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A STUDY OF NEW FIXED POINT RESULTS VIA HYBRID CONTRACTIONS

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Abstract. The available literature shows that the ideas of admissible mappings and that of Suzuki-type contractions on metric spaces have been well-investigated. However, a hybrid version of these results in connection with θ-contraction has not been adequately examined. On this basis therefore, the aim of this paper is to introduce a new concept under the name an admissible Jaggi-Suzuki-type hybrid (θ-ϕ)-contraction and to study new conditions for the existence of fixed point for this class of contractions on generalized or rectangular metric space. Applications and examples are provided to support the assumptions of our presented theorems. The results established herein extend some existing ideas in the corresponding literature. A few of these special cases are highlighted and discussed as corollaries.

Keywords: fixed point, metric space, θ-contraction, nonlinear integral equation.
2020 MSC: 47H10; 55M20; 54H25.

1. Introduction

One of the fundamental results in the evolution of fixed point (FP) theory is the Banach contraction principle [3]. Various generalizations of this principle have been made over the years. In 2014, Jleli and Samet [8] used the idea of Branciari distance or rectangular inequality introduced by Branciari [5] to define a new concept called θ-contraction in which the author showed that the Banach contraction is a particular kind of θ-contractions but they are θ-contraction that are not Banach contraction. Since then, several authors have been able to utilize the idea of θ-contraction in different directions, (e.g., see [15, 18]).

Metric FP theory has seen a huge increase in the number of publications over the previous few decades. Due to this circumstance, researchers now figured out ways to properly merge and harmonize the results that have already been obtained from the literature. In this regards, Aydi et al. [2] initiated the concept of interpolative Ćirić-Reich-Rus type contractions via the Branciari distance

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and proved some related FPs results for such mappings. Karapinar [11] introduced the notion of Kannan-type interpolative contraction that maximizes the rate of convergence, and unified some known results in the literature. Yelsilkaya [26] extended the idea in [11] to develop Hardy-Rogers contractive of the Suzuki-type mapping. Very recently, motivated by the outcome in [11]. Mitrovi´c et al. [16] introduced and investigated a hybrid contraction that combines a Reich-type contraction and interpolative-type contractions. Along the line, Karapinar and Fulga [12] provided a new hybrid contraction by combining Jaggi-type contraction and interpolative-type contraction. On similar development, Karapinar and Fulga [13] introduced a new hybrid contraction that unify several nonlinear and linear contractions in the set-up of a complete metric space. Recently, Shagari et al. [24] used the ideas of admissible mappings and θ-contractions to introduce some new notions under the name admissible hybrid (θ, ζ)-contractions and proved the existence of fixed points for such mappings. Yahaya et al. [25] employed the main ideas in [12] and established some hybrid fixed point theorems of Pata-type. For more results of such kind, (e.g., see [10]) and the reference therein.

Research from the related literature shows that there has been little or no work on Suzuki-hybrid-type-contraction in relation to θ-contraction. Hence, this research work aims to breach this gap by defining new concept of hybrid version of (θ-ϕ)-contraction and establish various FP results for such mappings in the setting of generalized metric space (GMS).

2. **Preliminaries**

In this section, some fundamental definitions, terminology and notations that will be deployed subsequently are recalled. Throughout this paper, every set Ω is considered non-empty, $\mathbb{R}_+$ is the set of non-negative real numbers and $\mathbb{N}$ is the set of natural numbers.

In 1969, Kannan [9] introduced a new type of contraction in the frame of complete metric spaces with a theorem given as:

Theorem 2.1. *Let (Ω, ϱ) be a complete metric spaces (CMS) and $\Upsilon : \Omega \longrightarrow \Omega$ be a Kannan contraction mapping, that is*

$$\varrho(\Upsilon\zeta, \Upsilon y) \leq \lambda[\varrho(\zeta, \Upsilon\zeta) + \varrho(y, \Upsilon y)],$$

for all $\zeta, y \in \Omega$, where $\lambda \in [0, \frac{1}{2})$. Then Υ has a unique FP in Ω.

Definition 2.2. [11] Let (Ω, ϱ) be a MS. A self mapping Υ on Ω is called an interpolative Kannan-type contraction if there exist a constant $\lambda \in [0, 1)$ and $\rho \in (0, 1)$ such that

$$\varrho(\Upsilon\zeta, \Upsilon y) \leq \lambda(\varrho(\zeta, \Upsilon\zeta))^{\rho}.(\varrho(y, \Upsilon y))^{1-\rho},$$

for all $\zeta, y \in \Omega$ with $\zeta \neq \Upsilon\zeta$.

In 1977, Jaggi [7] defined a new concept of a generalized Banach contraction principle called Jaggi contraction, which is one of the first known rational contractive inequalities.

Definition 2.3. [6] Let (Ω, ϱ) be a MS. A continuous self-mapping $\Upsilon : \Omega \longrightarrow \Omega$ is called Jaggi contraction if

$$\varrho(\Upsilon\zeta, \Upsilon y) \leq \rho_1 \frac{\varrho(\zeta, \Upsilon\zeta)\varrho(y, \Upsilon y)}{\varrho(\zeta, y)} + \rho_2 \varrho(\zeta, y),$$

for all $\zeta, y \in \Omega$, $\zeta \neq y$ and for some $\rho_1, \rho_2 \in [0, 1)$ with $\rho_1 + \rho_2 < 1$.

Recently, Jleli and Samet [8] introduced a new type of contraction called θ-contraction and established some new FP theorems for such contraction in the context of generalized metric spaces as introduced by Bianciari [5].

Theorem 2.4. *Let (Ω, ϱ) be a complete GMS. If the mapping $\Upsilon : \Omega \longrightarrow \Omega$ is a θ-contraction, that is*

$$\varrho(\Upsilon\zeta, \Upsilon y) \neq 0 \Rightarrow \theta(\varrho(\Upsilon\zeta, \Upsilon y)) \leq [\theta(\varrho(\zeta, y))]^r,$$

for all $\zeta, y \in \Omega$ and $\theta \in \Theta$, $r \in (0, 1)$, then Υ has a unique FP in Ω,

where Θ is the set of functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:

- (Θ_1) θ is non-decreasing;
- (Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{t\to\infty} (t_n) = 0^+$;
- (Θ_3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{r\to 0^+} \dfrac{\theta(t) - 1}{t^r} = \ell$.

It was shown in [8] that the Banach contraction is a particular case of θ-contraction while there are θ-contractions which are not Banach contractions. To be consistent with Jleli and Samet [8], we denote by Θ, the set of all functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the conditions (Θ_1-Θ_3).

Definition 2.5. [22] Let $\Upsilon : \Omega \longrightarrow \Omega$ and $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ be mappings. Then, Υ is called ρ-admissible if $\zeta, y \in \Omega$, $\rho(\zeta, y) \geq 1 \Rightarrow \rho(\Upsilon\zeta, \Upsilon y) \geq 1$.

Definition 2.6. [21] Let $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ and $\Upsilon : \Omega \longrightarrow \Omega$ be mappings. Then, Υ is called triangular ρ-orbital admissible if for all $\zeta, y, z \in \Omega$,

- $(\Upsilon 1)$ Υ is ρ-admissible;
- $(\Upsilon 2)$ $\rho(\zeta, z) \geq 1$ and $\rho(z, y) \geq 1 \Rightarrow \rho(\zeta, y) \geq 1$.

As a modification in the concept of ρ-admissible mappings, Popescu [20] introduced ρ-orbital admissible mappings as follows:

Definition 2.7. Let $\Upsilon : \Omega \longrightarrow \Omega$ be a mapping and let $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ be a function. Υ is said to be ρ-orbital admissible if for all $\zeta \in \Omega$, $\rho(\zeta, \Upsilon\zeta) \geq 1$ implies $\rho(\Upsilon\zeta, \Upsilon^2\zeta) \geq 1$.

Definition 2.8. Let $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ be a function. A mapping $\Upsilon : \Omega \longrightarrow \Omega$ is said to be triangular ρ-orbital admissible if for all $\zeta, y \in \Omega$,

(i) Υ is ρ-orbital admissible;
(ii) $\rho(\zeta, y) \geq 1$ and $\rho(y, \Upsilon y) \geq 1$ implies $\rho(\zeta, \Upsilon y) \geq 1$.

Definition 2.9. [1] A mapping $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called a (c)-comparison function if it satisfies the following conditions:

(a) ϕ is nondecreasing;
(b) the series $\sum_{n=1}^{\infty} \phi^n(z)$ is convergent for $z \geq 0$.

Lemma 2.10. *[4] Let Φ be the family of (c)-comparison functions and $\phi \in \Phi$. Then, the following conditions hold:*

(i) $\phi^n(z) \longrightarrow 0$ *as* $n \longrightarrow \infty$ *for all* $z \geq 0$*;*
(ii) $\phi(z) < z$ *for all* $z > 0$*;*
(iii) ϕ *is continuous;*
(iv) $\phi(z) = 0$ *if and only if* $z = 0$*;*
(v) *the series* $\sum_{n=1}^{\infty} \phi^n(z)$ *is convergent.*

Definition 2.11. [17] Let (Ω, ϱ) be a MS. A mapping $\Upsilon : \Omega \longrightarrow \Omega$ is called Jaggi-Suzuki-type hybrid contraction if there exist $\phi \in \Phi$ and a mapping $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ such that

$$\frac{1}{2}\varrho(\zeta, \Upsilon\zeta) \leq \varrho(\zeta, y) \Rightarrow \rho(\zeta, y)\varrho(\Upsilon\zeta, \Upsilon y) \leq \phi(M_{\lambda_i}(\zeta, y, s, \Upsilon)), \tag{1}$$

where $s \geq 0$, $\lambda_i \geq 0$ such that $\lambda_1 + \lambda_2 = 1$ and

$$M_{\rho_i}(\zeta, y, s, \Upsilon) = \begin{cases} \left[\rho_1 \left(\frac{\varrho(\zeta, \Upsilon\zeta).\varrho(y, \Upsilon y)}{\varrho(\zeta, y)}\right)^s + \rho_2(\varrho(\zeta, y))^s\right]^{\frac{1}{s}}; \text{ for } s > 0, \\ (\varrho(\zeta, \Upsilon\zeta))^{\rho_1}.(\varrho(y, \Upsilon y))^{\rho_2}; \text{ for } s = 0, \quad \zeta, y \in \Omega \backslash fix(\Upsilon). \end{cases}$$

Here, $fix(\Upsilon) = \{\zeta \in \Omega : \zeta = \Upsilon\zeta\}$.

Definition 2.12. [13] Let (Ω, ϱ) be a MS. A mapping $\Upsilon : \Omega \longrightarrow \Omega$ is said to be an admissible hybrid contraction if there exist $\phi \in \Phi$, and a mapping $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ such that

$$\rho(\zeta, y)\varrho(\Upsilon\zeta, \Upsilon y) \leq \phi(M_{\lambda_i}(\zeta, y, s, \Upsilon)), \tag{2}$$

where $s \geq 0$ and $\lambda_i \geq 0$; $i = 1, 2, \ldots, 5$, such that $\sum_{i=1}^{5} \lambda_i = 1$ and

$$M_{\lambda_i}(\zeta, y, s, \Upsilon) = \begin{cases} \Big[\lambda_1(\varrho(\zeta, y))^s + \lambda_2(\varrho(\zeta, \Upsilon\zeta))^s + \lambda_3(\varrho(y, \Upsilon y))^s \\ +\lambda_4\left(\frac{\varrho(y,\Upsilon y)(1+\varrho(\zeta,\Upsilon\zeta))}{1+\varrho(\zeta,y)}\right)^s + \lambda_5\left(\frac{\varrho(y,\Upsilon\zeta)(1+\varrho(\zeta,\Upsilon y))}{1+\varrho(\zeta,y)}\right)^s\Big]^{\frac{1}{s}} \\ \qquad \text{for some } s > 0,\ \zeta, y \in \Omega; \\ (\varrho(\zeta, y))^{\lambda_1}.(\varrho(\zeta, \Upsilon\zeta))^{\lambda_2}.(\varrho(y, \Upsilon y))^{\lambda_3}. \\ \left(\frac{\varrho(y,\Upsilon y)(1+\varrho(\zeta,\Upsilon\zeta))}{1+\varrho(\zeta,y)}\right)^{\lambda_4}.\left(\frac{\varrho(\zeta,\Upsilon y)(1+\varrho(y,\Upsilon\zeta))}{2}\right)^{\lambda_5} \\ \qquad \text{for } s = 0,\ \zeta, y \in \Omega\backslash fix(\Upsilon). \end{cases}$$

3. Main results

We begin this section by defining a new notion, namely admissible Jaggi-Suzuki-type hybrid (θ-ϕ)-contraction in GMS.

Definition 3.1. Let (Ω, ϱ) be a GMS. A self-mapping Υ on Ω is said to be an admissible Jaggi-Suzuki-type hybrid (θ-ϕ)-contraction if there exists $\theta \in \Theta$, $\phi \in \Phi$, $r \in (0, 1)$ and a mapping $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}^+$ such that

$$\frac{1}{2}\varrho(\zeta, \Upsilon\zeta) \leq \varrho(\zeta, y) \Rightarrow \theta(\rho(\zeta, y)\varrho(\Upsilon\zeta, \Upsilon y)) \leq [\theta(\phi(M_{\lambda_i}(\zeta, y, s, \Upsilon)))]^r, \tag{3}$$

where $M_{\lambda_i}(\zeta, y, s, \Upsilon)$ is as defined in (2).

Theorem 3.2. *Let (Ω, ϱ) be a complete GMS and Υ be an admissible Jaggi-Suzuki-type hybrid (θ-ϕ)-contraction. Assume further that:*

(i) Υ *is triangular ρ-orbital admissible;*
(ii) *there exists $\zeta_0 \in \Omega$ such that $\rho(\zeta_0, \Upsilon\zeta_0) \geq 1$;*
(iii) *either Υ is continuous or;*
(iv) Υ^2 *is continuous and $\rho(\Upsilon\zeta, \zeta) \geq 1$ for any $\zeta \in fix(\Upsilon)$.*

Then, Υ has a FP in Ω.

Proof. By hypothesis (ii), $\rho(\zeta_0, \Upsilon\zeta_0) \geq 1$ for some $\zeta_0 \in \Omega$. Define a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ in Ω by $\zeta_n = \Upsilon^n\zeta_0$, for all $n \in \mathbb{N}$. Suppose that we can find some $n_0 \in \mathbb{N}$ such that $\zeta_{n_0} = \zeta_{n_0+1} = \Upsilon\zeta_{n_0}$. This implies that ζ_{n_0} is a FP of Υ and hence, the proof.

Assume on the contrary that $\zeta_n \neq \zeta_{n-1}$ for all $n \in \mathbb{N}$. Since $\rho(\zeta_0, \Upsilon\zeta_0) \geq 1$ and Υ is triangular ρ-orbital admissible, then

$$\rho(\zeta_{n-1}, \zeta_n) \geq 1 \text{ for all } n \in \mathbb{N}. \tag{4}$$

Now, we consider the following cases:
Case 1: for $s > 0$, let $\zeta = \zeta_{n-1}$ and $y = \Upsilon\zeta_{n-1} = \zeta_n$ in (3).Then,

$$\frac{1}{2}\varrho(\zeta_{n-1}, \Upsilon\zeta_{n-1}) \leq \varrho(\zeta_{n-1}, \zeta_n) \Rightarrow$$
$$\theta(\rho(\zeta_{n-1}, \zeta_n)\varrho(\Upsilon\zeta_{n-1}, \Upsilon\zeta_n)) \leq [\theta(\phi(M_{\lambda_i}(\zeta_{n-1}, \zeta_n, s, \Upsilon)))]^r. \tag{5}$$

Combining (4) and (5) yields

$$\begin{aligned}\theta(\varrho(\zeta_n, \zeta_{n+1})) \leq& \theta(\rho(\zeta_{n-1}, \zeta_n)\varrho(\Upsilon\zeta_{n-1}, \Upsilon\zeta_n)) \\ \leq& [\theta(\phi(M_{\lambda_i}(\zeta_{n-1}, \zeta_n, s, \Upsilon)))]^r,\end{aligned} \tag{6}$$

where

$$\begin{aligned}M_{\lambda_i}(\zeta_{n-1}, \zeta_n) =& \Bigg[\lambda_1\varrho(\zeta_{n-1}, \zeta_n)^s + \lambda_2\varrho(\zeta_{n-1}, \Upsilon\zeta_{n-1})^s + \lambda_3\varrho(\zeta_n, \Upsilon\zeta_n)^s \\ &+ \lambda_4\left(\frac{\varrho(\zeta_n, \Upsilon\zeta_n)(1 + \varrho(\zeta_{n-1}, \Upsilon\zeta_{n-1}))}{1 + \varrho(\zeta_{n-1}, \zeta_n)}\right)^s \lambda_5\left(\frac{\varrho(\zeta_n, \Upsilon\zeta_{n-1})(1 + \varrho(\zeta_{n-1}, \Upsilon\zeta_n))}{1 + \varrho(\zeta_{n-1}, \zeta_n)}\right)^s\Bigg]^{\frac{1}{s}} \\ =& \Bigg[\lambda_1\varrho(\zeta_{n-1}, \zeta_n)^s + \lambda_2\varrho(\zeta_{n-1}, \zeta_n)^s + \lambda_3\varrho(\zeta_n, \zeta_{n+1})^s \\ &+ \lambda_4\left(\frac{\varrho(\zeta_n, \zeta_{n+1})(1 + \varrho(\zeta_{n-1}, \zeta_n))}{1 + \varrho(\zeta_{n-1}, \zeta_n)}\right)^s + \lambda_5\left(\frac{\varrho(\zeta_n, \zeta_n)(1 + \varrho(\zeta_{n-1}, \zeta_{n+1}))}{1 + \varrho(\zeta_{n-1}, \zeta_n)}\right)^s\Bigg]^{\frac{1}{s}} \\ =& \Bigg[\lambda_1\varrho(\zeta_{n-1}, \zeta_n)^s + \lambda_2\varrho(\zeta_{n-1}, \zeta_n)^s + \lambda_3\varrho(\zeta_n, \zeta_{n+1}))^s + \lambda_4\varrho(\zeta_n, \zeta_{n+1})^s\Bigg]^{\frac{1}{s}} \\ =& \Bigg[(\lambda_1 + \lambda_2)\varrho(\zeta_{n-1}, \zeta_n)^s + (\lambda_3 + \lambda_4)\varrho(\zeta_n, \zeta_{n+1})^s\Bigg]^{\frac{1}{s}}.\end{aligned} \tag{7}$$

Suppose that

$$\varrho(\zeta_{n-1}, \zeta_n) \leq \varrho(\zeta_n, \zeta_{n+1}).$$

Then, from (6) and (7),

$$\begin{aligned}\theta(\varrho(\zeta_n, \zeta_{n+1})) \leq& [\theta(\phi(M_{\lambda_i}(\zeta_{n-1}, \zeta_n, s, \Upsilon)))]^r \\ =& [\theta(\phi(\lambda_1 + \lambda_2)\varrho(\zeta_{n-1,\zeta_n})^s + (\lambda_3 + \lambda_4)\varrho(\zeta_n, \zeta_{n+1})^s)^{\frac{1}{s}}]^r \\ \leq& [\theta(\phi(\lambda_1 + \lambda_2)\varrho(\zeta_n, \zeta_{n+1})^s + (\lambda_3 + \lambda_4)\varrho(\zeta_n, \zeta_{n+1})^s)^{\frac{1}{s}}]^r \\ =& [\theta(\phi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^{\frac{1}{s}}\varrho(\zeta_n, \zeta_{n+1}))]^r \\ \leq& [\theta(\phi(\varrho(\zeta_n, \zeta_{n+1})))]^r. \\ <& [\theta(\varrho(\zeta_n, \zeta_{n+1}))]^r.\end{aligned}$$

That is, $\theta(\varrho(\zeta_n,\zeta_{n+1})) < [\theta(\varrho(\zeta_n,\zeta_{n+1}))]^r$, which is a contradiction for all $r \in (0,1)$. Hence,

$$\begin{aligned}\theta(\varrho(\zeta_n,\zeta_{n+1})) &\leq [\theta(\phi(\varrho(\zeta_{n-1},\zeta_n)))]^r \\ &\leq [\theta(\phi(\phi(\varrho(\zeta_{n-2},\zeta_{n-1}))))]^{r^2} \\ &= [\theta(\phi^2(\varrho(\zeta_{n-2},\zeta_{n-1})))]^{r^2} \\ &\leq \\ &\vdots \\ &\leq [\theta(\phi^n(\varrho(\zeta_0,\zeta_1)))]^{r^n}, \quad \text{for all } n \in \mathbb{N}.\end{aligned} \tag{8}$$

Thus, we have

$$1 \leq \theta(\varrho(\zeta_n,\zeta_{n+1})) \leq [\theta(\phi^n(\varrho(\zeta_0,\zeta_1)))]^{r^n}. \tag{9}$$

Letting $n \longrightarrow \infty$ in (9) and using Sandwich theorem, yields $\theta(\varrho(\zeta_n,\zeta_{n+1})) \to 1$ as $n \longrightarrow \infty$, which implies from (Θ_2) that $\lim_{n\to\infty} \varrho(\zeta_n,\zeta_{n+1}) = 0$. From Condition (Θ_3), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(\varrho(\zeta_n,\zeta_{n+1})) - 1}{[\varrho(\zeta_n,\zeta_{n+1})]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \frac{\ell}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\theta(\varrho(\zeta_n,\zeta_{n+1})) - 1}{[\varrho(\zeta_n,\zeta_{n+1})]^r} - \ell\right| \leq B \text{ for all n} \geq \text{n}_0.$$

This implies that

$$\frac{\theta(\varrho(\zeta_n,\zeta_{n+1})) - 1}{(\varrho(\zeta_n,\zeta_{n+1}))^r} \geq \ell - B = B, \text{ for all } n \geq n_0.$$

Then,

$$n[\varrho(\zeta_n,\zeta_{n+1})]^r \leq An[\theta(\varrho(\zeta_n,\zeta_{n+1})) - 1], \text{ for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(\varrho(\zeta_n,\zeta_{n+1})) - 1}{(\varrho(\zeta_n,\zeta_{n+1}))^r} \geq B, \text{ for all } n \geq n_0.$$

This implies that

$$n[\varrho(\zeta_n,\zeta_{n+1})]^r \leq An[\theta(\varrho(\zeta_n,\zeta_{n+1})) - 1], \text{ for all } n \geq n_0,$$

where $A = \frac{1}{B}$.
Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[\varrho(\zeta_n,\zeta_{n+1})]^r \leq An[\theta(\varrho(\zeta_n,\zeta_{n+1})) - 1], \text{ for all } n \geq n_0.$$

Using (9), we obtain

$$n[\varrho(\zeta_n,\zeta_{n+1})]^r \le An([\theta(\varrho(\zeta_0,\zeta_1))]^{r^n}-1), \quad \text{for all } n \ge n_0.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n[\varrho(\zeta_n,\zeta_{n+1})]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$\varrho(\zeta_n,\zeta_{n+1}) \le \frac{1}{n^{\frac{1}{r}}}, \text{ for all } n \ge n_1. \tag{10}$$

Now, we shall prove that Υ has a periodic point. Suppose that it is not the case, then $\zeta_n \neq \zeta_m$ for every $n, m \in \mathbb{N}$ such that $n \neq m$. Using (8), we obtain

$$\begin{aligned}\theta(\varrho(\zeta_n,\zeta_{n+2})) &\le [\theta(\phi(\varrho(\zeta_{n-1},\zeta_{n+1})))]^r\\ &\le [\theta(\phi^2(\varrho(\zeta_{n-2},\zeta_n)))]^{r^2}\\ &\le\\ &\vdots\\ &\le [\theta(\phi^n(\varrho(\zeta_0,\zeta_2)))]^{r^n}.\end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality and using (Θ_2), we have

$$\lim_{n\to\infty} \varrho(\zeta_n,\zeta_{n+2}) = 0.$$

Similarly, from condition (Θ_3), there exists $n_2 \in \mathbb{N}$ such that

$$\varrho(\zeta_n,\zeta_{n+2}) \le \frac{1}{n^{\frac{1}{r}}}, \text{ for all } n \ge n_2. \tag{11}$$

Let $H = \max\{n_0, n_1\}$. We consider two instances:

(A) If $m > 2$ is odd, then writing $m = 2l+1$, $l \ge 1$ and using (10) , for all $n \ge H$, gives

$$\begin{aligned}\varrho(\zeta_n,\zeta_{n+m}) \le \varrho(\zeta_n,\zeta_{n+1}) &+ \varrho(\zeta_{n+1},\zeta_{n+2})\\ &+ \cdots + \varrho(\zeta_{n+2l},\zeta_{n+2l+1})\\ &\le \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+1)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2l)^{\frac{1}{r}}}\\ &= \sum_{i=n}^{n+2l} \frac{1}{i^{\frac{1}{r}}} \le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

(B) If $m > 2$ is even, then writing $m = 2l$, $l \geq 2$ and using (10) and (11), yields

$$\begin{aligned}\varrho(\zeta_n, \zeta_{n+m}) \leq \varrho(\zeta_n, \zeta_{n+2}) &+ \varrho(\zeta_{n+2}, \zeta_{n+3}) \\ &+ \cdots + \varrho(\zeta_{n+2l-1}, \zeta_{n+2l}) \\ &\leq \frac{1}{n^{\frac{1}{r}}} + \frac{1}{(n+2)^{\frac{1}{r}}} + \cdots + \frac{1}{(n+2l-1)^{\frac{1}{r}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}.\end{aligned}$$

Thus, combining all the instances, leads to

$$\varrho(\zeta_n, \zeta_{n+m}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}} \text{ for all } n \geq H, \; m \in \mathbb{N}.$$

The series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ converges since $\frac{1}{r} > 1$, which shows that $\{\zeta_n\}$ is a Cauchy sequence. Since (Ω, ϱ) is complete, there exists $u \in \Omega$ such that $\zeta_n \longrightarrow u$ as $n \longrightarrow \infty$. Using assumption (iii) that Υ is continuous, then

$$\varrho(u, \Upsilon u) = \lim_{n\to\infty} \varrho(\zeta_n, \Upsilon\zeta_n) = \lim_{n\to\infty} \varrho(\zeta_n, \zeta_{n+1}) = \varrho(u, u) = 0.$$

This implies that $u = \Upsilon u$.
Alternatively, by the assumption that (iv) holds, we have $\Upsilon^2 u = \lim_{n\to\infty} \Upsilon^2\zeta_n = u$. To see that $\Upsilon u = u$, Suppose on the contrary that $\Upsilon u \neq u$ Then, (3) yields

$$\frac{1}{2}\varrho(\Upsilon u, \Upsilon^2 u) = \frac{1}{2}\varrho(\Upsilon u, u) \leq \varrho(\Upsilon u, u)$$

.

$$\Rightarrow \theta(\rho(\Upsilon u, u)\varrho(\Upsilon^2 u, \Upsilon u)) \leq [\theta(\phi(M_{\lambda_i}(\Upsilon u, u)))]^r \tag{12}$$

Using same idea as in (6) yields

$$\theta(\varrho(\Upsilon u, u)) \leq [\theta(\phi(M_{\lambda_i}(\Upsilon u, u)))]^r$$

where

$$
\begin{aligned}
M_{\lambda_i}(\Upsilon u,u) &= \bigg[\lambda_1\varrho(\Upsilon u,u)^s+\lambda_2\varrho(\Upsilon u,\Upsilon^2u)^s+\lambda_3\varrho(u,\Upsilon u)^s\\
&+\lambda_4\bigg(\frac{\varrho(u,\Upsilon u)(1+\varrho(\Upsilon u,\Upsilon^2 u))}{1+\varrho(\Upsilon u,u)}\bigg)^s\\
&+\lambda_5\bigg(\frac{\varrho(u,\Upsilon^2 u)(1+\varrho(\Upsilon u,\Upsilon u))}{1+\varrho(\Upsilon u,u)}\bigg)^s\bigg]^{\frac{1}{s}}\\
&=\bigg[\lambda_1\varrho(\Upsilon u,u))^s+\lambda_2\varrho(\Upsilon u,u)^s+\lambda_3\varrho(u,\Upsilon u)^s\\
&+\lambda_4\bigg(\frac{\varrho(u,\Upsilon u)(1+\varrho(\Upsilon u,u))}{1+\varrho(\Upsilon u,u)}\bigg)^s\bigg)^s\bigg]^{\frac{1}{s}}\\
&=\bigg[(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\varrho(\Upsilon u,u)^s\bigg]^{\frac{1}{s}}\le\varrho(\Upsilon u,u).
\end{aligned}
$$

Hence, (12) becomes $\theta(\varrho(u,\Upsilon u))<[\theta(\varrho(u,\Upsilon u))]^r$, which is a contraction for all $r\in(0,1)$. Therefore, $\Upsilon u=u$.
Case 2: for $s=0$, let $\zeta=\zeta_{n-1}$ and $y=\Upsilon\zeta_{n-1}=\zeta_n$ in (3), we obtain

$$
\begin{aligned}
M_{\lambda_i}(\zeta_{n-1},\zeta_n)=&\varrho(\zeta_{n-1},\zeta_n)^{\lambda_1}.\varrho(\zeta_{n-1},\Upsilon\zeta_{n-1})^{\lambda_2}.\varrho(\zeta_n,\Upsilon\zeta_n)^{\lambda_3}.\\
&\bigg(\frac{\varrho(\zeta_n,\Upsilon\zeta_n)(1+\varrho(\zeta_{n-1},\Upsilon\zeta_{n-1}))}{1+\varrho(\zeta_{n-1},\zeta_n)}\bigg)^{\lambda_4}.\bigg(\frac{\varrho(\zeta_{n-1},\Upsilon\zeta_n)(1+\varrho(\zeta_n,\Upsilon\zeta_{n-1}))}{2}\bigg)^{\lambda_5}\\
\le&\varrho(\zeta_{n-1},\zeta_n)^{\lambda_1}.\varrho(\zeta_{n-1},\Upsilon\zeta_{n-1})^{\lambda_2}.\varrho(\zeta_n,\zeta_{n+})^{\lambda_3}.\\
&\bigg(\frac{\varrho(\zeta_n,\zeta_{n+})(1+\varrho(\zeta_{n-1},\Upsilon\zeta_{n-1}))}{1+\varrho(\zeta_{n-1},\zeta_n)}\bigg)^{\lambda_4}.\bigg(\frac{\varrho(\zeta_{n-1},\zeta_{n+1})(1+\varrho(\zeta_n,\zeta_n))}{2}\bigg)^{\lambda_5}\\
\le&\varrho(\zeta_{n-1},\zeta_n)^{(\lambda_1+\lambda_2)}.\varrho(\zeta_n,\zeta_{n+1})^{(\lambda_3+\lambda_4)}.\bigg(\frac{\varrho(\zeta_{n-1},\zeta_n)+\varrho(\zeta_n,\zeta_{n+1}))}{2}\bigg)^{\lambda_5}\\
\le&\varrho(\zeta_{n-1},\zeta_n)^{(\lambda_1+\lambda_2)}.\varrho(\zeta_n,\zeta_{n+1})^{(\lambda_3+\lambda_4)}.\frac{(\varrho(\zeta_{n-1},\zeta_n))^{\lambda_5}+(\varrho(\zeta_n,\zeta_{n+1}))^{\lambda_5}}{2}.
\end{aligned}
$$

Suppose that $\varrho(\zeta_{n-1},\zeta_n)\le\varrho(\zeta_n,\zeta_{n+1})$, then,

$$
\begin{aligned}
M_{\lambda_i}(\zeta_n,\zeta_{n+1})\le&\varrho(\zeta_{n-1},\zeta_n)^{(\lambda_1+\lambda_2)}.\varrho(\zeta_n,\zeta_{n+1})^{(\lambda_3+\lambda_4)}.(\varrho(\zeta_n,\zeta_{n+1}))^{\lambda_5}\\
=&\varrho(\zeta_n,\zeta_{n+1})^{(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5)}\\
=&\varrho(\zeta_n,\zeta_{n+1}).
\end{aligned}
$$

Hence, (6) can be written as

$$
\begin{aligned}
\theta(\varrho(\zeta_n,\zeta_{n+1}))\le&[\theta(\phi(\varrho(\zeta_{n-1},\zeta_n)))]^r\\
<&[\theta(\varrho(\zeta_n,\zeta_{n+1}))]^r,
\end{aligned}
$$

which is a contradiction for all $r \in (0,1)$. Therefore by (6), we have

$$\begin{aligned}\theta(\varrho(\zeta_n,\zeta_{n+1})) &\leq [\theta(\phi(\varrho(\zeta_{n-1},\zeta_n)))]^r\\ &\leq [\theta(\phi(\phi(\varrho(\zeta_{n-2},\zeta_{n-1}))))]^{r^2}\\ &= [\theta(\phi^2(\varrho(\zeta_{n-2},\zeta_{n-1})))]^{r^2}\\ &\leq\\ &\vdots\\ &\leq [\theta(\phi^n(\varrho(\zeta_0,\zeta_1))))]^{r^n}.\end{aligned}$$

Using the same argument as the case of $s=0$, it can easily be obtained that $\{\zeta_n\}_{n\in\mathbb{N}}$ is a cauchy sequence and since (Ω,ϱ) being complete, there exists a point say u such that $\lim_{n\to\infty}\zeta_n = u$. To see that u is a FP of Υ, from assumption (iii), we have

$$\varrho(u,\Upsilon u) = \lim_{n\to\infty}\varrho(\zeta_n,\Upsilon\zeta_n) = \lim_{n\to\infty}\varrho(\zeta_n,\zeta_{n+1}) = \varrho(u,u) = 0.$$

This implies that $u = \Upsilon u$. Again, from (iv) under the assumption that Υ^2 is continuous as in case (i), we have

$$\begin{aligned}\theta(\varrho(u,\Upsilon u)) = \theta(\varrho(\Upsilon^2 u,\Upsilon u)) &\leq \theta(\rho(\Upsilon u,u)\varrho(\Upsilon u,u))\\ &\leq [\theta(\phi(M_{\lambda_i}(\Upsilon u,u)))]^r\\ &< \theta(M_{\lambda_i}(\Upsilon u,u))]^r\\ &= [\theta(\varrho(u,\Upsilon u))^{\lambda_1+\lambda_2+\lambda_3+\lambda_+4+\lambda_5}]^r\\ &= [\theta(\varrho(u,\Upsilon u))]^r.\end{aligned}$$

That is, $\theta(\varrho(u,\Upsilon u)) < [\theta(\varrho(u,\Upsilon u))]^r$ which is a contradiction for all $r\in(0,1)$. It follows that $u=\Upsilon u$. □

Theorem 3.3. *If in Theorem 3.2, we assume an additional condition that $\rho(\zeta,y)\geq 1$ for all $\zeta,y\in fix(\Upsilon)$, then the FP of Υ is unique.*

Proof. Let ζ,u be two FP of Υ such that $\Upsilon\zeta=\zeta\neq u=\Upsilon u$, then $\varrho(\zeta,u) = \varrho(\Upsilon\zeta,\Upsilon u)\neq 0$ and using same idea in (6), gives

$$\begin{aligned}&\frac{1}{2}\varrho(\zeta,\Upsilon\zeta)\leq\varrho(\zeta,u)\\ &\quad\Rightarrow \theta(\varrho(\Upsilon\zeta,\Upsilon u)) \leq \theta(\rho(\zeta,u)\varrho(\Upsilon\zeta,\Upsilon y))\\ &\qquad\qquad \leq [\theta(\phi(M_{\lambda_i}(\zeta,u,s,\Upsilon)))]^r.\end{aligned} \tag{13}$$

Case 1: for $s > 0$

$$\begin{aligned} M_{\lambda_i}(\zeta,u) = &\Bigg[\lambda_1\varrho(\zeta,u))^s + \lambda_2\varrho(\zeta,\Upsilon\zeta)^s + \lambda_3\varrho(u,\Upsilon u)^s \\ &+ \lambda_4\left(\frac{\varrho(u,\Upsilon u)(1+\varrho(\zeta,\Upsilon\zeta))}{1+\varrho(\zeta,u)}\right)^s + \lambda_5\left(\frac{\varrho(u,\Upsilon\zeta)(1+\varrho(\zeta,\Upsilon u))}{1+\varrho(\zeta,u)}\right)^s\Bigg]^{\frac{1}{s}} \\ = &\Bigg[\lambda_1\varrho(\zeta,u)^s + \lambda_2\varrho(\zeta,\zeta)^s + \lambda_3\varrho(u,u)^s \\ &+ \lambda_4\left(\frac{\varrho(u,u)(1+\varrho(\zeta,\zeta))}{1+\varrho(\zeta,u)}\right)^s + \lambda_5\left(\frac{\varrho(u,\zeta)(1+\varrho(\zeta,u))}{1+\varrho(\zeta,u)}\right)^s\Bigg]^{\frac{1}{s}} \\ = &\Bigg[\lambda_1\varrho(\zeta,u)^s + \lambda_5\varrho(\zeta,u)^s\Bigg]^{\frac{1}{s}} \\ = &(\lambda_1+\lambda_5)^{\frac{1}{s}}\varrho(\zeta,u) \le \varrho(\zeta,u). \end{aligned}$$

Hence, (13) can be written as

$$\theta(\varrho(\zeta,u)) \le [\theta(\phi((\varrho(\zeta,u))))]^r < [\theta(\varrho(\zeta,u))]^r,$$

which is a contradiction for all $r \in (0,1)$. Thus, $\zeta = u$.
case 2: for $s = 0$

$$\begin{aligned} M_{\lambda_i}(\zeta,u) =&\varrho(\zeta,u)^{\lambda_1}.\varrho(\zeta,\Upsilon\zeta)^{\lambda_2}.\varrho(u,\Upsilon u)^{\lambda_3}. \\ &\left(\frac{\varrho(u,\Upsilon u)(1+\varrho(\zeta,\Upsilon\zeta))}{1+\varrho(\zeta,u)}\right)^{\lambda_4}.\left(\frac{\varrho(\zeta,\Upsilon u)(1+\varrho(u,\Upsilon\zeta))}{2}\right)^{\lambda_5} \\ =&\varrho(\zeta,u)^{\lambda_1}.\varrho(\zeta,\zeta)^{\lambda_2}.\varrho(u,u)^{\lambda_3}. \\ &\left(\frac{\varrho(u,u)(1+\varrho(\zeta,\zeta))}{1+\varrho(\zeta,u)}\right)^{\lambda_4}.\left(\frac{\varrho(\zeta,u)(1+\varrho(u,\zeta))}{2}\right)^{\lambda_5} \\ =&0. \end{aligned}$$

Therefore, (13) becomes $\theta(\varrho(\zeta,u)) \le [\theta(\phi((0)))]^r < [\theta(0)]^r$, which is a contradiction, since θ is non-decreasing. Hence, $\zeta = u$ which shows that the FP of Υ is unique. □

Corollary 3.4. *Let (Ω,ϱ) be a complete GMS and $\Upsilon : \Omega \to \Omega$ be an admissible Jaggi-Suzuki-type hybrid contraction such that*

$$\frac{1}{2}\varrho(\zeta,\Upsilon\zeta) < \varrho(\zeta,y) \Rightarrow \theta(\varrho(\Upsilon\zeta,\Upsilon y)) \le [\theta(\phi(M_{\lambda_i}(\zeta,y,s,\Upsilon)))]^r.$$

Then, Υ possesses a unique fixed in Ω.

Proof. From Theorem 3.3, take $\rho(\zeta,y) = 1$, then the proof is immediate. □

Corollary 3.5. *Let (Ω,ϱ) be a complete GMS and $\Upsilon : \Omega \to \Omega$ an admissible Jaggi-Suzuki-type hybrid contractions such that*

$$\frac{1}{2}\varrho(\zeta,\Upsilon\zeta) < \varrho(\zeta,y) \Rightarrow \theta(\varrho(\Upsilon\zeta,\Upsilon y)) \le [\theta(\phi(\varrho(\zeta,y,)))]^r.$$

Then, Υ possesses a unique fixed in Ω.

Proof. The conclusion follows from Corollary 3.4 by taken $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. □

Note: take Θ' the set of function $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Θ'_1) θ is non-decreasing and continuous;
(Θ'_2) $\inf_{t\in(0,\infty)} \theta(t) = 1$,

which are all deducible from the set of functions Θ. For more studies, see [lemma 1.6 in [14] and Lemma 3.1 in [23]].

Corollary 3.6. *[14] (see [Theorem 2.1]) Let (Ω, ϱ) be a CMS and $\Upsilon : \Omega \to \Omega$ be a mapping. If there exist $r \in (0,1)$ and $\theta \in \Theta'$ such that for all $\zeta, y \in \Omega$,*

$$\frac{1}{2}\varrho(\zeta, \Upsilon\zeta) < \varrho(\zeta, y) \Rightarrow \theta(\varrho(\Upsilon\zeta, \Upsilon y)) \le [\theta(M(\zeta, y))]^k,$$

where

$$M(\zeta, y) = \max\left\{\varrho(\zeta, y), \varrho(\zeta, \Upsilon\zeta), \varrho(y, \Upsilon y), \frac{1}{2}\varrho(\zeta, \Upsilon y), \varrho(y, \Upsilon\zeta)\right\}.$$

Then, Υ possesses a unique fixed in Ω.

Remark 3.7. We can deduce many other corollaries from the literature for example, by defining the mapping $H : \mathbb{R}^{+4} \longrightarrow (0,1) \subset [0,\infty)$ as $H(t) = \lambda$ for all $t \in \mathbb{R}^{+4}$, [2.1 in [19]] can easily be obtained from (3).

In the following, we construct an example to show the assertions of Theorem 3.3

Example 3.8. *Let $\Omega = [0,2]$, $\varrho : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ be the usual metric on $\mathbb{R}$ and the mapping $\Upsilon : \Omega \longrightarrow \Omega$ be defined by*

$$\Upsilon(\zeta) = \begin{cases} \frac{1}{5}, & \text{if } \zeta \in [0,1]; \\ \frac{\zeta}{5}, & \text{if } \zeta \in (1,2]. \end{cases}$$

Take

$$\rho(\zeta, y) = \begin{cases} 5, & \text{if } \zeta, y \in [0,1]; \\ 1, & \text{if } \zeta = 0,\ y = 2; \\ 0, & \text{otherwise}, \end{cases}$$

and $\phi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined by $\phi(t) = \frac{t}{3}$. It is easy to observe that ϕ is a comparison function, Υ is continuous and $\Upsilon^2(\zeta) = \frac{1}{5}$ for all $\zeta, y \in [0,1]$ is also continuous. Moreover, we examine the following cases:
case 1: for any $\zeta, y \in [0,1]$, $\varrho(\Upsilon\zeta, \Upsilon y) = 0$ and so the inequality (3) holds.

Case 2: for $\zeta = 0$ and $y = 2$, we have

$$\frac{1}{2}\varrho(0, \Upsilon 0) = \frac{1}{2}\varrho(0, \frac{1}{5}) = \frac{1}{10} < \varrho(0, 2) = 2$$
$$\Rightarrow \rho(0, 2)\varrho(\Upsilon 0, \Upsilon 2) = \varrho(\frac{1}{5}, \frac{2}{5}) = \frac{1}{5} \leq \sqrt{\frac{29}{75}}$$
$$= \left[\frac{1}{3}\left(\frac{1}{4}\left(2 + \frac{1}{5} + \frac{8}{5} + \frac{63}{75}\right)\right)\right]^{\frac{1}{2}}$$
$$= \left[\frac{1}{3}\left(\frac{1}{4}\varrho(0, 2) + \frac{1}{4}\varrho(0, \Upsilon 0) + \frac{1}{4}\varrho(2, \Upsilon 2) + \frac{1}{4}\left(\frac{\varrho(2, \Upsilon 2)(1 + \varrho(0, \Upsilon 2))}{1 + \varrho(0, 2)}\right)\right)\right]^{\frac{1}{2}}.$$

Since θ is non-decreasing, this implies that $\frac{1}{2}\varrho(\zeta, \Upsilon\zeta) < \varrho(\zeta, y)$

$$\Rightarrow \theta(\rho(\zeta, y)\varrho(\Upsilon\zeta, \Upsilon y)) \leq [\theta(\phi(M_{\lambda_i}(\zeta, y)))]^r.$$

Here, for the case where $s > 0$, take $s = 1$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = \frac{1}{4}, \lambda_4 = 0$ and $r = \frac{1}{2}$.

Case 3: for $\rho(\zeta, y) = 0$, there is nothing to show. Thus, it follows that Υ is an admissible Jaggi-Suzuki-type hybrid (θ-ϕ)-contraction which satisfies all the assumption of Theorem 3.3 and then $\zeta = 0$ is the FP of Υ.

Definition 3.9. Let (Ω, ϱ) be a GMS. A mapping $\Upsilon : \Omega \longrightarrow \Omega$ is called Jaggi-Suzuki-type hybrid (θ-ϕ)-contraction if there exist $\theta \in \Theta$, $\phi \in \Phi$, $r \in (0, 1)$ and a mapping $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ such that

$$\frac{1}{2}\varrho(\zeta, \Upsilon\zeta) \leq \varrho(\zeta, y) \Rightarrow \theta(\rho(\zeta, y)\varrho(\Upsilon\zeta, \Upsilon y)) \leq [\theta(\phi(M_{\lambda_i}(\zeta, y, s, \Upsilon)))]^r,$$

where $M_{\lambda_i}(\zeta, y, s, \Upsilon)$ is as defined in (1).

Theorem 3.10. *Let (Ω, ϱ) be a complete GMS and Υ be a Jaggi-Suzuki-type hybrid (θ-ϕ)-contraction. Assume further that:*

(i) *Υ is triangular ρ-orbital admissible;*
(ii) *there exists $\zeta_0 \in \Omega$ such that $\rho(\zeta_0, \Upsilon\zeta_0) \geq 1$;*
(iii) *either Υ is continuous or;*
(iv) *Υ^2 is continuous and $\rho(\Upsilon\zeta, \zeta) \geq 1$ for any $\zeta \in fix(\Upsilon)$.*

Then, Υ possesses a FP in Ω. Moreover, Υ has a unique FP when $\rho(\zeta, y) \geq 1$ for all $\zeta, y \in fix(\Upsilon)$.

Proof. The argument of the proof follows analogously from the proof of Theorem 3.2 and 3.3 □

Corollary 3.11. *(see [27] Theorem 2.3) Let (Ω, ϱ) be a CMS and $\Upsilon : \Omega \to \Omega$ be a mapping satisfying:*

$$\frac{1}{2}\varrho(\zeta, \Upsilon\zeta) < \varrho(\zeta, y) \Rightarrow \theta(\varrho(\Upsilon\zeta, \Upsilon y)) \leq \phi[\theta(N(\zeta, y))],$$

where $N(\zeta, y) = \max\{\varrho(\zeta, y), \varrho(\zeta, \Upsilon\zeta), \varrho(y, \Upsilon y)\}$.

Then Υ has a unique FP in Ω.

4. Applications to a solution of an integral equation

Motivated by the idea in [28], one of our obtained results is applied in this section to analyze conditions for the existence of solution of the integral equation:

$$\zeta(t) = h(t) + \int_a^b \mathcal{L}(t,s)\tau(s,\zeta(s))ds; \quad t \in [a,b], \tag{14}$$

where $h : [a,b] \longrightarrow \mathbb{R}$, $\mathcal{L} : [a,b]^2 \longrightarrow \mathbb{R}_+$ and $\tau : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ are given continuous functions. Let $\Omega = C([a,b],\mathbb{R})$ be the set of all continuous real valued functions defined on $[a,b]$. We equip Ω with the metric defined by $\varrho(\zeta,y) = \max_{t\in[a,b]} |\zeta(t) - y(t)|$. One can easily see that (Ω,ϱ) is a CMS. Consider the mapping $\Upsilon : \Omega \longrightarrow \Omega$ defined by

$$\Upsilon\zeta(t) = h(t) + \int_a^b \mathcal{L}(t,s)\tau(s,\zeta(s))ds; \quad t \in [a,b]. \tag{15}$$

Then, u* is a unique FP of Υ if and only if it is a solution to (14). Now, we study conditions for the existence and uniqueness solution of integral equation (14) under the following hypotheses:

Theorem 4.1. *Assume that the following assumption hold:*

(i) $\max_{t\in[a,b]} \int_a^b |\mathcal{L}(t,s)|ds \leq \frac{r}{b-a}$, *where* $r \in (0,1)$*;*

(ii) *there exist* $\eta > 1$ *and* $\phi \in \Phi$ *such that* $\frac{1}{2}\varrho(\zeta,\Upsilon\zeta) \leq \varrho(\zeta,y) \Rightarrow |\tau(s,\zeta) - \tau(s,y)| \leq \phi(\frac{1}{\eta}|\zeta(t) - y(t)|)$.

Then the integral equation (14) *has a unique solution in* Ω.

Proof. Employing Conditions (i) and (ii) of Theorem 4.1, we have

$$\begin{aligned}
\varrho(\Upsilon\zeta,\Upsilon y) &= \max_{t\in[a,b]} |\Upsilon\zeta(t) - \Upsilon y(t)| \\
&= \max_{t\in[a,b]} \left| h(t) + \int_a^b \mathcal{L}(t,s)\tau(s,\zeta(s))ds - h(t) - \int_a^b \mathcal{L}(t,s)\tau(s,y(s))ds \right| \\
&= \max_{t\in[a,b]} \left| \int_a^b \big(\mathcal{L}(t,s)\tau(s,\zeta(s)) - \mathcal{L}(t,s)\tau(s,y(s))\big)ds \right| \\
&= \max_{t\in[a,b]} \left| \int_a^b \mathcal{L}(t,s)[\tau(s,\zeta(s)) - \tau(s,y(s))]ds \right| \\
&\leq \max_{t\in[a,b]} \left[\int_a^b |\mathcal{L}(t,s)ds| \int_a^b |\tau(s,\zeta(s)) - \tau(s,y(s))|ds \right] \\
&= \frac{r}{b-a}\phi\left(\frac{1}{\eta}\max_{t\in[a,b]} |\zeta(t) - y(t)|\right) \int_a^b ds \\
&\leq r\phi(\varrho(\zeta,y)).
\end{aligned}$$

This implies that

$$\varrho(\Upsilon\zeta, \Upsilon y) \leq r\phi(\varrho(\zeta, y)). \tag{16}$$

Taking exponential of both side of (16) and define a mapping $\theta(t) = e^t$, then we get

$$\theta(\varrho(\Upsilon\zeta, \Upsilon y)) \leq [\theta(\phi(\varrho(\zeta, y)))]^r.$$

Hence,

$$\frac{1}{2}\varrho(\zeta, \Upsilon\zeta) < \varrho(\zeta, y) \Rightarrow \theta(\varrho(\Upsilon\zeta, \Upsilon y)) \leq [\theta(\phi(\varrho(\zeta, y)))]^r.$$

Thus, all the hypothesis of Corollary 3.5 are satisfied and it follows that Υ has a unique FP in Ω, which amounts to say that the integral equation (14) has a unique solution which belongs to Ω. □

Example 4.2. *Consider the following integral equation in $\Omega = C([0, 1], \mathbb{R})$:*

$$\zeta(t) = \frac{t^3}{3+t} + \frac{1}{6}\int_0^1 \frac{s^3}{(3+t)}\frac{1}{(3+\zeta(s))}ds; \quad t \in [0, 1]. \tag{17}$$

In order to find the solution of (17), we will prove that $\zeta(t)$ is a solution of the mapping Υ, that is $\zeta(t) = \Upsilon\zeta(t)$, where

$$\Upsilon\zeta(t) = \frac{t^3}{3+t} + \frac{1}{6}\int_0^1 \frac{s^3}{(3+t)}\frac{1}{(3+\zeta(s))}ds; \quad t \in [0, 1].$$

Observe that the integral equation (17) is a special case of (14) in which $\tau(s, t) = \frac{1}{(3+\zeta(s))}$; $\mathcal{L}(t, s) = \frac{s^3}{(3+t)}$; $h(t) = \frac{t^3}{3+t}$. Indeed, the functions $\tau(s, t)$, $\mathcal{L}(t, s)$ and $h(t)$ are continuous. Thus, the assumption that either Υ is continuous or Υ^2 is continuous are satisfied. Also, for all $\zeta, y \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq |\tau(s, \zeta) - \tau(s, y)| &\leq \left|\frac{1}{6(3+\zeta)} - \frac{1}{6(3+y)}\right| \\ &\leq \frac{1}{6}|\zeta - y| \leq \phi(|\zeta - y|), \end{aligned}$$

where $\phi(t) = \frac{t}{2}$. Hence, Condition (ii) of Theorem 4.1 is verified. For Condition (i), we have

$$\max_{t\in[0,1]}\int_0^1 |\mathcal{L}(t, s)|ds = \max_{t\in[0,1]}\int_0^1 \frac{s^3}{(3+t)}ds = \max_{t\in[0,1]}\frac{1}{4(3+t)} \leq \frac{r}{b-a} = \frac{1}{12(1)}, \text{ where } r = \frac{1}{12}.$$

Hence, Condition (i) is also proved for all $t \in [0, 1]$. Consequently, all the conditions of Theorem 4.1 are satisfied. Thus, the integral equation (17) has a solution in Ω.

5. Conclusion

The concept of θ-contraction was introduced by Jlelli and Samet [8], which is an improvement of the famous Banach contraction principle in the context of generalized/rectangular metric space (see [5]). Since the introduction of rational contractive inequality by Jaggi [7], a lot of results in this regard have been presented by different researchers. Following this development, this manuscript introduced an idea of hybrid contraction, using Jaggi-Suzuki-type inequalities in addition with θ-contraction. Using this new contractive operators, three fixed point theorems (Theorems 3.2, 3.3,3.10) were formulated and proved. Example 3.8 was constructed to indicate that the hypotheses of the established theorems are well-stated. To show a possible utility of the ideas of this research, an application concerning the existence criteria of an integral equation was presented in Section 4.

This work was approached from a theoretical aspect of mathematics which might result to some limitation on its applicability. The existence results for the integral equation has been developed analytically, based on the abstract assumptions of our theorems. However, it is important to note that this paper's concept directs scholars toward other studies and applications such as partial metric space and multi-valued mappings.

6. Author Contributions

All the authors equally conceived the idea of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

7. Data Availability Statement

Not applicable

8. Aknowledgement

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11. Conflict of interest

The authors declare that they have no competing interests.

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