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CHARACTERIZATION OF n-JORDAN MULTIPLIERS ON RINGS

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Abstract. Our main result states that every Jordan multiplier T from a commutative ring $\mathcal{R}$ into a faithful $\mathcal{R}$-bimodule M with characteristic different from 2, is a multiplier. We also generalize this result for all $n \geq 2$ with a suitable condition. Furthermore, we investigate some illuminating properties of such maps.

Keywords: n-Jordan multiplier, n-multiplier, faithful, commutative ring.
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1. Introduction

Let $\mathcal{R}$ be a ring, M be an $\mathcal{R}$-bimodule and let $n \geqslant 2$ be an integer. The additive map $T : \mathcal{R} \longrightarrow M$ is called a *right n-multiplier* (*left n-multiplier*) if for all $a_1, a_2, ..., a_n \in \mathcal{R}$,

$$T(a_1a_2...a_n) = a_1T(a_2...a_n), \quad \big(T(a_1a_2...a_n) = T(a_1a_2...a_{n-1})a_n\big),$$

and T is called an *n-multiplier* if it is both a left and right n-multiplier.

Moreover, T is called a *right n-Jordan multiplier* (*left n-Jordan multiplier*) if

$$T(a^n) = aT(a^{n-1}), \quad \big(T(a^n) = T(a^{n-1})a\big),$$

for all $a \in \mathcal{R}$. If T is a left and right n-Jordan multiplier, then it is natural to call T an *n-Jordan multiplier*.

The concepts of n-multiplier and n-Jordan multiplier were introduced and studied for Banach algebras in [6] and [1], respectively. In particular, if $n = 2$ then T is called simply a multiplier or Jordan multiplier.

It should be pointed out that the term multiplier was introduced by Helgason in [3]. Even nowadays some authors use the term centralizer instead of multiplier. This terminology was introduced by Wendel in [10], who characterized multipliers on group algebras $L^1(G)$. The general theory of (centralizers) multipliers on Banach algebras has been developed by Johnson [5].

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Clearly, every left (right) n-multiplier is a left (right) n-Jordan multiplier, but in general, the converse is not true, (see Example 2.1 below). For characterization of the n-Jordan multiplier on rings and Banach algebras, we refer the reader to [2, 11–14] and the references therein.

Let $n \geqslant 2$ be an integer. We say that an $\mathcal{R}$-bimodule M is of characteristic not n ($\text{char}(M) \neq n$) if $na = 0$ implies $a = 0$ for every $a \in M$, and M is of characteristic greater than n ($\text{char}(M) > n$), if $n!a = 0$ implies $a = 0$ for every $a \in M$.

A ring $\mathcal{R}$ is called a prime ring if for every $a, b \in \mathcal{R}$, $a\mathcal{R}b = \{0\}$ implies that $a = 0$ or $b = 0$, and it is semiprme if $a = 0$, whenever $a\mathcal{R}a = \{0\}$.

A classical result due to Zalar [11] asserts that every right (left) Jordan multiplier on a 2-torsion free semiprime ring is a right (left) multiplier.

Another approach of Zalar's result obtained by Vukman as follows.

Theorem 1.1 ([8, Theorem 1])**.** *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $T : \mathcal{R} \longrightarrow \mathcal{R}$ be such an additive mapping that $2T(a^2) = T(a)a + aT(a)$ holds for all $a \in \mathcal{R}$. In this case, T is a left and right multiplier.*

It is shown [9, Theorem 1] that if $\mathcal{R}$ is a 2-torsion free semiprime ring and $T : \mathcal{R} \longrightarrow \mathcal{R}$ is an additive mapping that $T(aba) = aT(b)a$ holds for all $a, b \in \mathcal{R}$, then T is a multiplier. See also [14] for it generalization. Molnar in [7] proves that if $\mathcal{R}$ is a 2-torsion free prime ring and $T : \mathcal{R} \longrightarrow \mathcal{R}$ is an additive mapping that $T(aba) = T(a)ba$ holds for all $a, b \in \mathcal{R}$, then T is a left multiplier.

The purpose of this paper is to show that every n-Jordan multiplier T from commutative ring $\mathcal{R}$ into $\mathcal{R}$-bimodule M with the property that $\text{char}(M) > n$ and $aT(b) = bT(a)$ for all $a, b \in \mathcal{R}$, is an n-multiplier.

In this note, we will focus just on the right n-Jordan multipliers, the left versions can be proved analogously.

Throughout the paper, $\mathcal{R}$ is a ring, M is an $\mathcal{R}$-bimodule and for $a, b \in \mathcal{R}$, $[a, b]$ stands the commutator $ab - ba$.

2. n-Jordan Multipliers on rings

The following example which obtained by the author in [12, Example 1.3], provided that we cannot assert that n-Jordan multipliers of rings are always n-multipliers.

Example 2.1. *Let*

$$\mathcal{R} = \left\{ \begin{bmatrix} 0 & a & x & c \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix} : \quad a, b, c, x, y \in \mathbb{C} \right\},$$

and define $T : \mathcal{R} \longrightarrow \mathcal{R}$ *via*

$$T\left(\begin{bmatrix} 0 & a & x & c \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a & y & c \\ 0 & 0 & b & x \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, for all $A \in \mathcal{R}$, $T(A^2) = A^2$ and $T(A^3) = A^3$. Therefore, T is a 3-Jordan multiplier, but T is not 3-multiplier.

For all $a, b, c \in \mathcal{R}$, we set

$$\Delta_1(a, b) = aT(b) - bT(a), \quad \textit{and} \quad \Delta_2(a, b, c) = aT(bc) - bT(ac).$$

An $\mathcal{R}$-bimodule M is called left (right) *faithful* if the condition $am = 0$ ($ma = 0$) for $m \in M$ implies that $m = 0$. If $\mathcal{R}$ is unital with unit e and M is unitary, i.e., $em = m = me$, then M is faithful.

Theorem 2.2. *Let* $n \in \{2, 3\}$ *be fixed and* $\mathcal{R}$ *be a commutative ring. Suppose that* M *is faithful and characteristic of* M *different from* 2 *and* 3*. Then every right* n*-Jordan multiplier* $T : \mathcal{R} \longrightarrow M$ *is a right* n*-multiplier.*

Proof. First assume that $n = 2$ and we intend to prove that $\Delta_1(a, b) = 0$ for all $a, b \in \mathcal{R}$. By our assumption

$$T(a^2) = aT(a), \qquad a \in \mathcal{R}.$$

Replacing a by $a + b$ we get

$$2T(ab) = aT(b) + bT(a), \qquad a, b \in \mathcal{R}. \tag{1}$$

Interchanging b by bc in (1), we obtain

$$2T(abc) = aT(bc) + bcT(a). \tag{2}$$

Plugging (1) into (2) to get

$$4T(abc) = abT(c) + acT(b) + 2bcT(a). \tag{3}$$

Similarly,

$$4T(abc) = 4T(bac) = baT(c) + bcT(a) + 2acT(b). \tag{4}$$

Comparing (3) and (4), we arrive at

$$c\Delta_1(a, b) = 0, \qquad a, b, c \in \mathcal{R}.$$

Since M is faithful, $\Delta_1(a, b) = 0$ for all $a, b \in \mathcal{R}$. As $\text{char}(M) > 2$ it follows from (1) that $T(ab) = aT(b)$ for all $a, b \in \mathcal{R}$ and hence T is a right multiplier.

Now let $n = 3$ and $T(a^3) = aT(a^2)$ for all $a \in \mathcal{R}$. Replacing a by $a + b$ we arrive at

$$3T(a^2b + ab^2) = 2(a + b)T(ab) + aT(b^2) + bT(a^2). \tag{5}$$

Switching b by $-b$ in (5) and plus the result by (5), we obtain

$$3T(ab^2) = 2bT(ab) + aT(b^2). \tag{6}$$

Replacing b by $b+c$ in (6) and simplifying the result to get

$$3T(abc) = aT(bc) + bT(ac) + cT(ab), \tag{7}$$

for all $a, b, c \in \mathcal{R}$. Interchanging c by xy in (7), we have

$$3T(abxy) = aT(bxy) + bT(axy) + xyT(ab). \tag{8}$$

It follows from (7) and (8) that

$$\begin{aligned} 9T(abxy) = &\, abT(xy) + axT(by) + ayT(bx) \\ &+ baT(xy) + bxT(ay) + byT(ax) \\ &+ 3xyT(ab). \end{aligned}$$

Similarly,

$$\begin{aligned} 9T(bxya) = &\, bxT(ya) + byT(xa) + baT(xy) \\ &+ xyT(ba) + xaT(by) + xbT(ya) \\ &+ 3yaT(bx). \end{aligned}$$

Comparing the above two equalities, we arrive at

$$2xyT(ab) + abT(xy) = 2yaT(bx) + xbT(ya). \tag{9}$$

Replacing b by y in (9), we obtain

$$y\Delta_2(x, a, y) = 0,$$

for all $a, x, y \in \mathcal{R}$. Since M is faithful, we get

$$\Delta_2(x, a, y) = 0, \qquad a, x, y \in \mathcal{R}.$$

Consequently, by (7), $T(abc) = aT(bc)$ for all $a, b, c \in \mathcal{R}$ and so T is a right 3-multiplier. □

The next corollary follows immediately from Theorem 2.2.

Corollary 2.3. *Let $n \in \{2, 3\}$ be fixed, $\mathcal{R}$ be a commutative faithful ring and characteristic of $\mathcal{R}$ different from 2 and 3. Then each right n-Jordan multiplier $T : \mathcal{R} \longrightarrow \mathcal{R}$ is a right n-multiplier.*

Theorem 2.2 leads to the following conjecture:

Let $\mathcal{R}$ be a commutative ring, M be a faithful $\mathcal{R}$-bimodule with char$(M) > n$. Then every n-Jordan multiplier $T : \mathcal{R} \longrightarrow M$ is an n-multiplier, for all $n \in \mathbb{N}$.

Unfortunately, we were unable to prove this conjecture for $n > 3$, but we prove that if the faithfully of M replaced by $\Delta_1(a, b) = 0$ for all $a, b \in \mathcal{R}$, then the result follows.

The following lemma which was appeared in [4, Lemma 1], is useful for characterization of n-Jordan multipliers.

Lemma 2.4. *Let $\phi(a_1, a_2, ..., a_n)$ be multi-additive from $\mathcal{R}^n$ into M and let $\phi(a, a, ..., a) = 0$ for all $a \in \mathcal{R}$. Then*

(i)
$$\sum_{\sigma \in S_n} \phi(a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}) = 0,$$
where S_n is the set of permutations on $\{1, 2, ..., n\}$.

(ii) *If char$(M) > n-1$, then*
$$\phi(a, b, ..., b) + \phi(b, a, ..., b) + ... + \phi(b, b, ..., a) = 0,$$
for all $a, b \in \mathcal{R}$.

Our first main theorem is the following.

Theorem 2.5. *Let $n \geqslant 2$ be an integer. Suppose that $\mathcal{R}$ is a commutative ring, char$(M) > n$ and $T : \mathcal{R} \longrightarrow M$ is a right n-Jordan multiplier. If $\Delta_1(a, b) = 0$ for all $a, b \in \mathcal{R}$, then T is a right n-multiplier.*

Proof. Define the map $\phi : \mathcal{R}^n \longrightarrow M$ by
$$\phi(a_1, a_2, ..., a_n) = T(a_1a_2...a_n) - a_1T(a_2...a_n),$$
for all $a_1, a_2, ..., a_n \in \mathcal{R}$. Then ϕ is multi-additive and $\phi(a, a, ..., a) = 0$ for all $a \in \mathcal{R}$. It follows from Lemma 2.4, that
$$\sum_{\sigma \in S_n} \phi(a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}) = 0.$$
Since $\mathcal{R}$ is commutative the above equation yields that
$$\begin{aligned} n!T(a_1a_2...a_n) &= (n-1)!a_1T(a_2a_3...a_n) \\ &+ (n-1)!a_2T(a_1a_3...a_n) + ... + (n-1)!a_nT(a_1a_2...a_{n-1}), \end{aligned} \tag{10}$$
for all $a_1, a_2, ..., a_n \in \mathcal{R}$. On the other hand, by our assumption
$$\begin{aligned} a_1T(a_2a_3...a_n) = (a_2a_3...a_n)T(a_1) &= (a_3...a_n)a_2T(a_1) \\ &= (a_3...a_n)a_1T(a_2) \\ &= (a_1a_3...a_n)T(a_2) = a_2T(a_1a_3...a_n). \end{aligned}$$
Hence
$$a_1T(a_2a_3...a_n) = a_2T(a_1a_3...a_n).$$
Similarly,
$$a_1T(a_2a_3...a_n) = a_3T(a_1a_2...a_n) = ... = a_nT(a_1a_2...a_{n-1}).$$
Now it follows from (10) that
$$n!T(a_1a_2...a_n) = n(n-1)!a_1T(a_2a_3...a_n).$$
Since char$(M) > n$, we obtain
$$T(a_1a_2...a_n) = a_1T(a_2a_3...a_n),$$
for every $a_1, a_2, ..., a_n \in \mathcal{R}$. Consequently, T is a right n-multiplier. □

It is shown in [12, Theorem 2.8] that every $(n+1)$-Jordan multiplier from unital Banach algebra A into unitary A-bimodule X is an n-Jordan multiplier. Here by using Lemma 2.4 we give an alternative proof for the mentioned result.

Theorem 2.6. *Let $\mathcal{R}$ be an unital ring with unit e, M be unitary $\mathcal{R}$-bimodule and char$(M) > (n+1)$. Then each right $(n+1)$-Jordan multiplier $T : \mathcal{R} \longrightarrow M$ is a right n-Jordan multiplier.*

Proof. Define the map $\phi : \mathcal{R}^{n+1} \longrightarrow M$ by

$$\phi(a_1, a_2, ..., a_n, a_{n+1}) = T(a_1 a_2 ... a_n a_{n+1}) - a_1 T(a_2 ... a_n a_{n+1}),$$

for all $a_1, a_2, ..., a_n, a_{n+1} \in \mathcal{R}$. It follows from Lemma 2.4, that

$$\sum_{\sigma \in S_{n+1}} \phi\big(a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}, a_{\sigma(n+1)}\big) = 0,$$

where S_{n+1} is the set of permutations on $\{1, 2, ..., n, n+1\}$. Let $x, y \in \mathcal{R}$. Then by taking $a_1 = x$ and $a_2 = a_3 = ... = a_n = a_{n+1} = y$ in the above summation, we get

$$n!T(xy^n + yxy^{n-1} + ... + y^n x) = n!\big(xT(y^n) + yT(xy^{n-1}) + ... + yT(y^{n-1}x)\big).$$

Interchanging x by e to get

$$n!(n+1)T(y^n) = n!(n+1)yT(y^{n-1}).$$

Since char$(M) > (n+1)$, the above equality forces $T(y^n) = yT(y^{n-1})$ for all $y \in \mathcal{R}$ and so T is a right n-Jordan multiplier. □

Lemma 2.7. *Let $T : \mathcal{R} \longrightarrow M$ be a right Jordan multiplier, char$(M) > 2$ and let $f(a,b) = T(ab) - aT(b)$. Then for all $a, b, c, x \in \mathcal{R}$,*

(i) $T(aba) = abT(a)$,
(ii) $T(abc + cba) = abT(c) + cbT(a)$,
(iii) $[a,b]f(a,b) = 0$,
(iv) $[a,b]xf(a,b) = 0$.

Proof. (i) Suppose that $T(a^2) = aT(a)$ for all $a \in \mathcal{R}$. Replacing a by $a+b$ we get

$$T(ab + ba) = aT(b) + bT(a), \qquad a, b \in \mathcal{R}. \tag{11}$$

Replacing b by $ab + ba$ in (11), we arrive at

$$T(a^2 b + 2aba + ba^2) = aT(ab + ba) + (ab + ba)T(a). \tag{12}$$

Using (11) and (12), we obtain

$$\begin{aligned} T(a^2 b + ba^2) + 2T(aba) &= aT(ab + ba) + (ab + ba)T(a) \\ &= a\big(aT(b) + bT(a)\big) + abT(a) + b(aT(a)) \\ &= a^2 T(b) + 2abT(a) + bT(a^2) \\ &= \big(a^2 T(b) + bT(a^2)\big) + 2abT(a). \end{aligned}$$

Since char$(M) > 2$, hence we have $T(aba) = abT(a)$ for every $a, b \in \mathcal{R}$.

(ii) The equality in (ii) follows from (i) by replacing a by $a + c$.

(iii) It follows from (ii) that

$$T(abab + ba^2b) = abaT(b) + baT(ab), \qquad a, b \in \mathcal{R}.$$

Thus, it follows from (i) and the equation above that

$$\begin{aligned}
[a, b]f(a, b) &= abT(ab) + ba^2T(b) - abaT(b) - baT(ab) \\
&= T((ab)^2) + T(ba^2b) - abaT(b) - baT(ab) \\
&= T((ab)^2) + T(ba^2b) - T(abab + ba^2b) \\
&= T\big(abab + ba^2b - abab - ba^2b\big) \\
&= T(0) = 0.
\end{aligned}$$

(iv) From (11), we get

$$f(a, b) + f(b, a) = 0, \qquad a, b \in \mathcal{R}.$$

By using (ii), we conclude that

$$\begin{aligned}
0 &= T\big((ab)x(ba) + (ba)x(ab)\big) - T\big(a(bxb)a + b(axa)b\big) \\
&= abxT(ba) + baxT(ab) - abxbT(a) - baxaT(b) \\
&= abxf(b, a) + baxf(a, b) \\
&= -abxf(a, b) + baxf(a, b) \\
&= [a, b]xf(a, b).
\end{aligned}$$

Consequently, $[a, b]xf(a, b) = 0$ for all $a, b, x \in \mathcal{R}$. □

Example 2.8. Let

$$\mathcal{R} = \left\{ \begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix} : \quad z, w \in \mathbb{C} \right\}.$$

We make $M = \mathbb{C}$ an $\mathcal{R}$-bimodule by defining

$$a\lambda = 0, \quad \lambda a = \lambda z, \quad \lambda \in \mathbb{C},\ a \in \mathcal{R}.$$

Define $T : \mathcal{R} \longrightarrow M$ via

$$T\Big(\begin{bmatrix} z & w \\ 0 & 0 \end{bmatrix} \Big) = w.$$

Then $T(a^2) = T(a)a$ for all $a \in \mathcal{R}$ and hence T is a left Jordan multiplier. Now it is easy to check that

$$T(aba) = T(a)ba, \qquad a, b \in \mathcal{R}.$$

Therefore the condition (i) of Lemma 2.7 is satisfied. Similarly, the conditions (ii), (iii) and (iv) are fulfilled.

Lemma 2.9. *Let G_1 and G_2 be groups, char$(\mathcal{R}) > 2$ and $\phi, \psi : G_1 \times G_2 \longrightarrow \mathcal{R}$ be bi-additive maps. Suppose that*

(i) $\phi(a, b)\psi(a, b) = 0$, *or*

(ii) $\psi(a,b)\phi(a,b)=0$,

for all $a\in G_1$, $b\in G_2$. *If there exists a non-zero divisor* $\psi(x,y)$ *for some* $x\in G_1$ *and* $y\in G_2$, *then* $\phi(a,b)=0$ *for all* $(a,b)\in G_1\times G_2$.

Proof. Suppose that (i) holds and $\psi(x,y)$ is a non-zero divisor for some $x\in G_1$ and $y\in G_2$. Therefore $\phi(x,y)=0$. Replacing a by $a+x$, we get

$$\phi(a,b)\psi(x,b)+\phi(x,b)\psi(a,b)=0. \tag{13}$$

Replacing b by $b+y$ in (13), and simplifying the result, we arrive at

$$\phi(a,b)\psi(x,y)+\phi(a,y)\psi(x,b)+\phi(x,b)\psi(a,y)+\phi(x,y)\psi(a,b)=0. \tag{14}$$

Taking $a=x$ in (14) and using $\phi(x,y)=0$ we get $2\phi(x,b)\psi(x,y)=0$. Noticing that $\text{char}(\mathcal{R})>2$ and $\psi(x,y)$ is non-zero divisor, so $\phi(x,b)=0$ for all $b\in G_2$ and hence from (13), $\phi(a,b)\psi(x,b)=0$. If we take $b=y$ in the last equality, then we obtain $\phi(a,y)=0$ for all $a\in G_1$. Now it follows from (14) that $\phi(a,b)=0$ for all $a\in G_1$ and $b\in G_2$. The result follows in a similar way, if (ii) holds. □

Corollary 2.10. *Let* $\mathcal{R}'$ *be a ring with char*$(\mathcal{R}')>2$. *Let* $T:\mathcal{R}\longrightarrow\mathcal{R}'$ *be a right Jordan multiplier. If there exist* $x,y\in\mathcal{R}$ *such that* $[x,y]$ *is a non-zero divisor, then* T *is a right multiplier.*

Proof. By Lemma 2.7 (iii), we have $[a,b]f(a,b)=0$ for every $a,b\in\mathcal{R}$. Suppose that $[x,y]$ is a non-zero divisor for some $x,y\in\mathcal{R}$. Then $f(x,y)=0$. Now define

$$\psi(a,b)=[a,b] \quad \textit{and} \quad \phi(a,b)=f(a,b),$$

for every $a,b\in\mathcal{R}$. Then it follows from Lemma 2.9 that $\phi(a,b)=0$ for all $a,b\in\mathcal{R}$ and hence we reach the desired result. □

Theorem 2.11. *Suppose that* $T:\mathcal{R}\longrightarrow M$ *is a right Jordan multiplier and char*$(M)>2$. *Then for all* $a,b,c\in\mathcal{R}$,

$$cf(a,b)=[b,a]T(c)+T\big([a,b]c\big), \tag{15}$$

where $f(a,b)=T(ab)-aT(b)$.

Proof. By Lemma 2.7 (ii), we have

$$\begin{aligned} cf(a,b) &= cT(ab)-caT(b)\\ &= cT(ab)+baT(c)-T(cab+bac)\\ &= cT(ab)+baT(c)-T\big(c(ab)+(ab)c\big)+T\big((ab-ba)c\big)\\ &= cT(ab)+baT(c)-\big(cT(ab)+abT(c)\big)+T\big([a,b]c\big)\\ &= baT(c)-abT(c)+T\big([a,b]c\big)\\ &= [b,a]T(c)+T\big([a,b]c\big), \end{aligned}$$

as required. □

As a consequence of Theorem 2.11, we get the next result which is Theorem 2.2 for the case $n = 2$.

Corollary 2.12. *Let $\mathcal{R}$ be a commutative ring and M be a faithful $\mathcal{R}$-bimodule with char$(M) > 2$. Then every right Jordan multiplier $T : \mathcal{R} \longrightarrow M$ is a right multiplier.*

Proposition 2.13. *Let $\mathcal{R}$ be an unital ring with unit e, M be an $\mathcal{R}$-bimodule with char$(M) > 2$ and $T : \mathcal{R} \longrightarrow M$ be a right Jordan multiplier. Then the following conditions are equivalent.*

(i) *$T(a) = aT(e)$, for all $a \in \mathcal{R}$.*
(ii) *T is a right Jordan multiplier.*

Moreover, if M is unitary and char$(M) > n-1$, then (i) and (ii) are equivalent with

(iii) *T is a right n-Jordan multiplier.*
(iv) *T is a right n-multiplier.*

Proof. $(i) \Longrightarrow (ii)$ is clear. Assume that (ii) holds. Then by Lemma 2.7, $T(aba) = abT(a)$ for all $a, b \in \mathcal{R}$. Taking $a = e$, we get $T(b) = bT(e)$, for all $b \in \mathcal{R}$.

$(ii) \Longrightarrow (iii)$ is [12, Lemma 2.1].

$(iii) \Longrightarrow (i)$. Suppose that M is unitary and T is a right n-Jordan multiplier. Define the map $\phi : \mathcal{R}^n \longrightarrow M$ by

$$\phi(a_1, a_2, ..., a_n) = T(a_1a_2...a_n) - a_1T(a_2...a_n),$$

for all $a_1, a_2, ..., a_n \in \mathcal{R}$. Then ϕ is multi-additive and $\phi(a, a, ..., a) = 0$ for all $a \in \mathcal{R}$. It follows from Lemma 2.4, that

$$\phi(a, b, ..., b) + \phi(b, a, ..., b) + ... + \phi(b, b, ..., a) = 0, \quad a, b \in \mathcal{R}.$$

If we take $b = e$ in the above equality, then we conclude that $T(a) = aT(e)$ for all $a \in \mathcal{R}$.

$(i) \Longrightarrow (iv)$ and $(iv) \Longrightarrow (iii)$ are obvious. □

3. Conclusion

The study of Jordan multipliers and their characterization on rings goes back to Zalar. He proved that each Jordan multiplier on semiprime ring $\mathcal{R}$ with characteristic different from 2, is exact a multiplier [11]. Vukman [8] improved this result by proving that every additive mapping $T : \mathcal{R} \longrightarrow \mathcal{R}$ on 2-torsion free semiprime ring $\mathcal{R}$ that satisfies in $2T(a^2) = T(a)a + aT(a)$ for all $a \in \mathcal{R}$ is a multiplier. In this note we gave a new characterization of n-Jordan multipliers from a commutative ring $\mathcal{R}$ into a faithful $\mathcal{R}$-bimodule M with characteristic different from 2 and 3. We also proved that any $(n+1)$-Jordan multiplier from unital ring $\mathcal{R}$ into unitary $\mathcal{R}$-bimodule M is an n-Jordan multiplier.

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4. Conflict of interest

The authors declare no conflict of interest.

References

[1] Fozouni, M. (2018). n-Jordan multipliers, Surveys Math. Appl., 13, 121–129.
[2] Gahramani, H. (2015). On centralizers of Banach algebras, Bull. Malays. Math. Sci. Soc., 38(1), 155–164. https://doi.org/10.1007/s40840-014-0011-2
[3] Helgason, S. (1956). Multipliers of Banach algebras, Ann. Math., 64(2), 240–254. https://doi.org/10.2307/1969971
[4] Herstein, I. N. (1956). Jordan homomorphisms, Trans. Amer. Math. Soc., 81(2), 331–341. https://doi.org/10.2307/1992920
[5] Johnson, B. E. (1964). An introduction to the theory of centralizers, Proc. London Math. Soc., 14(2), 299–320. https://doi.org/10.1112/plms/s3-14.2.299
[6] Laali, J., & Fozouni, M. (2017). n-multipliers and their relations with n-homomorphisms, Vietnam J. Math., 45, 451–457. https://doi.org/10.1007/s10013-016-0216-9
[7] Molnar, L. (1995). On centralizers of an H^*-algebra, Publ. Math. Debrecen., 46(2), 89–95. DOI: 10.5486/PMD.1995
[8] Vukman, J. (1999). An identity related to centralizers in semiprime rings, Comment. Math. Univ. Carol., 40(3), 447–456. http://dml.cz/dmlcz/119101
[9] Vukman, J. (2001). Centralizers on semiprime rings, Comment. Math. Univ. Carol., 42(2), 237–245. http://dml.cz/dmlcz/119239
[10] Wendel, J. G. (1952). Left centralizers and isomorphisms on group algebras, Pacific J. Math., 2(4), 251–261.
[11] Zalar, B. (1991). On centralizers of semiprime rings, Comment. Math. Univ. Carol., 32(4), 609–614. http://dml.cz/dmlcz/118440
[12] Zivari-Kazempour, A. (2022). Characterization of n-Jordan multipliers, Vietnam J. Math., 50(1), 87–94. https://doi.org/10.1007/s10013-021-00479-7
[13] Zivari-Kazempour, A., & Valaei, M. (2022). Characterization of n-Jordan multipliers through zero products, J. Analysis, 30(3), 1059-1067. https://doi.org/10.1007/s41478-022-00395-0
[14] Zivari-Kazempour, A. (2025). An identity related to θ-centralizers in semiprime rings, J. Algebraic Systems., 12(2), 367–377. https://doi.org/10.22044/jas.2023.11856.1607

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