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THE SMALL CONDITION FOR MODULES WITH NOETHERIAN DIMENSION

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Abstract. A module M with Noetherian dimension is said to satisfy the small condition, if for any small submodule S of M the Noetherian dimension of S is strictly less than the Noetherian dimension of M. For an Artinian module M, this is equivalent to that M is semisimple. In this article, we introduce and study this concept and observe some basic facts for modules with this condition. As a main result, it is shown that if M is a module with finite hollow dimension which satisfies the small condition, then $\alpha \leq n\text{-dim}\, M \leq \alpha + 1$, where $\alpha = \sup\{n\text{-dim}\, S : S \ll M\}$. Furthermore, if M is a module with Krull dimension and finite hollow dimension, then $\alpha \leq k\text{-dim}\, M \leq \alpha + 1$, where $\alpha = \sup\{k\text{-dim}\, S : S \ll M\}$. Also, we study the projective cover of modules satisfying the small condition or with finite hollow dimension.

Keywords: Large condition , Small condition, Semiatomic modules, Projective cover.
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1. Introduction

Throughout this paper, all rings are associative with $1 \neq 0$, and all modules are unital right modules. The Krull dimension of modules, introduced by Rentschler and Gabriel [17], then it was extended to modules over noncommutative rings for each ordinal number, by Krause [13]. The dual of this concept, that is, the Noetherian dimension almost simultaneously, is introduced and investigated by Karamzadeh [11] and Lemonnier [14]. In 1978, Boyle in [3], introduced the concept of the large condition and investigated the relationship between it and the injective hull of a module. Then she introduced the concept of the semicritical module and obtained some related results about it, see [4]. In this paper, motivated by Boyle works, we define the concepts of small condition and semiatomic modules and provide dual of the existing theorems as much as possible. In what follows, we list some of the interesting properties of semicritical modules or modules with the large condition, which are to be dualized for semiatomic modules or modules with the small condition.

(1) Every non-zero submodule of a semicritical module is semicritical.

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(2) Every non-zero submodule of a semicritical module has a critical composition series.
(3) Every semicritical module satisfies the large condition.
(4) Every uniform submodule of a semicritical module is critical.
(5) If M is an R-module with $k\text{-}\dim M = \alpha$, M_0 a closed submodule of M with $k\text{-}\dim M_0 < \alpha$, and $\frac{M}{M_0}$ satisfies the large condition, then M satisfies the large condition.
(6) If M is a module with Krull dimension satisfying the large condition and $\{E_i : i \in I\}$ is the collection of essential submodules of M. Then $k\text{-}\dim M = \sup\{k\text{-}\dim \frac{M}{E_i} : i \in I\} + 1$.
(7) If M is a module with finite Goldie dimension and M_0 a closed submodule of M, then $E(\frac{M}{M_0}) \simeq \frac{E(M)}{E(M_0)}$.

Now, let us give a brief outline of this article. In Section 2, we review some necessary preliminaries. Section 3 is devoted to a brief study of modules satisfying the large condition. It is shown that every dual-local module which satisfies the large condition is Artinian. Also, if M satisfies the large condition, then so does every essential submodule of M. In the special case, if $E(M)$ satisfies the large condition, then so does M. Moreover, if E is an essential submodule of M with $k\text{-}\dim \frac{M}{E} < k\text{-}\dim E$, then M satisfies the large condition if and only if E satisfies the large condition. Furthermore, it is proved that an R-module M is Artinian and satisfies the large condition if and only if it is semisimple with finite Goldie dimension. In Section 4, we first introduce the concept of the small condition and semiatomic module and dualize almost all the above 7 basic results for modules which satisfy the small condition. For instance, it is shown that if M satisfies the small condition, then so does every small quotient of M. Also, if A is a small submodule of M with $n\text{-}\dim A < n\text{-}\dim \frac{M}{A}$, then M satisfies the small condition if and only if $\frac{M}{A}$ satisfies the small condition. It is proved that every local module with the small condition is Noetherian. Also, if M is Noetherian and satisfies the small condition, then $J(M) = 0$. Moreover, if M_0 is a coclosed submodule of M which $n\text{-}\dim \frac{M}{M_0} < n\text{-}\dim M$ and M_0 satisfies the small condition, then so does M. In particular, it is shown that if M is a module with finite hollow dimension which satisfies the small condition, then $\alpha \leq n\text{-}\dim M \leq \alpha + 1$, where $\alpha = \sup\{n\text{-}\dim S : S \ll M\}$. Furthermore, if M is a module with Krull dimension and finite hollow dimension, then $\alpha \leq k\text{-}\dim M \leq \alpha + 1$, where $\alpha = \sup\{k\text{-}\dim S : S \ll M\}$. Finally, we study the projective cover of modules satisfying the small condition or with finite hollow dimension. It is proved that if M_0 is a coclosed submodule of an R-module M with finite hollow dimension and projective cover $P(M)$, then $P(\frac{M}{M_0}) \simeq \frac{P(M)}{P(M_0)}$.

If M is an R-module, then $k\text{-}\dim M$ and $n\text{-}\dim M$ denote the Krull dimension and Noetherian dimension of M, respectively. It is convenient that, when we are dealing with the latter dimensions, we may begin our list of ordinals with -1. Now, we recall some definitions that we need throughout this article. If

α is an ordinal number, an R-module M is called α-critical if k-dim $M = \alpha$ and k-dim $\frac{M}{N} < \alpha$, for every non-zero submodule N of M. An R-module M is said to be *critical* if it is α-critical for some α, see [6]. Also, if an R-module M has Noetherian dimension and α is an ordinal, then M is called α-atomic if n-dim $M = \alpha$ and n-dim $N < \alpha$, for every proper submodule N of M. An R-module M is said to be *atomic* if it is α-atomic for some α, see [12]. A submodule E of an R-module M is said to be an *essential (or a large)* submodule of M, denoted by $E \subseteq_e M$, if for each non-zero submodule A of M, $E \cap A \neq 0$. If E is an essential submodule of M, then M is referred to as an essential extension of E. For any module M, there is a unique (up to isomorphism) essential injective extension $E(M)$ of M, which is called the *injective hull* of M, see [5]. A submodule S of M is called *small* in M if $S + L \neq M$ for every proper submodule L of M. We will indicate that S is a small submodule of M by the notation $S \ll M$. Also, a non-zero module M is called hollow if every proper submodule N of M is small in M, see [15]. A submodule N of M is said to be *closed* in M, if it has no proper essential extension in M, see [5]. Moreover, submodule N of a module M is said to be *coclosed* in M, if N has no proper submodule K such that $\frac{N}{K} \ll \frac{M}{K}$, see [15]. An R-module M is called *weakly supplemented* if for every submodule N of M, there exists a submodule such as L of M, such that $N + L = M$ and $N \cap L \ll M$, see [15]. The reader is referred to [2–4] for more details and undefined terms and notations.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed in the sequel.

Lemma 2.1. *[15, Lemma 1.1.1] Let M be a module, $K \subseteq N \subseteq M$ and $H \subseteq M$. Then*

1. *$N \ll M$ if and only if $K \ll M$ and $\frac{N}{K} \ll \frac{M}{K}$.*
2. *If $f : M \to M'$ is a homomorphism and $N \ll M$, then $f(N) \ll M'$.*
3. *If $K \ll N$, then $K \ll M$.*
4. *$H + K \ll M$ if and only if $H \ll M$ and $K \ll M$.*
5. *If $M = M_1 \oplus M_2$ and $K_1 \subseteq M_1$ and $K_2 \subseteq M_2$, then $K_1 \oplus K_2 \ll M_1 \oplus M_2$ if and only if $K_1 \ll M_1$ and $K_2 \ll M_2$.*
6. *Let N be a direct summand of M. Then $K \ll M$ if and only if $K \ll N$.*

Definition 2.2. [15, 1.4] A non-empty family $\{E_i\}_{i \in I}$ of proper submodules of an R-module M is called coindependent, if for any $k \in I$ and any finite subset $F \subseteq I \setminus \{k\}$, $E_k + \bigcap_{j \in F} E_j = M$.

Proposition 2.3. *[15, Theorem 3.1.2] For a non-zero module M, the following are equivalent:*

1. *M does not contain an infinite coindependent family of submodules.*

(2) *For some* $n \in \mathbb{N}$, *M contains a coindependent family of submodules* $\{E_1, E_2, ..., E_n\}$ *such that* $\bigcap_{i=1}^{n} E_i$ *is small in M and* $\frac{M}{E_i}$ *is a hollow module for every* $1 \leq i \leq n$.
(3) $\sup\{k : M$ *contains a coindependent family of submodules of cardinality* $k\} = n$, *for some* $n \in \mathbb{N}$.
(4) *For any descending chain* $M_1 \supseteq M_2 \supseteq \cdots$ *of submodules of M there exists j, such that* $\frac{M_j}{M_k} \ll \frac{M}{M_k}$ *for all* $k \geq j$.
(5) *There exists a small epimorphism from M to a finite direct sum of n hollow factor modules. That is, there exists an epimorphism* $f : M \to \bigoplus_{i=1}^{n} \frac{M}{N_i}$ *such that* $\frac{M}{N_i}$ *is hollow module for all i, and* $\mathrm{Ker}(f) \ll M$.

Definition 2.4. [15, 3.1] An R-module M is said to have finite hollow dimension if it satisfies one of the conditions in the previous proposition. In particular, if M satisfies condition (2) or (3), then M is said to have hollow dimension n, written as $h\text{-dim}\, M = n$. If $M = 0$, we define $h\text{-dim}\, M = 0$ and if M does not have finite hollow dimension, we write $h\text{-dim}\, M = \infty$.

Lemma 2.5. *[15, Theorem 3.1.10] Let N be a submodule of an R-module M.*
(1) $h\text{-dim}\, \frac{M}{N} \leq h\text{-dim}\, M$.
(2) *If* $N \ll M$, *then* $h\text{-dim}\, M = h\text{-dim}\, \frac{M}{N}$.
(3) *If M has finite hollow dimension and* $h\text{-dim}\, M = h\text{-dim}\, \frac{M}{N}$, *then* $N \ll M$.
(4) *If M has finite hollow dimension, then M is weakly supplemented.*

Proposition 2.6. *[15, Proposition 1.2.1] Let N be a submodule of an R-module M. Consider the following assertions:*
(1) *N is a supplement in M.*
(2) *N is coclosed in M.*
(3) *For all* $K \subseteq N$, $K \ll M$ *implies* $K \ll N$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) *and, if N is a weak supplement in M, then* (3) $\Rightarrow$ (1).

We also cite the following fact from part $(2')$ of the comment which follows [7, Corollary 13].

Proposition 2.7. *If M is an R-module and* $h\text{-dim}\, M = n$, *then there exists coindependent family of submodules* $\{E_1, E_2, \cdots, E_n\}$, *such that* $\bigcap_{i=1}^{n} E_i \ll M$ *and* $\frac{M}{\bigcap_{i=1}^{n} E_i} \simeq \bigoplus_{i=1}^{n} \frac{M}{E_i}$ *such that* $\frac{M}{E_i}$ *is hollow for all* $i = 1, 2, \cdots, n$.

Definition 2.8. [20, 1.4] Let M be an R-module. The Krull dimension of M, denoted by $k\text{-dim}\, M$ is defined by transfinite recursion as follows: If $M = 0$, $k\text{-dim}\, M = -1$. If α is an ordinal number and $k\text{-dim}\, M \not< \alpha$, then $k\text{-dim}\, M = \alpha$ provided there is no infinite descending chain of submodules of M such as $M_0 \supseteq M_1 \supseteq M_2 \supseteq ...$ such that for each $i = 1, 2, ...$, $k\text{-dim}\, \frac{M_{i-1}}{M_i} \not< \alpha$. In other words, $k\text{-dim}\, M = \alpha$, if $k\text{-dim}\, M \not< \alpha$ and for each chain of submodules to M such as $M_0 \supseteq M_1 \supseteq M_2 \supseteq ...$ there exists an integer t, such that for each

$i \geq t$, $k\text{-dim}\,\frac{M_{i-1}}{M_i} < \alpha$. A ring R has the Krull dimension, if as an R-module has the Krull dimension. It is possible that there is no ordinal α such that $k\text{-dim}\,M = \alpha$, in this case we say M has no Krull dimension.

The dual Krull dimension, that is Noetherian dimension, of a module is defined similarly with ascending chains. It is well known that a module has Krull dimension if and only if it has Noetherian dimension.

We now go on to detail some results on rings and modules with Krull dimension (resp., Noetherian dimension) which we will use later in this paper. All of these are well known and we omit the proofs, instead referring to the literature for further information.

Lemma 2.9. *[6, Lemma 1.1(i)] Let N be a submodule of an R-module M. Then $k\text{-dim}\,M = \sup\{k\text{-dim}\,N, k\text{-dim}\,\frac{M}{N}\}$, if either side exists.*

Lemma 2.10. *[20, Lemma 1.4.5] Let M be an R-module with Krull dimension such that M is a sum of submodules each of which has Krull dimension at most α, for some ordinal α. Then $k\text{-dim}\,M \leq \alpha$.*

Lemma 2.11. *[12, Lemma 1.2] Let N be a submodule of an R-module M. Then $n\text{-dim}\,M = \sup\{n\text{-dim}\,N, n\text{-dim}\,\frac{M}{N}\}$, if either side exists.*

Corollary 2.12. *Let $M = \bigoplus_{i=1}^{n} M_i$ be an R-module with Noetherian dimension. Then $n\text{-dim}\,M = \sup\{n\text{-dim}\,M_i : 1 \leq i \leq n\}$.*

Proposition 2.13. *[12, Proposition 2.2(1)] Let M be an α-atomic R-module. Then every non-zero factor module of M is α-atomic.*

3. The large condition for modules

An R-module M with Krull dimension is said to satisfy the large condition if $k\text{-dim}\,\frac{M}{E} < k\text{-dim}\,M$, for any essential submodule E of M. The ring R is said to satisfy the large condition if it satisfies the large condition, as an R-module, see [3]. We note that if M satisfies the large condition, then $k\text{-dim}\,M = k\text{-dim}\,E$, for every essential submodule E of M. An R-module M is termed semicritical provided there exists a finite collection of submodules $K_1, ..., K_n$ such that $\frac{M}{K_i}$ is a critical module for all i and $\bigcap_{k=1}^{n} K_i = 0$, see [4].

Note that every semicritical module M has Krull dimension and every non-zero submodule N of M is semicritical, see [4, Theorem 2.1].

Example 3.1. (1) *Every semiprime ring R with Krull dimension satisfies the large condition, since $k\text{-dim}\,R = \sup\{k\text{-dim}\,\frac{R}{E} : E \leq_e R\} + 1$, see [6, Proposition 6.1]. More generally, every semiprime FQS module (i.e., finitely generated, quasi-projective and self-generator) with Krull dimension satisfies the large condition, because if M is a semiprime FQS module with Krull dimension, then $k\text{-dim}\,M = \sup\{k\text{-dim}\,\frac{M}{E} : E \subseteq_e M\} + 1$, see [8, Theorem 4.14]. But the converse of this fact is* *not true in general. For example, $R = \mathbb{Z} \times \mathbb{Z}_4$ is not a semiprime ring. But, if $E \leq_e R$, then either $E = n\mathbb{Z}\times\mathbb{Z}_4$ or $E = \mathbb{Z}\times\mathbb{Z}_2$ or $E = n\mathbb{Z}\times\mathbb{Z}_2$, so $\frac{R}{E} = \frac{\mathbb{Z}\times\mathbb{Z}_4}{n\mathbb{Z}\times\mathbb{Z}_4} \simeq \mathbb{Z}_n$ or $\frac{R}{E} = \frac{\mathbb{Z}\times\mathbb{Z}_4}{\mathbb{Z}\times\mathbb{Z}_2} \simeq \mathbb{Z}_2$ or $\frac{R}{E} = \frac{\mathbb{Z}\times\mathbb{Z}_4}{n\mathbb{Z}\times\mathbb{Z}_2} \simeq \mathbb{Z}_n \times \mathbb{Z}_2$, which is Artinian in any case. Therefore, $0 = k\text{-dim}\,\frac{R}{E} < k\text{-dim}\,R = 1$ and consequently, R satisfies the large condition.*

(2) *For every prime number p, $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module does not satisfy the large condition, because every its factor module is isomorphic to itself.*

(3) *$\mathbb{Z}$ as a $\mathbb{Z}$-module satisfies the large condition, because every non-zero submodule $n\mathbb{Z}$ of $\mathbb{Z}$ is essential in it and $k\text{-dim}\,\frac{\mathbb{Z}}{n\mathbb{Z}} = k\text{-dim}\,\mathbb{Z}_n = 0 < k\text{-dim}\,\mathbb{Z} = 1$.*

The following conditions are equivalent for semisimple modules.

Lemma 3.2. *Let M be an R-module. The following statements are equivalent.*

(1) *M is Artinian and satisfies the large condition.*

(2) *M is semisimple with finite Goldie dimension.*

Proof. (1) $\Longrightarrow$ (2) Assume (1). It suffices to show that $\text{Soc}(M) = M$, where $\text{Soc}(M)$ is the sum of all simple submodules of M. For this, let E be an essential submodule of M. Thus, $k\text{-dim}\,\frac{M}{E} < k\text{-dim}\,M = 0$, so $\frac{M}{E} = 0$, and this implies that $E = M$. Thus, M is the only essential submodule of M, that is $\text{Soc}(M) = M$.
(2) $\Longrightarrow$ (1) Assume (2). Then M is Artinian and has not proper essential submodule. Hence, M satisfies the large condition.

□

Corollary 3.3. *Every semiprime right Artinian ring is a semisimple ring. More generally, every semiprime Artinian FQS module is a semisimple module.*

Proof. Let R be a semiprime right Artinian ring. In view of Example 3.1(1), R satisfies the large condition, so by the previous lemma, it is a semisimple ring. Now, if M is a semiprime Artinian FQS module, then by Example 3.1(1), M satisfies the large condition, so by the previous lemma, it is a semisimple module. □

Corollary 3.4. *Let M be an Artinian module which is not Noetherian. Then M does not satisfy the large condition.*

It is known that, M is an Artinian module with $J(M) = 0$, if and only if M is semisimple and Noetherian, see [1, Proposition 10.15], thus we have the following result.

Corollary 3.5. *Let M be an Artinian module with $J(M) = 0$. Then M satisfies the large condition.*

Remark 3.6. Let M be an R-module which satisfies the large condition. Every submodule of M does not necessarily satisfy the large condition. For instance, according to Example 3.1(1), $R = \mathbb{Z} \times \mathbb{Z}_4$ as an R-module satisfies the large condition but $\mathbb{Z}_4$ as an R-submodule does not satisfy the large condition (note that, $\mathbb{Z}_4$ is an Artinian non-semisimple module).

In what follows, we show that essential submodules of a module with the large condition satisfy the large condition.

Proposition 3.7. *Let M be an R-module. If M satisfies the large condition, then so does every essential submodule of M.*

Proof. Assume that M satisfies the large condition and E is an essential submodule of M. We show that E satisfies the large condition. To see this, let A be an essential submodule of E. It suffices to show that $k\text{-dim}\,\frac{E}{A} < k\text{-dim}\,E$. According to [5, Proposition 5.6(a)], A is an essential submodule of M, so $k\text{-dim}\,\frac{E}{A} \leq k\text{-dim}\,\frac{M}{A} < k\text{-dim}\,M = k\text{-dim}\,E$. □

Corollary 3.8. *Let M be an R-module. If $E(M)$ satisfies the large condition, then so does M.*

Theorem 3.9. *Let E be an essential submodule of M which $k\text{-dim}\,\frac{M}{E} < k\text{-dim}\,E$. Then M satisfies the large condition if and only if E satisfies the large condition.*

Proof. In view of Proposition 3.7, it suffices to show that if E satisfies the large condition, then so does M. To see this, let A be an essential submodule of M, we must show that $k\text{-dim}\,\frac{M}{A} < k\text{-dim}\,M$. According to [5, Proposition 5.6(a),(b)], $A \cap E \subseteq_e E$. As E satisfies the large condition thus, $k\text{-dim}\,\frac{E}{A\cap E} < k\text{-dim}\,E$, and by assumption, $k\text{-dim}\,E = k\text{-dim}\,M$. Hence, $k\text{-dim}\,\frac{E}{A\cap E} < k\text{-dim}\,M$. We infer that $k\text{-dim}\,\frac{A+E}{A} = k\text{-dim}\,\frac{E}{A\cap E} < k\text{-dim}\,M$ and $k\text{-dim}\,\frac{M}{A+E} \leq k\text{-dim}\,\frac{M}{E} < k\text{-dim}\,E = k\text{-dim}\,M$. Therefore, $k\text{-dim}\,\frac{M}{E} = \sup\{k\text{-dim}\,\frac{M}{A+E}, k\text{-dim}\,\frac{A+E}{E}\} < k\text{-dim}\,M$. This completes the proof. □

Corollary 3.10. *Let M be an R-module which $E(M)$ has Krull dimension and $k\text{-dim}\,\frac{E(M)}{M} < k\text{-dim}\,M$. Then M satisfies the large condition if and only if $E(M)$ satisfies the large condition.*

Proof. Sufficiency is clear, by Corollary 3.8. Conversely, suppose that M satisfies the large condition. Since $M \subseteq_e E(M)$ and $k\text{-dim}\,\frac{E(M)}{M} < k\text{-dim}\,M$, by the previous theorem, $E(M)$ satisfies the large condition. □

Recall that an R-module M is said to be *dual-local*, if it has exactly one minimal submodule A which is contained in all non-zero submodules of M, see [9, Definition 2.29]. So, A is simple and $A = \text{Soc}(M)$. Clearly, M is a uniform module.

Lemma 3.11. *Let M be a dual-local module which satisfies the large condition. Then M is Artinian.*

Proof. Let $0 \neq A$ be its only minimal submodule of M. Then $A = \mathrm{Soc}(M) \subseteq_e M$ implies that $k\text{-dim}\, \frac{M}{A} < k\text{-dim}\, M$, so by Lemma 2.9, $k\text{-dim}\, M = k\text{-dim}\, A = 0$. Therefore, M is Artinian. □

4. The small condition for modules

In this section, we consider the concepts of the small condition and semiatomic modules which are the dual concepts of the large condition and semicritical modules, respectively.

Definition 4.1. We say that an R-module M with Noetherian dimension satisfies the small condition if $n\text{-dim}\, S < n\text{-dim}\, M$, for any small submodule S of M. In particular, R is said to satisfy the small condition if as an R-module, it satisfies the small condition.

Recall that if S is a small submodule of M, then $\frac{M}{S}$ is called a *small quotient* module of M, see [18, Definition 3.12]. If M satisfies the small condition, then clearly $n\text{-dim}\, M = n\text{-dim}\, \frac{M}{S}$, for every small quotient $\frac{M}{S}$ of M.

Example 4.2. (1) *$\mathbb{Z}$ as a $\mathbb{Z}$-module satisfies the small condition. Because zero is its the only small submodule and $n\text{-dim}\, 0 = -1 < 0 = n\text{-dim}\, \mathbb{Z}$.*

(2) *Every atomic module satisfies the small condition. For example, $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module satisfies the small condition. But the converse of this fact is not true in general, for example $\mathbb{Z}$ as a $\mathbb{Z}$-module is not atomic. However, every hollow module with the small condition is atomic.*

Remark 4.3. In view of Lemma 3.2, if M is an Artinian module with the large condition, then M is Noetherian. Note that the dual of this fact is not true, in general. For instance, $\mathbb{Z}$ as a $\mathbb{Z}$-module is Noetherian with the small condition, but it is not Artinian.

Definition 4.4. We say that an R-module M is semiatomic if $M = \Sigma_{i=1}^{n} A_i$, for some finite collection of atomic submodules $A_1, \cdots, A_n$.

According to [4, Theorem 2.4], every semicritical module M satisfies the large condition. The following is the dual of this fact for semiatomic D-modules. Here, by a D-module, we mean a module M for which the lattice of submodules is distributive, that is, for all submodules N, K and L of M, $N \cap (K + L) = (N \cap K) + (N \cap L)$ or equivalently, $N + (K \cap L) = (N + K) \cap (N + L)$, see [10].

Theorem 4.5. *Every semiatomic D-module satisfies the small condition.*

Proof. Let M be a semiatomic D-module. Then there exists a finite collection of atomic modules $A_1, A_2, \cdots, A_n$ such that $M = \sum_{i=1}^{n} A_i$ and n is the smallest integer with this property. If S is a small submodule of M, then $S \cap A_i \subsetneq A_i$, for all i, because if $S \cap A_i = A_i$, then $A_i \subseteq S$ and since S is a small submodule of M, by Lemma 2.1(1), A_i is a small submodule of M, a contradiction. Hence, $n\text{-dim}\,(S \cap A_i) < n\text{-dim}\, A_i$, because A_i is atomic. But $S = S \cap M = S \cap \sum_{i=1}^{n} A_i = \sum_{i=1}^{n}(S \cap A_i)$, so $n\text{-dim}\, S = \sup\{n\text{-dim}\,(S \cap A_i) : 1 \leq i \leq n\} <$

n-dim $A_i \leq n$-dim M. Therefore, n-dim $S < n$-dim M and the proof is complete. □

Note that the converse of the previous theorem does not necessarily hold, in general. For example, it is easy to check that $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$-modules satisfy the small condition but they are not semiatomic.

Lemma 4.6. *Let N be a submodule of an R-module M which n-dim $N =$ n-dim M. If M satisfies the small condition, then so does N.*

Proof. Let S be a small submodule of N. By Lemma 2.1(3), S is a small submodule of M, so n-dim $S < n$-dim $M = n$-dim N. Therefore, N satisfies the small condition. □

Remark 4.7. In general, every quotient module of a module with the small condition does not need to satisfy the small condition, however for small quotients we have the following proposition. For example, it is easy to see that $\mathbb{Z}_{36}$ as a $\mathbb{Z}$-module does not satisfy the small condition, we note that $6\mathbb{Z}_{36}, 12\mathbb{Z}_{36}, 18\mathbb{Z}_{36}$ are the only non-zero small submodules of $\mathbb{Z}_{36}$.

Proposition 4.8. *Let M be an R-module. If M satisfies the small condition, then so does every small quotient of M.*

Proof. Let S be a small submodule of M. By Lemma 2.11, n-dim $M =$ n-dim $\frac{M}{S}$. By Lemma 2.1(1), if $\frac{N}{S} \ll \frac{M}{S}$, then $N \ll M$, so n-dim $N < n$-dim M. Hence n-dim $\frac{N}{S} \leq n$-dim $N < n$-dim $M = n$-dim $\frac{M}{S}$, which completes the proof. □

Theorem 4.9. *Let A be a small submodule of an R-module M with n-dim $A <$ n-dim $\frac{M}{A}$. Then M satisfies the small condition if and only if $\frac{M}{A}$ satisfies the small condition.*

Proof. In view of the previous proposition, it suffices to show that if $\frac{M}{A}$ satisfies the small condition, then so does M. To see this, let S be a small submodule of M. By Lemma 2.1(1,4), $\frac{A+S}{A} \ll \frac{M}{A}$ thus, n-dim $\frac{A+S}{A} < n$-dim $\frac{M}{A}$, then n-dim $\frac{S}{A \cap S} < n$-dim $\frac{M}{A}$. Moreover, n-dim $(A \cap S) \leq n$-dim $A < n$-dim $\frac{M}{A}$, by assumption, n-dim $\frac{M}{A} = n$-dim M. So n-dim $S = \sup\{n\text{-dim}\, \frac{S}{A \cap S}, n\text{-dim}\,(A \cap S)\} < n$-dim $\frac{M}{A} = n$-dim M, which completes the proof. □

Recall that if M is an R-module, $S = End_R(M)$ the ring of R-endomorphisms of M, then an R-submodule X of M is called *fully invariant* provided it is also an S-submodule of M, or equivalently, $f(X) \subseteq X$, for every $f \in S$, see [9]. Also, an R-module M is called a *duo* module provided every submodule of M is fully invariant. A source of duo modules is provided by uniserial modules (see [16, Theorem 1.2]). A module M is called uniserial if, for all submodules L and N of M, either $L \subseteq N$ or $N \subseteq L$, see [16].

In the following, we investigate one condition under which a finite direct sum of modules satisfying the small condition, also satisfies the small condition.

Theorem 4.10. *Let $M = \bigoplus_{i=1}^{n} M_i$ be a duo module. If every M_i satisfies the small condition, then so does M.*

Proof. It suffices to prove the theorem for $n = 2$. Let $M = M_1 \oplus M_2$, such that M_1 and M_2 be submodules of M with the small condition and K be a small submodule of M. Since M is a duo module, $K = A_1 \oplus A_2$, such that $A_1 = K \cap M_1 \subseteq M_1$ and $A_2 = K \cap M_2 \subseteq M_2$, see [16, Lemma 2.1]. Hence, $A_1 \ll M_1$ and $A_2 \ll M_2$, by Lemma 2.1 (5). Since M_1 and M_2 satisfy the small condition, so $n\text{-dim}\, A_1 < n\text{-dim}\, M_1$ and $n\text{-dim}\, A_2 < n\text{-dim}\, M_2$, thus $n\text{-dim}\, K = \sup\{n\text{-dim}\, A_1, n\text{-dim}\, A_2\} < \sup\{n\text{-dim}\, M_1, n\text{-dim}\, M_2\} = n\text{-dim}\, M_1 \oplus M_2 = n\text{-dim}\, M$. Therefore, $n\text{-dim}\, K < n\text{-dim}\, M$ and consequently, M satisfies the small condition. □

Theorem 4.11. *Let $M = \bigoplus_{i=1}^{n} M_i$ satisfies the small condition. Then:*

(1) *There exists at least one i such that M_i satisfies the small condition.*
(2) *For every $1 \leq i \leq n$, either M_i or $\frac{M}{M_i}$ satisfies the small condition.*
(3) *There exists a submodule N of M which N satisfies the small condition and $n\text{-dim}\, \frac{M}{N} < n\text{-dim}\, M$.*

Proof. (1) On the contrary, assume that none of the M_i's satisfy the small condition. By Lemma 4.6, $n\text{-dim}\, M_i < n\text{-dim}\, M$, so by Corollary 2.12, $n\text{-dim}\, M = \sup\{n\text{-dim}\, M_i : \text{for all } i,\ 1 \leq i \leq n\} < n\text{-dim}\, M$, that is, $n\text{-dim}\, M < n\text{-dim}\, M$, which is impossible.

(2) Let M_i does not satisfy the small condition. Then, by Lemma 4.6, $n\text{-dim}\, M_i < n\text{-dim}\, M$, so $n\text{-dim}\, M = \sup\{n\text{-dim}\, M_i,\ n\text{-dim}\, \frac{M}{M_i}\} = n\text{-dim}\, \frac{M}{M_i}$, by Lemma 2.11. Hence by Lemma 4.6, $\frac{M}{M_i}$ satisfies the small condition. Similarly, let $\frac{M}{M_i}$ does not satisfy the small condition. Then by Lemma 4.6, $n\text{-dim}\, \frac{M}{M_i} < n\text{-dim}\, M$, so $n\text{-dim}\, M = \sup\{n\text{-dim}\, M_i,\ n\text{-dim}\, \frac{M}{M_i}\} = n\text{-dim}\, M_i$, by Lemma 2.11. Hence, by Lemma 4.6, M_i satisfies the small condition.

(3) Put $X = \{M_i : n\text{-dim}\, M_i = n\text{-dim}\, M, 1 \leq i \leq n\}$ and $Y = \{M_i : n\text{-dim}\, M_i < n\text{-dim}\, M, 1 \leq i \leq n\}$. Let $N = \bigoplus_{M_i \in X} M_i$ and $K = \bigoplus_{M_i \in Y} M_i$, so $M = N \oplus K$. By Corollary 2.12, $n\text{-dim}\, N = n\text{-dim}\, M$ and $n\text{-dim}\, K < n\text{-dim}\, M$. Therefore, by Lemma 4.6, N satisfies the small condition. Also, $n\text{-dim}\, \frac{M}{N} = n\text{-dim}\, K < n\text{-dim}\, M$.

□

Recall that *Jacobson radical* $J(M)$ of an R-module M is the sum of all its small submodules. In particular, the Jacobson radical $J(R)$ of R, is the intersection of all maximal right ideals of R. If $J(R) = 0$, then R is said to be a Jacobson semisimple ring (Jacobson semisimple rings are also referred to as J-semisimple rings and they are sometimes called semiprimitive rings).
It is clear that every module M with $J(M) = 0$, satisfies the small condition.

Lemma 4.12. *Let M be a Noetherian module which satisfies the small condition. Then $J(M) = 0$.*

Proof. Let S be a small submodule of M. Then $n\text{-dim}\, S < n\text{-dim}\, M$, but M is Noetherian, that is, $n\text{-dim}\, M = 0$ and this implies that $n\text{-dim}\, S = -1$. Thus, the only small submodule of M is zero, therefore $J(M) = 0$. □

Recall that an R-module M is *local* if it has exactly one maximal submodule that contains all of its proper submodules. Also, a ring R is called local if it has only one maximal right ideal, see [9].

Theorem 4.13. *Let R be a ring.*

(1) *If R is right Noetherian, then R satisfies the small condition if and only if R is a Jacobson semisimple ring.*
(2) *If R is local and non-Noetherian, then it does not satisfy the small condition.*

Proof. (1) Using the previous lemma, it is clear.

(2) Let $\mathfrak{m}$ be the only maximal right ideal of R. Clearly, $n\text{-dim}\, \frac{R}{\mathfrak{m}} = 0$. Moreover, $n\text{-dim}\, R \neq 0$, because R is not Noetherian, according to Lemma 2.11, $n\text{-dim}\, R = n\text{-dim}\, \mathfrak{m}$. But $\mathfrak{m} = J(R)$ is a small right ideal of R, so R does not satisfy the small condition.

□

Corollary 4.14. *Let M be a local module which satisfies the small condition. Then M is Noetherian.*

Let M be a module with $k\text{-dim}\, M = \alpha$ and M_0 be a closed submodule of M with $k\text{-dim} M_0 < \alpha$. If $\frac{M}{M_0}$ satisfies the large condition, then so does M, see [3]. In what follows, we prove the dual of this fact.

Proposition 4.15. *Let M be an R-module and M_0 be a coclosed submodule of M with $n\text{-dim}\, \frac{M}{M_0} < n\text{-dim}\, M$. If M_0 satisfies the small condition, then so does M.*

Proof. Let S be a small submodule of M and $n\text{-dim}\, M = \alpha$. By Lemma 2.1(1), $S \cap M_0 \ll M$ and hence, by Proposition 2.6, $S \cap M_0 \ll M_0$, thus $n\text{-dim}\,(S \cap M_0) < n\text{-dim}\, M_0$. But, $n\text{-dim}\, M_0 = n\text{-dim}\, M = \alpha$, by Lemma 2.11, so $n\text{-dim}\,(S \cap M_0) < \alpha$. As $\frac{S}{S \cap M_0} \simeq \frac{S+M_0}{M_0}$, it follows that $n\text{-dim}\, \frac{S}{S \cap M_0} = n\text{-dim}\, \frac{S+M_0}{M_0} \leq n\text{-dim}\, \frac{M}{M_0} < \alpha$. Therefore, according to Lemma 2.11, $n\text{-dim}\, S < \alpha$. Consequently, M satisfies the small condition. □

We recall that if M is an R-module, then a finite chain of submodules $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = (0)$ of M is said to be atomic if each factor module $\frac{M_i}{M_{i+1}}$ is atomic, see [12].

Theorem 4.16. *Let M be a semiatomic R-module. Then:*

(1) *M has Noetherian dimension.*

(2) *Every non-zero factor module of M is semiatomic.*
(3) *Every non-zero factor module of M has atomic chain.*

Proof. Since M is semiatomic, there exists a finite collection of atomic modules $A_1, A_2, \cdots, A_n$, such that $M = \sum_{i=1}^{n} A_i$.

(1) It is obvious.
(2) Let B be a proper submodule of M. By Proposition 2.13, $\frac{A_i}{A_i \cap B}$ is atomic, and hence $\frac{A_i+B}{B}$ is atomic, because $\frac{A_i}{A_i \cap B} \simeq \frac{A_i+B}{B}$. Now, we have $\frac{M}{B} = \frac{\sum_{i=1}^{n} A_i+B}{B} = \sum_{i=1}^{n} \frac{A_i+B}{B}$. This implies that $\frac{M}{B}$ is semiatomic.
(3) Let N be a proper submodule of M and consider the chain $0 \neq \frac{M}{N} = \frac{\sum_{i=1}^{n} A_i+N}{N} \supseteq \frac{\sum_{i=2}^{n} A_i+N}{N} \supseteq \cdots \supseteq \frac{N}{N} = 0$. Now, if $\frac{N+A_1+\cdots+A_j}{N} \neq \frac{N+A_1+\cdots+A_j+A_{j+1}}{N}$, then $\frac{N+A_1+\cdots+A_j+A_{j+1}/N}{N+A_1+\cdots+A_j/N} \simeq \frac{N+A_1+\cdots+A_j+A_{j+1}}{N+A_1+\cdots+A_j} \simeq \frac{A_{j+1}}{(N+A_1+\cdots+A_j)\cap A_{j+1}}$. By Proposition 2.13, $\frac{A_{j+1}}{(N+A_1+\cdots+A_j)\cap A_{j+1}}$ is atomic. Thus, after identifying equal members of the above sequence, we have an atomic chain.

□

It is easy to check that every quotient module of a D-module is a D-module. Also, every uniform submodule of a semicritical module is critical and every non-zero submodule of a semicritical module satisfies the large condition, see [4, Theorems 2.1, 2.4, Corollary 2.3]. The following result is the dual of this fact for a semiatomic D-module.

Theorem 4.17. *Let M be a semiatomic* D-module. *Then:*

(1) *Every quotient module of M satisfies the small condition.*
(2) *Every hollow quotient module of M is atomic.*

Proof. (1) In view of Theorems 4.16(2) and 4.5, it is evident.
(2) Let N be a submodule of M such that $\frac{M}{N}$ is hollow. It suffices to prove the n-dim $\frac{L}{N} <$ n-dim $\frac{M}{N}$, for every proper submodule $\frac{L}{N}$ of $\frac{M}{N}$. Since $\frac{M}{N}$ is hollow, $\frac{L}{N}$ is a small submodule of it. As M is a semiatomic D-module, by the part (1), $\frac{M}{N}$ satisfies the small condition. Hence n-dim $\frac{L}{N} <$ n-dim $\frac{M}{N}$ and the proof is complete.

□

It is known that if M is a module with Krull dimension and $\alpha = \sup\{k\text{-dim}\frac{M}{E} : E \subseteq_e M\}$, then $\alpha \leq k\text{-dim}\, M \leq \alpha + 1$, see [6, Corollary 1.5]. The following is devoted to modules with Krull dimension which have finite hollow dimension.

Theorem 4.18. *Let M be an R-module with Krull dimension which has finite hollow dimension. Then $\alpha \leq k\text{-dim}\, M \leq \alpha + 1$, where $\alpha = \sup\{k\text{-dim}\, S : S \ll M\}$.*

Proof. Since k-dim $M = \sup\{k\text{-dim}\, N : N$ is a submodule of $M\}$, it follows that $\alpha \leq k\text{-dim}\, M$. It suffices to show that $k\text{-dim}\, M \leq \alpha + 1$. Let $M_1 \supseteq M_2 \supseteq \cdots$ be a descending chain of submodules of M. As M has finite hollow dimension, by Proposition 2.3(4), there exists an integer k such that $\frac{M_k}{M_i} \ll \frac{M}{M_i}$, for every $i \geq k$. But $\frac{M_i}{M_{i+1}} \subseteq \frac{M_k}{M_{i+1}}$ implies that $\frac{M_i}{M_{i+1}} \ll \frac{M}{M_{i+1}}$. By Lemma 2.5(4), M is weakly supplemented, so there exists a submodule A of M such that $M_{i+1}+A = M$ and $M_{i+1} \cap A \ll M$. Let $B = M_{i+1} \cap A$, so $\frac{M}{M_{i+1}} = \frac{M_{i+1}+A}{M_{i+1}} \simeq \frac{A}{B}$. By applying Lemma 2.1(1, 2, 3), for all small submodules $\frac{S}{M_{i+1}}$ of $\frac{M}{M_{i+1}}$ there exists a small submodule $\frac{S'}{B} \ll \frac{A}{B} \subseteq \frac{M}{B}$ such that $\frac{S}{M_{i+1}} \simeq \frac{S'}{B}$, then $S' \ll M$. Therefore, $k\text{-dim}\, \frac{S}{M_{i+1}} = k\text{-dim}\, \frac{S'}{B} \leq k\text{-dim}\, S' \leq \alpha$ for all $\frac{S}{M_{i+1}} \ll \frac{M}{M_{i+1}}$ specially, $k\text{-dim}\, \frac{M_i}{M_{i+1}} \leq \alpha < \alpha + 1$. It follows that $k\text{-dim}\, M \leq \alpha + 1$.

□

Proposition 4.19. *Let M be an R-module with finite hollow dimension and $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ be any ascending chain of submodules of M. Then there exists an integer k such that $\frac{M_i}{M_k} \ll \frac{M}{M_k}$ for all $i \geq k$.*

Proof. By Lemma 2.5(1), $h\text{-dim}\, \frac{M}{M_{i+1}} \leq h\text{-dim}\, M < \infty$. Thus, there exists an integer k such that $h\text{-dim}\, \frac{M}{M_k} = h\text{-dim}\, \frac{M}{M_i} = h\text{-dim}\, \frac{M/M_k}{M_i/M_k}$ for all $i \geq k$. Therefore, by Lemma 2.5(3), $\frac{M_i}{M_k} \ll \frac{M}{M_k}$. □

Theorem 4.20. *Let M be an R-module with Noetherian dimension and finite hollow dimension. Then $n\text{-dim}\, M \leq \alpha + 1$, where $\alpha = \sup\{n\text{-dim}\, S : S \ll M\}$.*

Proof. Assume, to the contrary, that $n\text{-dim}\, M > \alpha + 1$. So, there exists an ascending chain $M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M such that $n\text{-dim}\, \frac{M_{i+1}}{M_i} \geq \alpha + 1$, for all i. By the previous proposition, there exists an integer k such that $\frac{M_i}{M_k} \ll \frac{M}{M_k}$ for every $i \geq k$. If $M_k \ll M$, then $M_i \ll M$ for every $i \geq k$. Thus, $n\text{-dim}\, M_i \leq \alpha$, so $n\text{-dim}\, \frac{M_{i+1}}{M_i} \leq \alpha$, which is impossible. Now, assume that $M_k \not\ll M$. By applying Lemma 2.5(4), M is weakly supplemented, hence there exists a submodule A of M such that, $A + M_k = M$ and $B = A \cap M_k \ll M$. We have $\frac{M}{M_k} \simeq \frac{A}{B}$. Now, by Lemma 2.1(1, 2, 3), we infer that, for all small submodules $\frac{S}{M_k}$ of $\frac{M}{M_k}$, there exists a submodule $\frac{S'}{B} \ll \frac{A}{B} \subseteq \frac{M}{B}$ such that $\frac{S}{M_k} \simeq \frac{S'}{B}$, then $S' \ll M$. Consequently, $n\text{-dim}\, \frac{S}{M_k} = n\text{-dim}\, \frac{S'}{B} \leq n\text{-dim}\, S' \leq \alpha$, that is, every small submodule of $\frac{M}{M_k}$ has Noetherian dimension less than or equal to α. Hence, $n\text{-dim}\, \frac{M_{i+1}}{M_i} = n\text{-dim}\, \frac{M_{i+1}/M_k}{M_i/M_k} \leq n\text{-dim}\, \frac{M_{i+1}}{M_k} \leq \alpha$, which is a contradiction. □

The following is a dual version [6, Corollary 1.5].

Corollary 4.21. *Let M be an R-module with finite hollow dimension which satisfies the small condition. Then $\alpha \leq n\text{-dim}\, M \leq \alpha+1$, where $\alpha = \sup\{n\text{-dim}\, S : S \ll M\}$.*

Corollary 4.22. *Let M be an R-module with finite hollow dimension which satisfies the small condition. If the Noetherian dimension of M is a limit ordinal, then $n\text{-}\dim M = \sup\{n\text{-}\dim S : S \ll M\}$.*

Recall that an epimorphism $f : M \to N$ is called small if $\mathrm{Ker}(f) \ll M$. Obviously, $K \ll M$ if and only if the canonical projection $g : M \to \frac{M}{K}$ is a *small epimorphism*, see [15]. A *projective cover* of an R-module M is a projective module $P(M)$ together with a small epimorphism $f : P(M) \to M$, see [19, Definition 19.4]. We note that every module does not necessarily have a projective cover. For example, $\mathbb{Z}_{p^\infty}$ and $\mathbb{Q}$ as $\mathbb{Z}$-modules have no projective cover, because they have no maximal submodule, see [2, Proposition 7.2.8]. According to Theorem 4.16(2), it is clear that if $P(M)$ is a semiatomic module, then so does M.

Proposition 4.23. *Let M be an R-module with projective cover $P(M)$. There is a one-one correspondence between small submodules of M and small submodules of $P(M)$.*

Proof. Since $P(M)$ is the projective cover of M, then there exists an epimorphism $f : P(M) \to M$ such that $\frac{P(M)}{\mathrm{Ker}(f)} \simeq M$ and $\mathrm{Ker}(f) \ll P(M)$. Let S be a small submodule of M. It can be defined an isomorphism $g : M \to \frac{P(M)}{\mathrm{Ker}(f)}$ such that $g(S) = \frac{N}{\mathrm{Ker}(f)} \ll \frac{P(M)}{\mathrm{Ker}(f)}$, and hence, by Lemma 2.1(1,2), $N \ll P(M)$. Now, we assume that $L \ll P(M)$. Since $\mathrm{Ker}(f)$ is a small submodule of $P(M)$, by Lemma 2.1(1, 4), $L + \mathrm{Ker}(f) \ll P(M)$ and $\frac{L+\mathrm{Ker}(f)}{\mathrm{Ker}(f)} \ll \frac{P(M)}{\mathrm{Ker}(f)} \simeq M$, and hence we are done. □

Lemma 4.24. *Let M and M' be isomorphic modules with projective covers $P(M)$ and $P(M')$, respectively. Then $P(M) \simeq P(M')$.*

Proof. Let $f : P(M) \to M$ is a small epimorphism and $g : M \to M'$ is an isomorphism. Thus $gf : P(M) \to M'$ is a small epimorphism, see [19, 19.3(1)]. It follows that $P(M)$ is a projective cover of M', and by [2, Proposition 7.2.2], $P(M) \simeq P(M')$. □

Using Proposition 4.8, can be adapted to prove the following result.

Proposition 4.25. *Let M be an R-module with projective cover $P(M)$.*

(1) *If $P(M)$ satisfies the small condition, then so does M.*
(2) *Let $\frac{P(M)}{S} \simeq M$ for small submodule S of $P(M)$ and $n\text{-}\dim S < n\text{-}\dim M$. Then M satisfies the small condition if and only if $P(M)$ satisfies the small condition.*

Proposition 4.26. *Let M be an R-module with projective cover $P(M)$. Then $P(M) = P(\frac{M}{S})$, for every small submodule S of M.*

Proof. Let $f : P(M) \to M$ be a small epimorphism and $g : M \to \frac{M}{S}$ be the canonical projection. Thus, by [19, 19.3(1)], $gf : P(M) \to \frac{M}{S}$ is a small epimorphism. Consequently, $P(M)$ is a projective cover of $\frac{M}{S}$.

□

By Lemma 2.5(2) and the previous proposition, the following fact is evident.

Proposition 4.27. *Let M be an R-module with projective cover $P(M)$. Then $h\text{-}\dim M = h\text{-}\dim P(M)$.*

Theorem 4.28. *Let M be an R-module with finite hollow dimension and projective cover $P(M)$. Then $P(M) \simeq \bigoplus_{i=1}^{n} P(K_i)$ in which K_i is hollow for all $1 \leq i \leq n$.*

Proof. Since M has finite hollow dimension, by Proposition 2.7, there exists a coindependent collection of submodules $\{N_1, N_2, \cdots, N_n\}$ such that $S = \bigcap_{i=1}^{n} N_i \ll M$, $\frac{M}{S} \simeq \bigoplus_{i=1}^{n} \frac{M}{N_i}$ and $\frac{M}{N_i}$ is hollow. Now, by Proposition 4.26 and [19, 19.5 (5)], we have $P(M) = P(\frac{M}{S}) \simeq P(\bigoplus_{i=1}^{n} \frac{M}{N_i}) = \bigoplus_{i=1}^{n} P(\frac{M}{N_i})$ and the proof is complete.

□

Corollary 4.29. *Let M be an R-module with finite hollow dimension and projective cover $P(M)$.*

(1) *If every hollow projective module has Noetherian dimension, then so does M.*
(2) *If every hollow projective module is atomic, then M is semiatomic. Moreover, if M is a D-module, then M satisfies the small condition.*

Proof. In view of the previous theorem, we have $P(M) \simeq \bigoplus_{i=1}^{n} P(K_i)$ in which K_i is hollow for all $1 \leq i \leq n$.

(1) By hypothesis $P(K_i)$ has Noetherian dimension, so by Corollary 2.12, $P(M)$ has Noetherian dimension. Since $\frac{P(M)}{S} \simeq M$ for some $S \ll P(M)$, thus M has Noetherian dimension.
(2) By hypothesis $P(K_i)$ is atomic, thus $P(M)$ is semiatomic. Consequently, by Theorem 4.16(2), $\frac{P(M)}{S} \simeq M$ is semiatomic for some $S \ll P(M)$. Now, if M is D-module, by Theorem 4.5, M satisfies the small condition.

□

If M is a module with finite Goldie dimension and M_0 is a closed submodule of M, then $E(\frac{M}{M_0}) \simeq \frac{E(M)}{E(M_0)}$, see [3, Lemma 2.3]. The following is a dual of this fact for modules with finite hollow dimension.

Proposition 4.30. *Let M be an R-module with finite hollow dimension and projective cover $P(M)$ and M_0 be a coclosed submodule of M. Then $P(\frac{M}{M_0}) \simeq \frac{P(M)}{P(M_0)}$.*

Proof. By Lemma 2.5(4), M is weakly supplemented, so there exists a submodule K of M such that $M_0+K=M$ and $M_0\cap K \ll M$, hence $P(M)=P(\frac{M}{M_0\cap K})$, by Proposition 4.26. Moreover, by Proposition 2.6, $M_0\cap K \ll M_0$, so $P(M_0)=P(\frac{M_0}{M_0\cap K})$. Now, we have $\frac{M}{M_0\cap K}=\frac{M_0+K}{M_0\cap K}\simeq\frac{M_0}{M_0\cap K}\oplus\frac{K}{M_0\cap K}\simeq\frac{M}{K}\oplus\frac{M}{M_0}$. Hence, Lemma 4.24 and [19, 19.5 (5)] imply that $P(M)=P(\frac{M}{M_0\cap K})\simeq P(\frac{M}{K})\oplus P(\frac{M}{M_0})$. But, by Lemma 4.24, $P(\frac{M}{K})=P(\frac{M_0+K}{K})\simeq P(\frac{M_0}{M_0\cap K})=P(M_0)$. Therefore, $P(M)\simeq P(M_0)\oplus P(\frac{M}{M_0})$ and we infer that $P(\frac{M}{M_0})\simeq\frac{P(M)}{P(M_0)}$. □

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7. Conflict of interest

The authors declare no conflict of interes.

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