then $B \ll_\mu L$, but the converse is not true in general, see [17, Examples and Remarks 2.10].

Let R be a ring and L an R-module. We now state a few well-known preliminary results:

Remark 1.1. (1) Let R be a commutative ring and L an R-module. Then L is monoform if and only if L is uniform prime [15].

(2) Let R be a commutative ring. Then every compressible R-module is monoform [15].
(3) It is clear that every monoform R-module is a small monoform module. However, the converse in general is not true. $\mathbb{Z}_4$ is a small monoform $\mathbb{Z}$-module while it is not monoform [11].
(4) Let L be a semisimple R-module. Then the following are equivalent [11].
  (a) L is small monoform;
  (b) L is monoform;
  (c) L is simple.

We list some properties of μ-small submodules that will be used in the paper.

Lemma 1.2. *([17]) Let L be an R-module.*

(1) *Let $X \leq Y \leq L$. Then $Y \ll_\mu L$ iff $X \ll_\mu L$ and $Y/X \ll_\mu L/X$.*
(2) *Let $X \leq L$ and $Y \leq L$. Then $X + Y \ll_\mu L$ iff $X \ll_\mu L$ and $Y \ll_\mu L$. Moreover, if X_1, X_2, ...,X_n are submodules of L with $X_i \ll_\mu L$, $\forall$ $i = 1, ..., n$, then $\sum_{i=1}^{n} X_i \ll_\mu L$.*
(3) *Let $f : L \to N$ be a homomorphism. If $X \ll_\mu L$, then $f(X) \ll_\mu N$.*
(4) *Let $L = L_1 \oplus L_2$ be a module, $X_1 \leq L_1$ and $X_2 \leq L_2$. Then $X_1 \oplus X_2 \ll_\mu L_1 \oplus L_2$ iff $X_1 \ll_\mu L_1$ and $X_2 \ll_\mu L_2$.*
(5) *Let L be a module and let $X \leq Y \leq L$. If $Y \leq^{\oplus} L$ and $X \ll_\mu L$, then $X \ll_\mu Y$.*

Lemma 1.3. *([5]) Let K be a submodule of a module L. Then the following statements are equivalent.*

(1) *$K \ll_\mu L$.*
(2) *If $X + K = L$, then $X \leq^{\oplus} L$ and L/X is semisimple injective.*

2. Modules in which every partial endomorphism has a μ-small kernel

Definition 2.1. An R-module L is called $G - M$ if for every non-zero submodule N of L and for each $f \in Hom_R(N, L)$, $f \neq 0$ implies $Kerf \ll_\mu N$.

Example 2.2. *Let $H = \mathbb{Z}_{q^\infty}$. Since H is a hollow group, each proper subgroup is μ-small, hence H is a $G - M$ group. But H is not monoform because the multiplication by q induces an endomorphism of H which is not a monomorphism.*

Theorem 2.3. *The following are equivalent for an R-module L:*

(1) *L is $G - M$.*
(2) *For every non-zero partial endomorphism $f \in Hom(N, L)$ where $0 \neq N \leq L$, if there exists $P \leq N$ such that $f(P) = f(N)$, then there exists an injective semisimple direct summand H of N such that $N = H \oplus P$.*

Proof. (1) $\Rightarrow$ (2) Assume that $f \in Hom(N, L)$ where $0 \neq N \leq L$ is a non-zero partial endomorphism. If there exists $P \leq N$ such that $f(P) = f(N)$, then $Kerf + P = N$. Since L is $G - M$, $Kerf \ll_{\mu} N$. By Lemma 1.3, $N = H \oplus P$ for some injective semisimple $H \leq N$.

(2) $\Rightarrow$ (1) Let $f \in Hom(N, L)$ where $0 \neq N \leq L$ be a non-zero partial endomorphism and $Ker(f) + P = N$ for some $P \leq N$, where N/P is cosingular. Then $f(P) = f(N)$. By (2), there exists an injective semisimple direct summand H of N such that $N = H \oplus P$, then N/P is noncosingular by [13, Lemma 4]. Thus $N/P = 0$. Therefore $N = P$ and $Ker(f) \ll_{\mu} N$.

□

Proposition 2.4. *Every $G - M$ cosingular module is uniform.*

Proof. Let L be a $G - M$ cosingular R-module. Suppose there exists a non-zero submodule N of L such that N is not essential in L. So, there exists a relative complement K of N in L such that $N \oplus K$ is essential in L. Let $f : N \oplus K \to L$ define by $f(n + k) = k$ for all $n + k \in N \oplus K$. It is clear that f is well defined and $f \neq 0$. Since L is $G - M$, $Kerf = \{0\} \oplus K \ll_{\mu} N \oplus K$. So, by Lemma 1.3, K is injective semisimple. Thus K is noncosingular by [13, Lemma 4]. Since L is cosingular, K is also cosingular, then K must be zero. This implies that N is essential in L, contradiction.

□

Recall that a non-zero right R-module L is called prime, if whenever N is a non-zero submodule of L and A is an ideal of R such that $NA = 0$, then $LA = 0$.

Example 2.5. *It is clear that a simple module is $G - M$ prime. But in general, the converse is not true. For example, $\mathbb{Z}$ is a $G - M$ prime $\mathbb{Z}$-module. However, $\mathbb{Z}$ is not simple.*

Definition 2.6. [10]. Let L be an R-module. Then L is called weakly co-Hopfian if every its injective endomorphism has an essential image.

Recall that an Artinian principal ideal ring is a left and right Artinian, left and right principal ideal ring.

Theorem 2.7. *Let R be an Artinian principal ideal ring and L be a cosingular R-module. Then the following statements are equivalent:*

(1) *L is $G - M$ prime;*
(2) *L is simple.*

Proof. (1) $\Rightarrow$ (2) Suppose L is a $G - M$ prime module. Since L is cosingular, so L is uniform by Proposition 2.4. Thus L is weakly co-Hopfian. As R is an Artinian principal ideal ring, then by [1, Theorem 3.8], L is finitely generated. Hence there exists an epimorphism $g : R \to L$ such that $R/ann_R(L) \cong L$. Since L is a prime module, $ann_R(L)$ is a prime ideal of R. Hence $ann_R(L)$ is maximal in R as R is Artinian. Thus L is a simple module.

(2) ⇒ (1) It is clear.

□

Example 2.8. *Every compressible R-module is $G-M$. In general, the converse is not true. For example, $\mathbb{Q}$ is a $G-M$ $\mathbb{Z}$-module. But it is not compressible because $Hom_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = \{0\}$.*

Corollary 2.9. *Let R be an Artinian principal ideal ring and L be a prime cosingular R-module. Then the following statements are equivalent:*

(1) *L is $G-M$;*
(2) *L is monoform;*
(3) *L is small monoform;*
(4) *L is compressible;*
(5) *L is uniform;*
(6) *L is weakly co-Hopfian.*

Proof. (1) ⇒ (2) By Theorem 2.7, L is a simple module, then L is monoform.

(2) ⇒ (1) It is clear.

(1) ⇔ (3) By [17], if L is cosingular and if $K \leq L$, then $K \ll L$ if and only if $K \ll_{\mu} L$. Therefore L is $G-M$ iff L is a small monoform module.

(1) ⇒ (4) By (1), we obtain that L is a uniform prime finitely generated module hence, by [15, Lemma 26.2.9], L is compressible.

(4) ⇒ (1) By Remark 1.1, every compressible module is monoform, then it is $G-M$.

(2) ⇒ (5) It is clear.

(5) ⇒ (2) Suppose that L is a uniform module. According to the proof of (1) ⇒ (2), L is simple. Thus, L is monoform.

(5) ⇒ (6) It is clear.

(6) ⇒ (5) Assume that L is a weakly co-Hopfian module. Then L is simple. Thus, L is uniform.

□

Corollary 2.10. *Let R be an Artinian principal ideal ring and L a $G-M$ cosingular R-module. Then* $\mathrm{End}(L)$ *is a local ring.*

Proof. Since L is a finitely generated module over an Artinian ring, by Theorem 2.7, L is of finite length. And since L is uniform, L is indecomposable of finite length. Thus, $\mathrm{End}(L)$ is a local ring.

□

Proposition 2.11. *Let R be a principal quasi-Frobenius ring and L a $G-M$ cosingular R-module. Then the following statements hold:*

(1) *L is reflexive.*
(2) *L^* and $E(L^*)$ are finitely generated.*

Proof. 1) According to Theorem 2.7, L is a finitely generated R-module. Thus, by [12, Theorem 15.11], L is reflexive.

2) Since R is Artinian and L^* is finitely generated, $E(L^*)$ is finitely generated.

□

Proposition 2.12. *Let L be a $G-M$ module and f be a surjective endomorphism of L, $N \leq L$. Then $f(N) \ll_\mu L$ if and only if $N \ll_\mu L$.*

Proof. $\Rightarrow$) Let $N + Y = L$ with L/Y cosingular for some $Y \leq L$. Then $f(N) + f(Y) = L$. Since the class of cosingular R-modules is closed under homomorphic images, $L/f(Y)$ is cosingular. Then $f(Y) = L$ as $f(N) \ll_\mu L$. This implies that $Kerf + Y = L$. Being L a $G-M$ module implies $Kerf \ll_\mu L$. Hence $Y = L$. Therefore $N \ll_\mu L$.

$\Leftarrow$) By Lemma 1.2. □

Definition 2.13. ([14]) A module L is said to be fully retractable if for any non-zero submodule N of L and every non-zero $g \in Hom_R(N, L)$ we have $Hom_R(L, N)g \neq 0$.

Example 2.14. *According to [14], $\mathbb{Z}_4$ is a fully retractable $\mathbb{Z}$-module.*

Proposition 2.15. *Let L be a fully retractable R-module such that for every non-zero submodule N of L, the kernel of any non-zero endomorphism of N is μ-small. Then L is $G-M$.*

Proof. Let $0 \neq N \leq L$ and $f : N \to L$ such that $f \neq 0$. Since L is fully retractable, there exists $g : L \to N$, $g \neq 0$. Consider

$$N \xrightarrow{\quad f \quad} L \xrightarrow{\quad g \quad} N$$

We have $gf \neq 0$ as L is fully retractable. By hypothesis, $Ker(gf) \ll_\mu N$. Since $Kerf \subseteq Ker(gf)$, according to Lemma 1.2, $Kerf \ll_\mu N$. It follows that L is $G-M$.

□

Proposition 2.16. *Let L be a semisimple quasi-injective R-module. Then the following statements are equivalent:*

(1) *L is $G-M$;*
(2) *L is μ-Hopfian.*

Proof. (1) $\Rightarrow$ (2) Is clear.

(2) $\Rightarrow$ (1) Let $0 \neq N \leq L$ and $f : N \to L$ such that $f \neq 0$. Since L is quasi-injective, there exists $g \in End_R(L)$ such that $gi = f$ where i is the inclusion map. Hence, $g(x) = f(x)$ for each $x \in N$ and so $Kerf \leq Kerg$. Since L is μ-Hopfian, $Kerg \ll_\mu L$. So $Kerf \ll_\mu L$. On the other hand, $Kerf \leq N$ and L is semisimple, then N is a direct summand of L. Hence, by Lemma 1.2, $Kerf \ll_\mu N$. This shows that L is $G-M$.

□

Corollary 2.17. *If R is a semisimple ring, then every R-module is $G - M$.*

Proof. By [5, Theorem 2.10] and Proposition 2.16. □

It is obvious that every small monoform module is $G - M$. The following example shows that the converse is false, in general.

Example 2.18. *Let R be a semisimple ring. According to Corollary 2.17, $R^{(\mathbb{N})}$ is $G - M$. But the kernel of every non-zero endomorphism of $R^{(\mathbb{N})}$ is not small by [5, Example 2.11]. Thus $R^{(\mathbb{N})}$ is not a small monoform module.*

A ring R is a right GV -ring if every simple R-module is either projective or injective. In [16], the authors proved that every simple module is small if and only if R is a right GV-ring. It follows directly from [16, Theorem 3.1], that R is a right GV-ring if and only if R has no simple (semisimple) injective R-module.

Corollary 2.19. *Let R be a right GV-ring and L an R-module. Then the following statements are equivalent:*

(1) *L is $G - M$;*
(2) *L is small monoform.*

Proof. (1) $\Rightarrow$ (2) Let L be a $G-M$ module, N be a non-zero submodule of L and $f \in Hom(N, L)$ be a non-zero partial endomorphism. Assume $Kerf + K = N$ for some $K \leq N$. As L is a $G-M$ module, $Kerf \ll_\mu N$. Then by Theorem 2.3, $N = K \oplus H$ for some injective semisimple submodule H of N. By hypothesis, $H = 0$. This implies that $N = K$ and so $Kerf \ll N$. Hence L is a small monoform module.

(2) $\Rightarrow$ (1) Is clear.

□

Definition 2.20. ([9]) A right Goldie ring is a ring R that has finite uniform dimension as a right module over itself, and satisfies the ascending chain condition on right annihilators of subsets of R

Proposition 2.21. *Let R be a prime right Goldie ring which is not a right primitive (e.g. a commutative domain which is not a field) and L a semisimple R-module. Then the following assertions are equivalent:*

(1) *L is $G - M$;*
(2) *L is monoform;*
(3) *L is small monoform;*
(4) *L is simple.*

Proof. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) By Remark 1.1.

(1) $\Rightarrow$ (2) Let N be a non-zero submodule of L and $f \in Hom(N, L)$ a non-zero partial endomorphism. Since L is $G - M$, $Kerf \ll_\mu N$. As R is a prime right Goldie ring which is not a right primitive ring, N is cosingular

by [13, Corollary 9], and it is semisimple as L is semisimple, hence the only μ-small submodule of N is zero. So $Ker f = 0$, this completes the proof.

(2) $\Rightarrow$ (1) Is clear.

□

Lemma 2.22. *For an R-module L, consider the following statements.*

(1) *L is $G - M$.*

(2) *For every right R-module Y, if there is an epimorphism $L \to L \oplus Y$, then Y is semisimple injective.*

Then (1) $\Rightarrow$ (2).

Proof. (1) $\Rightarrow$ (2) Let $g : L \to L \oplus Y$ be a surjective homomorphism, and let $\pi : L \oplus Y \to L$ the natural projection. It is obvious that $Ker(\pi g) = g^{-1}(0 \oplus Y)$. Since L is $G - M$, $Ker(\pi g) \ll_\mu L$. According to Lemma 1.2, $0 \oplus Y = g[g^{-1}(0 \oplus Y)] = g(Ker(\pi g)) \ll_\mu L \oplus Y$. Thus $Y \ll_\mu Y$ by Lemma 1.2. Therefore, Y is semisimple injective by Lemma 1.3.

□

In the following, we characterize the class of rings R for which every (free) R- module is $G - M$.

Theorem 2.23. *Let R be a ring. The following assertions are equivalent:*

(1) *R is semisimple;*
(2) *Any R-module is $G - M$;*
(3) *Any projective R-module is $G - M$;*
(4) *Any free R-module is $G - M$.*

Proof. (1) $\Rightarrow$ (2) By Corollary 2.17

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) Let $L = R^{(\mathbb{N})}$, by (4) L is a $G - M$ R-module. Since $L \cong L \oplus L$, L is semisimple injective by Lemma 2.22. Therefore R is a semisimple ring.

□

Proposition 2.24. *Every non-zero submodule of a $G - M$ module is $G - M$.*

Proof. Let N be a non-zero submodule of a $G - M$ module L. For any $0 \neq K \leq N$, let $f : K \to N$ be a non-zero partial endomorphism of N, then $if \neq 0$ where $i : N \to L$ is the inclusion mapping. Since L is $G - M$, $Ker(if) \ll_\mu K$, hence $Ker f \ll_\mu K$, and so N is $G - M$. □

Remark 2.25. Let $\pi : \mathbb{Z} \to \mathbb{Z}/12\mathbb{Z}$, where π is the natural projection. However $\mathbb{Z}/12\mathbb{Z}$ is not a $G - M$ $\mathbb{Z}$-module as $\overline{0} \neq f = 4\overline{x} \in \mathrm{End}(\mathbb{Z}/12\mathbb{Z})$ and $Ker f =< \overline{3} >$ is not μ-small in $\mathbb{Z}/12\mathbb{Z}$.

(1) Let $L = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. Each of $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$ is a $G - M$ module (because every of each is small monoform). Since $L \cong \mathbb{Z}/12\mathbb{Z}$. Then the direct sum of $G - M$ modules is not necessarily $G - M$.

(2) Since $\mathbb{Z}$ is a $G-M$ $\mathbb{Z}$-module and $\mathbb{Z}/12\mathbb{Z}$ is not $G-M$ $\mathbb{Z}$-module. Then the homomorphic image of $G-M$ module is not necessarily $G-M$.

Proposition 2.26. *Let L be a Noetherian R-module. Then L is $G-M$ iff any non-zero 3-generated submodule of L is $G-M$.*

Proof. $\Rightarrow$) Clear from Proposition 2.24.

$\Leftarrow$) Suppose that any non-zero 3-generated submodule of L is $G-M$. Let N be a non-zero submodule of L and $f: N \to L$ such that $f \neq 0$. If $Kerf = 0$ then $Kerf \ll_{\mu} N$. If $Kerf \neq 0$, let $x \in Kerf$. Let $y \in N$ and $z = f(y)$. Consider $P = Rx + Ry + Rz$ is a 3-generated submodule of L. Let $H = Rx + Ry$ and $h = f\mid_H: H \to P$. By hypothesis P is $G-M$, hence $Kerh \ll_{\mu} H \leq N$. But $x \in Kerh$, so $< x > \subseteq Kerh \ll_{\mu} N$. Since L is Noetherian, $Kerf$ is finitely generated, hence $Kerf = \sum_{i=1}^{n} Rx_i$, for some $x_i \in L$, $1 \leq i \leq n$. We have $< x_i > \ll_{\mu} N$ for every $1 \leq i \leq n$. Thus according to Lemma 1.2, $Kerf = \sum_{i=1}^{n} Rx_i \ll_{\mu} N$. Therefore L is $G-M$. □

Corollary 2.27. *Let R be an Artinian principal ideal ring and L a weakly co-Hopfian R-module. Then the following are equivalent:*

(1) *L is $G-M$.*
(2) *Any non-zero 3-generated submodule of L is $G-M$.*

Proof. By [1, Theorem 3.8], L must be finitely generated. Then L is a Noetherian since R is an Artinian principal ideal ring. Thus by Propostion 2.26 the result is obtained. □

3. Properties of Polynomial Extensions

In [6], we have recalled the definitions of the modules $L[x]$ and $L[x]/(x^{n+1})$.

Every element of $L[x]$ is a formal sum as $e_0 + e_1 x + ... + e_k x^k$ with $k \geq 0$ and $e_i \in L$. This sum that is denoted by $\sum_{i=1}^{k} e_i x^i$ ($e_0 x^0$, is the element $e_0 \in L$). The addition is defined by adding the corresponding coefficients. The structure of $R[x]$-module is given by

$$(\textstyle\sum_{i=0}^{k} \gamma_i x^i).(\sum_{j=0}^{z} e_j x^j) = \sum_{t=0}^{k+z} a_t x^t,$$

where $a_t = \sum_{i+j=t} \gamma_i e_j$, for any $\gamma_i \in R$, $e_j \in L$.

Any $P \in L[x]$ can be written under the form $(\sum_{i=k}^{l} e_i x^i)$ with $l \geq k \geq 0$, $e_i \in L$, $e_k \neq 0$ and $e_l \neq 0$. In this case we say that k is the order of P, l is the degree of P, e_k is the initial coefficient of P, and e_l as the leading coefficient of P.

Let n be any positive integer and

$$I_{n+1} = \{0\} \cup \{P; 0 \neq P \in R[x], \text{order of } P \geq n+1\}.$$

Hence $I_{n+1} \leq R[x]$. The quotient ring $R[x]/I_{n+1}$ is truncated at degree $n+1$. For that $R[x]/I_{n+1}$ is said to be the truncated polynomial ring. Since R has an identity element, $I_{n+1} = (x^{n+1})$. Even when R does not have an identity element, the ring $R[x]/I_{n+1}$ denoted by $R[x]/(x^{n+1})$. Every element of $R[x]/(x^{n+1})$ can be written under the form $(\sum_{i=0}^{n}\gamma_i x^i)$ with $\gamma_i \in R$.

Let

$$D_{n+1} = \{0\} \cup \{P; 0 \neq P \in L[x], \text{order of } P \geq n+1\}.$$

Hence $D_{n+1} \leq L[x]$. As $I_{n+1}L[x] \subset D_{n+1}$, we see that $R[x]/(x^{n+1})$ acts on $L[x]/D_{n+1}$. The module $L[x]/D_{n+1}$ denoted by $L[x]/(x^{n+1})$. The action of $R[x]/(x^{n+1})$ on $L[x]/(x^{n+1})$ is given by

$$(\textstyle\sum_{i=0}^{n}\gamma_i x^i).(\sum_{j=0}^{n} e_j x^j) = \sum_{\mu=0}^{n} a_\mu x^\mu,$$

where $a_\mu = \sum_{i+j=\mu}\gamma_i e_j$, for any $\gamma_i \in R$, $e_j \in L$.

Any non-zero element $P \in L[x]/D_{n+1}$ can be written uniquely under the form $(\sum_{i=k}^{n} e_i x^i)$ with $n \geq k \geq 0$, $e_i \in L$, $e_k \neq 0$. In this case we say that k is the order of P, e_k is the initial coefficient of P.

Similarly we define the $R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$-module $L[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$.

Lemma 3.1. *Let $g : N \to L[x]/(x^{n+1})$ be a non-zero partial endomorphism, where $0 \neq N \leq L[x]/(x^{n+1})$ and n is a positive integer. If $g(h(x)) \neq 0$ for $h(x) = \sum_{j=0}^{n} m_j x^j \in N$, then $o(h(x)) \leq o(g(h(x)))$, where $o(h(x))$ represent the order of $h(x)$. For instance when g is injective, we have that $o(h(x)) = o(g(h(x)))$.*

Proof. We have that $g(m) = \sum_{j=0}^{n} m_j x^j$ for any $m \in H$, where H is the non-zero submodule of N which is generated by the constant polynomials of N. Therefore $g(mx^k) = x^k(\sum_{j=0}^{n} m_j x^j) = \sum_{j=0}^{n-k} m_j x^{j+k}$, where $0 \leq k \leq n$. Clearly, $o(\sum_{j=0}^{n} m_j x^j) \leq o(g(\sum_{j=0}^{n} m_j x^j))$, that is $o(h(x)) \leq o(g(h(x)))$. In case g is injective, suppose $g(mx^k) = \sum_{j=k+1}^{n} m_j x^j$, then we get that $g(mx^n) = g(x^{n-k}(mx^k)) = x^{n-k}\sum_{j=k+1}^{n} m_j x^j = 0$, thus $m = 0$. So $o(h(x)) = o(g(h(x)))$.

□

Theorem 3.2. *$L[x]/(x^{n+1})$ is a monoform $R[x]/(x^{n+1})$-module iff L is monoform R-module.*

Proof. $\Longrightarrow$) Let $f : N \to L$ be any non-zero partial endomorphism of L where $0 \neq N \leq L$, then $g : N[x]/(x^{n+1}) \to L[x]/(x^{n+1})$ defined by $g(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i)x^i$ is a non-zero partial endomorphism of $L[x]/(x^{n+1})$ with $0 \neq N[x]/(x^{n+1}) \leq L[x]/(x^{n+1})$ and $Kerg = (Kerf)[x]/(x^{n+1})$. Since $L[x]/(x^{n+1})$ is monoform, $Kerg = 0$ then $Kerf = 0$. Hence L is monoform.

$\Longleftarrow$) Let $g : N \to L[x]/(x^{n+1})$ be a non-zero partial endomorphism of $L[x]/(x^{n+1})$, where N is a non-zero submodule of $L[x]/(x^{n+1})$ and $\tau : H \to N$ is the inclusion map, where H is the non-zero submodule of N generated by the constant polynomials of N. Define $p_i : L[x]/(x^{n+1}) \to L$ by $p_i(\sum_{j=0}^{n} m_j x^j) = m_i$, $i = 0, 1, ..., n$. We can prove that $p_i g\tau \neq 0$, else, there exists $0 \neq m \in H$ such that $p_i g\tau(m) = 0$, hence $p_i g\tau(m) = p_i g(m) = p_i(\sum_{j=0}^{n} m_j x^j) = m_i = 0$, then $o(m) = 0 < o(g(m))$, contradiction with Lemma 3.1. Since H is a non-zero submodule of L and L is monoform, then $Ker(p_i g\tau) = 0$.

Let $x \in Kerg$, so $g(x) = 0$ implies $p_i g\tau(x) = p_i g(x) = 0$. Hence $x \in Ker(p_i g\tau)$. It follows that $Kerg \subseteq Ker(p_i g\tau) = 0$. Thus $Kerg = 0$ and so $L[x]/(x^{n+1})$ is monoform.

□

Theorem 3.3. *$L[x]/(x^{n+1})$ is a small monoform $R[x]/(x^{n+1})$-module iff L is a small monoform R-module.*

Proof. $\Longrightarrow$) Let $f : N \to L$ be any non-zero partial endomorphism of L where $0 \neq N \leq L$, then $g : N[x]/(x^{n+1}) \to L[x]/(x^{n+1})$ defined by $g(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i)x^i$ is a non-zero partial endomorphism of $L[x]/(x^{n+1})$ with $0 \neq N[x]/(x^{n+1}) \leq L[x]/(x^{n+1})$ and $Kerg = (Kerf)[x]/(x^{n+1})$. Since $L[x]/(x^{n+1})$ is small monoform $Kerg \ll N[x]/(x^{n+1})$, then $Kerf \ll N$. Hence L is a small monoform module.

$\Longleftarrow$) Let $g : N \to L[x]/(x^{n+1})$ be a non-zero partial endomorphism of $L[x]/(x^{n+1})$, where N is a non-zero submodule of $L[x]/(x^{n+1})$ and $\tau : H \to N$ is the inclusion map, where H is the non-zero submodule of N generated by the constant polynomials of N. Define $p_i : L[x]/(x^{n+1}) \to L$ by $p_i(\sum_{j=0}^{n} m_j x^j) = m_i$, $i = 0, 1, ..., n$. We can prove that $p_i g\tau \neq 0$, else, there exists $0 \neq m \in H$ such that $p_i g\tau(m) = 0$, hence $p_i g\tau(m) = p_i g(m) = p_i(\sum_{j=0}^{n} m_j x^j) = m_i = 0$, then $o(m) = 0 < o(g(m))$, contradiction with Lemma 3.1. Since H is a non-zero submodule of L and L is small monoform, then $Kerp_i g\tau \ll H$.

Let $x \in Kerg$, so $g(x) = 0$ implies $p_i g\tau(x) = p_i g(x) = 0$. Hence $x \in Ker(p_i g\tau)$. It follows that $Kerg \subseteq Ker(p_i g\tau) \ll H \leq N$. Thus $Kerg \ll N$ and so $L[x]/(x^{n+1})$ is a small monoform module.

□

Lemma 3.4. *([8, Lemma 2.1]). Let $K \ll L$. Then $K[x]/(x^{n+1}) \ll L[x]/(x^{n+1})$ as $R[x]/(x^{n+1})$-modules, where $n \geq 0$.*

Theorem 3.5. *$L[x]/(x^{n+1})$ is a $G-M$ $R[x]/(x^{n+1})$-module iff L is a $G-M$ R-module.*

Proof. $\Longrightarrow$) Let $f : N \to L$ be any non-zero partial endomorphism of L where N be a non-zero submodule of L, then $g : N[x]/(x^{n+1}) \to L[x]/(x^{n+1})$ defined by $g(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} f(a_i)x^i$ is a non-zero partial endomorphism of $L[x]/(x^{n+1})$ with $0 \neq N[x]/(x^{n+1}) \leq L[x]/(x^{n+1})$ and $Kerg = (Kerf)[x]/(x^{n+1})$. Suppose that $Kerf + H = N$ for some $H \leq N$ with $Z^*(N/H) = N/H$, thus

$$(Ker(f))[x]/(x^{n+1}) + H[x]/(x^{n+1}) = N[x]/(x^{n+1}).$$

We show that $Z^*(\frac{N[x]/(x^{n+1})}{H[x]/(x^{n+1})}) = \frac{N[x]/(x^{n+1})}{H[x]/(x^{n+1})}$. Let $h = \sum_{i=0}^{n} m_i x^i \in \frac{N[x]/(x^{n+1})}{H[x]/(x^{n+1})}$. As N/H is cosingular, $m_i R$ is small for every $0 \leq i \leq n$. Then according to Lemma 3.4, $(m_i R)[x]/(x^{n+1})$ is small, hence $(hR)[x]/(x^{n+1})$ is small. Then

$Z^*(\frac{N[x]/(x^{n+1})}{H[x]/(x^{n+1})}) = \frac{N[x]/(x^{n+1})}{H[x]/(x^{n+1})}$. Since $Kerg \ll_\mu N[x]/(x^{n+1})$, $N[x]/(x^{n+1}) = H[x]/(x^{n+1})$ and so $N = H$. Thus $Kerf \ll_\mu N$ and L is $G-M$.

$\Longleftarrow$) Let $g : N \to L[x]/(x^{n+1})$ be a non-zero partial endomorphism of $L[x]/(x^{n+1})$, where N is a non-zero submodule of $L[x]/(x^{n+1})$ and $\tau : H \to N$ is the inclusion map, where H is the non-zero submodule of N generated by the constant polynomials of N. Define $p_i : L[x]/(x^{n+1}) \to L$ by $p_i(\sum_{j=0}^n m_j x^j) = m_i$, $i = 0, 1, ..., n$. We can prove that $p_i g\tau \neq 0$, else, there exists $0 \neq m \in H$ such that $p_i g\tau(m) = 0$, hence $p_i g\tau(m) = p_i g(m) = p_i(\sum_{j=0}^n m_j x^j) = m_i = 0$, then $o(m) = 0 < o(g(m))$, contradiction with Lemma 3.1. Since H is a non-zero submodule of L and L is $G-M$, then $Kerp_i g\tau \ll_\mu H$.

Let $x \in Kerg$, so $g(x) = 0$ implies $p_i g\tau(x) = p_i g(x) = 0$. Hence $x \in Ker(p_i g\tau)$. It follows that $Kerg \subseteq Ker(p_i g\tau) \ll_\mu H \leq N$. Thus by Lemma 1.2, $Kerg \ll_\mu N$ and so $L[x]/(x^{n+1})$ is $G-M$.

□

Corollary 3.6. *$L[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$ is a $G-M$ $R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$-module iff L is a $G-M$ R-module.*

Proof. We use the induction, the ring isomorphism

$(R[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq R[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$,

and

$(R[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1})$-module isomorphism

$(L[x_1, ..., x_{k-1}]/(x_1^{n_1+1}, ..., x_{k-1}^{n_{k-1}+1}))[x_k]/(x_k^{n_k+1}) \simeq L[x_1, ..., x_k]/(x_1^{n_1+1}, ..., x_k^{n_k+1})$

□

Open Problems

(1) What is the structure of rings whose finitely generated modules are $G-M$ modules?
(2) Let R be a ring with identity, and M a $G-M$ module. Is $M[X, X^{-1}]$ $G-M$ module in $R[X, X^{-1}]$-module?
(3) Let R be a $G-M$ ring and $n \geq 1$ an integer. Is the matrix ring $M_n(R)$ $G-M$?

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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