Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

THE ADJACENCY DIMENSION OF SOME PATH RELATED TREES

E. Hardani and A. Behtoei ✉

Article type: Research Article
(Received: 30 January 2024, Received in revised form 28 September 2024)
(Accepted: 01 November 2024, Published Online: 07 November 2024)

Abstract. Since the problem of computing the adjacency dimension of a graph is NP-hard, finding the adjacency dimension of special classes of graphs or obtaining good bounds on this invariant is valuable. In this paper we determine the properties of each adjacency resolving set of paths. Then, by using these properties, we determine the adjacency dimension of broom and double broom graphs.

Keywords: Adjacency resolving set, Adjacency dimension, Path, Broom, Tree.
2020 MSC: Primary 05C05, 05C12, 05C38.

1. Introduction

Throughout this paper, we only consider simple graphs. At first, we collect some standard graph-theoretic terminologies and notations in this section, see [10] and [21]. Locating or resolving sets are introduced as a graph-theoretic model of robot navigation and have different applications in diverse areas like network discovery, computer science and chemistry, see [18, 19, 22]. These applications lead to some graph parameters, like the metric dimension and the adjacency dimension. Let $\mathbb{N}$ denote the set of all positive integers. Given a connected graph $G = (V, E)$ with vertex set V and edge set E, consider the function $d_G : V \times V \to \mathbb{N} \cup \{0\}$ where $d_G(x, y)$ is the length of the shortest path between two vertices x and y in G. Clearly, (V, d_G) is a metric space and the diameter of G is understood in this metric. An ordered vertex set $S \subseteq V$ is said to be a metric generator for G if it is a generator of the metric space (V, d_G), i.e., each point of the space is uniquely determined by its distances from the elements of S. A minimum metric generator is called a metric basis, and its cardinality is the metric dimension of G, denoted by $dim(G)$. Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of the metric dimension of a graph was introduced by Slater in [24], where the metric generators were called locating sets. The concept of the metric dimension of a graph was also introduced independently by Harary and Melter in [11], where metric generators were called resolving sets. It is straightforward

✉ a.behtoei@sci.ikiu.ac.ir, ORCID: 0000-0003-2113-0210
https://doi.org/10.22103/jmmr.2024.22886.1576
Publisher: Shahid Bahonar University of Kerman

How to cite: E. Hardani, A. Behtoei, *The adjacency dimension of some path related trees*, J. Mahani Math. Res. 2025; 14(1): 369-386.

to see that when $n \geq 2$, for the complete graph K_n and the path P_n we have $dim(K_n) = n-1$ and $dim(P_n) = 1$, respectively. In [2], it is shown that for the wheel graph we have $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$. This graph parameter was studied further in a number of other papers including [3,5,12,13,22,25]. Several variations of metric generators including resolving dominating sets [1], independent resolving sets [6], local resolving sets [20], 1-locating dominating sets [4], strong resolving sets [23], etc. have since been introduced and studied. Now consider the distance function $d_2 : V \times V \to \mathbb{N} \cup \{0\}$, where $d_2(x, y) = min\{d_G(x, y), 2\}$. Let $S = \{s_1, s_2, ..., s_k\}$ be a non-empty ordered subset of $V = V(G)$. For each $v \in V(G)$, the k-tuple $r_a(v|S) = (d_2(v, s_1), d_2(v, s_2), ..., d_2(v, s_k))$ is called the adjacency representation of v with respect to S, and S is an adjacency resolving set (or an adjacency generator) for G if for each pair of distinct vertices $v_1, v_2 \in V(G)$ we have $r_a(v_1|S) \neq r_a(v_2|S)$. An adjacency resolving set with the minimum cardinality is called an adjacency basis and its cardinality is the adjacency dimension of G which is denoted by $adim(G)$, see [16]. It is easy to show that S is an adjacency resolving set for G if for each pair of different vertices $x, y \in V(G) \setminus S$ there exists $s_i \in S$ which is adjacent to exactly one of these two vertices, that is $|N_G(s_i) \cap \{x, y\}| = 1$, where $N_G(s_i)$ denotes the neighborhood of the vertex s_i in G. Therefore, S is an adjacency resolving set for G if and only if it is an adjacency resolving set for its complement $\overline{G}$, and consequently $adim(G) = adim(\overline{G})$. It is well known that almost all graphs have diameter two. Also, note that for each graph G of diameter at most two, we have $d_2(x, y) = d_G(x, y)$ and hence, $adim(G) = dim(G)$. Thus, for almost all graphs, we can determine their adjacency dimension instead of their metric dimension. Specially, $adim(K_n) = dim(K_n)$ and $adim(W_n) = dim(W_n)$. This concept has been studied further by many scientists. Fernau and Rodriguez-Velazquez in [8] and [9] show that the metric dimension of the corona product of a graph of order n and some nontrivial graph H equals n times the adjacency dimension of H, and they prove that the problem of computing the adjacency dimension is an NP-hard problem. This suggests finding the adjacency dimension for special classes of graphs or obtaining good bounds on this invariant. By using the fact $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ and by a short and non-constructive proof, Jannesari and Omoomi in [16] show that $adim(P_n) = \lfloor \frac{2n+2}{5} \rfloor$ for each $n \geq 2$. In the next section we will determine the properties of each adjacency resolving sets for a path and in a constructive way we will show that $adim(P_n) = \lfloor \frac{2n+2}{5} \rfloor$, which is a confirmation of the previous result and moreover, characteizes all of adjacency basis of each path. Then we use these properties to determine the adjacency dimension of broom and double broom graphs. Recall that for integers $n \geq 1$ and $k \geq 2$, the broom graph $B_{n,k}$ (see Figure 2) is a tree constructed by joining k new pendant vertices to a leaf of an n vertex path, i.e., we can assume that

$$V(B_{n,k}) = \{v_1, v_2, ..., v_n\} \cup \{x_1, x_2, ..., x_k\},$$

and

$$E(B_{n,k}) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\} \cup \{v_nx_1, v_nx_2, ..., v_nx_k\}.$$

Also, the double broom graph $B_{n,k,k}$ is obtained from a path P_n by joining k pendant vertices $\{x_1, x_2, ..., x_k\}$ to v_1 and k pendant vertices $\{x'_1, x'_2, ..., x'_k\}$ to v_n, see Figure 3. To see more results in this subject or related subjects, the reader is referred to [7, 9, 14–17, 21].

2. Adjacency resolving sets of paths

At first, by considering the novel ideas applied in [2], we provide the following definition which will be applied frequently through the paper.

Definition 2.1. Let P_n be a path with vertex set $V(P_n) = \{v_1, v_2, ..., v_n\}$ and edge set $E(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$, and $S = \{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ be a non-empty subset of $V(P_n)$ in which $i_1 < i_2 < \cdots < i_k$. Then, the set $G_0 = \{v_1, v_2, ..., v_{i_1-1}\}$ is called the left semi-gap, $G_k = \{v_{i_k+1}, v_{i_k+2}, ..., v_n\}$ is called the right semi-gap, and for each $1 \leq j \leq k-1$ the set $G_j = \{v_{i_j+1}, v_{i_j+2}, ..., v_{i_{j+1}-1}\}$ is called the j-th gap. Note that a *gap* or a semi-gap may be an empty set. Two consequtive gaps (or semi-gaps or a gap and a semi-gap) are called neighbor of each other.

For example, consider the path P_{17} in Figure 1 in which $S = \{v_1, v_4, v_6, v_9, v_{11}, v_{14}, v_{16}\}$ and the elements of S corresponds to the filled vertices. Thus, we have $G_0 = \emptyset$, $G_1 = \{v_2, v_3\}$, $G_2 = \{v_5\}$, $G_3 = \{v_7, v_8\}$, $G_4 = \{v_{10}\}$, $G_5 = \{v_{12}, v_{13}\}$, $G_6 = \{v_{15}\}$ and $G_7 = \{v_{17}\}$.

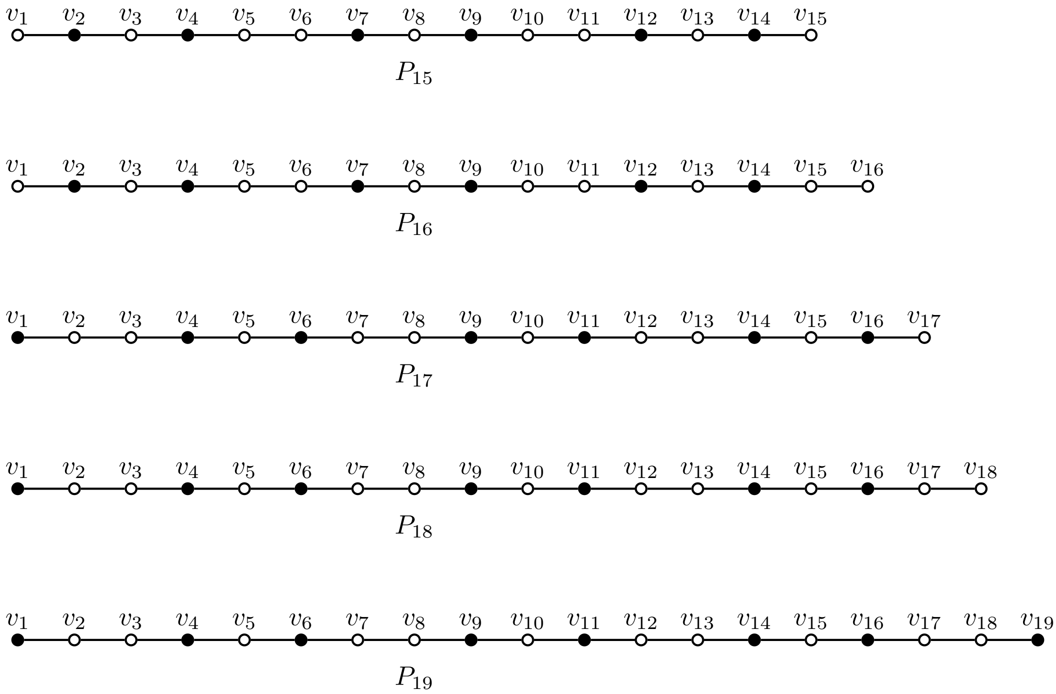

FIGURE 1. An adjacency basis for P_n, $15 \le n \le 19$.

Theorem 2.2. *A set $B \subseteq V(P_n) = \{v_1, v_2, ..., v_n\}$ is an adjacency resolving set for the path P_n if and only if it satisfies the following five properties:*

a) *each gap contains at most three vertices and there exists at most one gap containing three vertices.*
b) *a gap of size at least two, has no neighboring gap of size bigger than one nor a neighboring non-empty semi-gap.*
c) *if there exists a gap with three vertices, then each semi-gap contains at most one vertex.*
d) *each semi-gap contains at most two vertices and there is at most one semi-gap containing two vertices.*
e) *there exist no neighboring non-empty semi-gaps.*

Proof. First, suppose that $B \subseteq V(P_n)$ is an adjacency resolving set for P_n, we show that five properties (a) to (e) are satisfied.

a) If there exists a gap containing more than three vertices, namely $G_i = \{v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}\}$, then $r_a(v_{i+2}|B) = (2, 2, ..., 2) = r_a(v_{i+3}|B)$, which is a contradiction. Now if there exist two gaps containing three vertices, namely $G_i = \{v_{i+1}, v_{i+2}, v_{i+3}\}$ and $G_j = \{v_{j+1}, v_{j+2}, v_{j+3}\}$, then $r_a(v_{i+2}|B) = (2, 2, ..., 2) = r_a(v_{j+2}|B)$ which is contradiction again.

b) If there exist two neighboring gaps containing (at least) two vertices, say $G_i = \{v_{i+1}, v_{i+2}\}$ and $G_{i+1} = \{v_{i+4}, v_{i+5}\}$, then $N_{P_n}[v_{i+2}] \cap B = \{v_{i+3}\} = N_{P_n}[v_{i+4}] \cap B$ which implies that $r_a(v_{i+2}|B) = r_a(v_{i+4}|B)$, a contradiction. Now if there exists a gap containing (at least) two vertices which has a neighboring semi-gap, for convenient say $G_1 = \{v_3, v_4\}$ and $G_0 = \{v_1\}$, then $r_a(v_1|B) = r_a(v_3|B) = (1, 2, ..., 2)$, a contradiction.

c) If there exists a gap containing three vertices and a semi-gap containing two vertices, for instance $G_i = \{v_{i+1}, v_{i+2}, v_{i+3}\}$ and $G_0 = \{v_1, v_2\}$, then we get $r_a(v_1|B) = r_a(v_{i+2}|B) = (2, 2, ..., 2)$, contradiction.

d) If there exists a semi-gap containing three vertices, for example $G_0 = \{v_1, v_2, v_3\}$, then $r_a(v_1|B) = r_a(v_2|B) = (2, 2, ..., 2)$. Also, if there exist two semi-gaps containing two vertices, namely $G_0 = \{v_1, v_2\}$ and $G_k = \{v_{n-1}, v_n\}$, then $r_a(v_1|B) = r_a(v_n|B) = (2, 2, ..., 2)$. Both cases lead to a contradiction.

e) If there exist two neighboring non-empty semi-gaps (and hence, $|B|$=1), then the final vertex of the left semi-gap and the first vertex of the right semi-gap have the same adjacency representation with respect to B, a is contradiction.

Now for the converse, suppose that $B \subseteq V(P_n)$ satisfies five conditions (a) to (e). We show that B is an adjacency resolving set for P_n. Let $u, v \in V(P_n) \setminus B$ be two different vertices. By considering the properties (a) to (e), at most one vertex in P_n has no neighbor in B i.e., has the representation $(2, 2, ..., 2)$ with respect to B and hence, it is enough to check the situation that $N_{P_n}(u) \cap B \neq \emptyset$ and $N_{P_n}(v) \cap B \neq \emptyset$. If $d_{P_n}(u, v) \geq 3$, then $N_{P_n}(u) \cap N_{P_n}(v) = \emptyset$ and hence $N_{P_n}(u) \cap B \neq N_{P_n}(v) \cap B$. If $d_{P_n}(u, v) = 1$, then u, v are two adjacent vertices and hence, they form a gap of size two and their unique neighbors in B are different. Assume that $d_{P_n}(u, v) = 2$ and hence, u, v has a unique neighbor, say w. If $w \in B$, then u and v belong to two neighboring gaps (or a gap and a neighboring semi-gap) and by property (b) at least one of them has a neighbor in $B \setminus \{w\}$, which implies that $N_{P_n}(u) \cap B \neq N_{P_n}(v) \cap B$. If $w \notin B$, then u, v, w form a gap of size three and each of u and v has a unique (and different) neighbor in B, which completes the proof. □

Lemma 2.3. *If $n \geq 2$, then $adim(P_n) \leq \lfloor \frac{2n+2}{5} \rfloor$.*

Proof. We consider the following five cases (for convenient, Figure 1 provides a typical example for each case).

Case 1. $n \equiv 0 \pmod 5$.
Assume that $n = 5k$ in which $k \geq 1$. Let $B = \{v_{5i+2}, v_{5i+4} : 0 \leq i \leq k-1\}$. It is easy to check that B satisfies properties (a) to (e) in Theorem 2.2 (see P_{15} in Figure 1) and hence, it is an adjacency resolving set for P_n. Thus

$$adim(P_n) \leq |B| = 2k = \left\lfloor \frac{2(5k)+2}{5} \right\rfloor = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Case 2. $n \equiv 1 \pmod 5$.
Suppose that $n = 5k+1$ with $k \geq 1$. Let $B = \{v_{5i+2}, v_{5i+4} : 0 \leq i \leq k-1\}$. B satisfies the five properties of Theorem 2.2 (see P_{16} in Figure 1) and hence, it is an adjacency resolving set for P_n. Thus

$$adim(P_n) \leq |B| = 2k = \left\lfloor \frac{2(5k+1)+2}{5} \right\rfloor = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Case 3. $n \equiv 2 \pmod 5$.
Assume that $n = 5k+2$, $k \geq 1$ and let $B = \{v_{5i+1}, v_{5i+4} : 0 \leq i \leq k-1\} \cup \{v_{5k+1}\}$. By Theorem 2.2, B is an adjacency resolving set for P_n. Thus

$$adim(P_n) \leq |B| = 2k+1 = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Case 4. $n \equiv 3 \pmod 5$.
Suppose that $n = 5k+3$, $k \geq 1$, and let $B = \{v_{5i+1}, v_{5i+4} : 0 \leq i \leq k-1\} \cup \{v_{5k+1}\}$. Theorem 2.2 implies that B is an adjacency resolving set and, hence

$$adim(P_n) \leq |B| = 2k+1 = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Case 5. $n \equiv 4 \pmod 5$.
With the assumption $n = 5k+4$, $k \geq 1$, let $B = \{v_{5i+1}, v_{5i+4} : 0 \leq i \leq k\}$. Since B is an adjacency resolving set for P_n, see Theorem 2.2, we get

$$adim(P_n) \leq |B| = 2k+2 = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Now the proof is complete. □

Theorem 2.4. *Let P_n be a path of order $n \geq 2$. Then we have*

$$adim(P_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Proof. By using Lemma 2.3 we see that $adim(P_n) \leq \lfloor \frac{2n+2}{5} \rfloor$. Thus, in order to complete the proof, it is sufficient to show that $adim(P_n) \geq \lfloor \frac{2n+2}{5} \rfloor$. Assume that $B \subseteq V(P_n)$ is an adjacency basis for P_n. We want to show that $|B| \geq \lfloor \frac{2n+2}{5} \rfloor$. Consider the following two cases.

Case i. $|B|$ is an even integer.
Assume that $|B| = 2l$, where $l \in \mathbb{N}$. First assume that both relating semi-gaps are empty. Thus, we have $2l-1 = l+(l-1)$ gaps. By considering the property (b) in Theorem 2.2 and by the Pigeonhole Principle, there exist at most l gaps with at least 2 vertices. By property (a) in Theorem 2.2, at most one of them may contains three vertices. Thus,

$$n - 2l = n - |B| = |V(P_n) \setminus B| \leq l \times 2 + (l-1) \times 1 + 1 = 3l.$$

This implies that $n \leq 5l$ and hence,

$$|B| = 2l = \left\lfloor \frac{10l+2}{5} \right\rfloor \geq \left\lfloor \frac{2n+2}{5} \right\rfloor.$$

Next, assume that there exists just one non-empty semi-gap. Note that by property (d) in Theorem 2.2 this non-empty semi-gap contains at most two vertices, and if it contains two vertices then by property (c) each gap contains at most two vertices. By property (b) the neighboring gap of this semi-gap contains at most one vertex. Hence, there exist at most $l-1$ gaps of size bigger than one. Thus

$$n - 2l = n - |B| = |V(P_n) \setminus B| \leq (l-1) \times 2 + l \times 1 + 1 + 1 = 3l.$$

Therefore $n \leq 5l$ and hence, $\lfloor \frac{2n+2}{5} \rfloor \leq |B|$.

Finally, assume that both semi-gaps are non-empty. Similar to the previous situation, there exist at most $l-1$ gaps containing at least two vertices, and if there exist a semi-gap of size two, then no gap of size three exists. Thus,

$$n - 2l = n - |B| = |V(P_n) \setminus B| \leq 2(l-1) + 1(l) + 2 + 1 = 3l + 1,$$

which implies that $n \leq 5l + 1$ and

$$\left\lfloor \frac{2n+2}{5} \right\rfloor \leq \left\lfloor \frac{10l+2+2}{5} \right\rfloor = 2l = |B|.$$

Case ii. $|B|$ is an odd integer.
Assume that $|B| = 2l + 1$, where $l \in \mathbb{N}$ and hence, we have $2l$ gaps. First assume that both relating semi-gaps are empty.

By considering two properties (a) and (b) of Theorem 2.2 and by the Pigeonhole Principle, there exist at most l gaps with at least 2 vertices. Since at most one of them may contains three vertices, we have

$$n - (2l+1) = n - |B| = |V(P_n) \setminus B| \leq 2l + 1l + 1 = 3l + 1.$$

This means that $n \leq 5l + 2$ and hence,

$$\left\lfloor \frac{2n+2}{5} \right\rfloor \leq \left\lfloor \frac{10l+4+2}{5} \right\rfloor = 2l + 1 = |B|.$$

Next, assume that there exists just one non-empty semi-gap. Then, there exist at most l gaps containing at least two vertices. Also, a gap of size three and a semi-gap of size two may not occur at the same time. Thus

$$|V(P_n) \setminus B| = n - |B| \leq 2(l) + 1(l) + 1 + 1 = 3l + 2.$$

Thus $n \leq 5l + 3$ and $\lfloor \frac{2n+2}{5} \rfloor \leq |B|$.

Finally, assume that there exist two non-empty semi-gaps. Thus there exist at most $l-1$ gaps containing at least two vertices and we have

$$n-(2l+1)=n-|B|=|V(P_n)\setminus B|\leq 2(l-1)+1(l+1)+2+1=3l+2,$$

which implies that $n\leq 5l+3$ and $\lfloor\frac{2n+2}{5}\rfloor\leq|B|$.
Hence, in all of these situations we have $adim(P_n)=|B|\geq\lfloor\frac{2n+2}{5}\rfloor$ and therefore

$$adim(P_n)=\left\lfloor\frac{2n+2}{5}\right\rfloor,$$

as desired. □

Note that if we add a pendant edge to a non-leaf vertex of P_4, then the adjaceny dimension of the resulting 5-vertex graph is $2=\lfloor\frac{2\times5+2}{5}\rfloor$ but it is not (isomorphic to) P_5. The following result will be applied frequently for determining the adjacency dimension of broom and double broom trees.

Theorem 2.5. *Let P_n be a path with $n\geq 3$ vertices and $n\equiv 1$ or $3 \pmod 5$. Then for each (ordered) adjacency basis B of P_n, there exists a vertex in $V(P_n)$ whose adjacency representation with respect to B is $(2,2,...,2)$.*

Proof. Let B be an (ordered) adjacency basis of P_n and assume on the contrary that for each $v\in V(P_n)$, $r_a(v|B)\neq(2,2,...,2)$. Thus, by Theorem 2.2, each gap contains at most two vertices and each semi-gap contains at most one vertex. We first investigate the case $n\equiv 1 \pmod 5$. Let $n=5k+1$, where $k\geq 1$. Therefore,

$$|B|=adim(P_n)=\left\lfloor\frac{2n+2}{5}\right\rfloor=2k.$$

By considering Theorem 2.2, and similar to the proof of Theorem 2.4, we see that:
i) If both semi-gaps are empty, then

$$n-2k=n-|B|=|V(P_n)\setminus B|\leq 2(k)+1(k-1)=3k-1,$$

which implies that $|B|\geq 2k+2$. This contradicts the fact $|B|=2k$.
ii) If exactly one semi-gap is empty, then

$$n-2k=|V(P_n)\setminus B|\leq 2(k-1)+1(k)+1=3k-1,$$

which implies that $|B|\geq 2k+2$, a contradiction.
iii) If both semi-gaps are non-empty, then

$$n-2k\leq 2(k-1)+1(k)+2=3k,$$

and hence, $|B|\geq 2k+1$, a contradiction.
Since each of these situations leads to a contradiction, the proof is completed for the case $n\equiv 1 \pmod 5$.

Now suppose that $n \equiv 3 \pmod 5$. Let $n = 5k + 3$ where $k \geq 0$. Hence,

$$|B| = adim(P_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor = 2k + 1.$$

Similarly, By using Theorem 2.2 and the proof of Theorem 2.4, we see that:
i) If both semi-gaps are empty, then

$$n - 2k - 1 \leq 2(k) + 1(k) = 3k,$$

and hence, $|B| \geq 2k + 3$, which contradicts the fact $|B| = 2k + 1$.
ii) If exactly one semi-gap is empty, then $n - 2k - 1 \leq 2(k) + 1(k) + 1 = 3k + 1$ and hence, $|B| \geq 2k + 2$, a contradiction.
iii) If both semi-gaps are non-empty, then $n - 2k - 1 \leq 2(k-1) + 1(k+1) + 2 = 3k + 1$ and $|B| \geq 2k + 2$, a contradiction.
The proof is completed. □

Note that if $n \not\equiv 1$ or $3 \pmod 5$, then there exists an (ordered) adjacency basis for the path P_n in which no vertex received the adjacency representation $(2, 2, ..., 2)$, see Figure 1. Also, in these cases, if we replace the first element of B with v_3, then the adjacency representation of v_1 will be $(2, 2, ..., 2)$. Thus, the converse of Theorem 2.5 is not valid in general.

3. Adjacency dimension of $B_{n,k}$ and $B_{n,k,k'}$

By using the previous results, in the following we will determine the adjacency dimension of brooms and double brooms.

Lemma 3.1. *If $n \geq 1$ and $k \geq 2$, then*

$$adim(B_{n,k}) \leq k + \left\lfloor \frac{2n+2}{5} \right\rfloor - 1.$$

Proof. By using the previous notations, since $x_1, x_2, ..., x_k$ are twin vertices (i.e., vertices with the same neighbors), at least $k - 1$ of them must belong to each adjacency resolving set (in fact, two twin vertices not in an ordered set S will have the same adjacency representations with respect to S). Let $W_0 = \{x_1, x_2, ..., x_{k-1}\}$. By the division algorithm, there exist integers t, r such that $n = 5t + r$ with $t \geq 0$ and $0 \leq r \leq 4$. Now regarding the value of r, consider the following cases (Figure 2 provides an illustration for each case):
if $n = 5t$, then let $W_1 = \{v_{5i+1}, v_{5i+3} \ : \ 0 \leq i \leq t-1\}$,
if $n = 5t + 1$, then let $W_1 = \{v_{5i+2} \ : \ 0 \leq i \leq t-1\} \cup \{v_{5i-1} \ : \ 1 \leq i \leq t\}$,
if $n = 5t + 2$, then let $W_1 = \{v_{5i+1}, v_{5i+4} \ : \ 0 \leq i \leq t-1\} \cup \{v_{n-1}\}$,
if $n = 5t + 3$, then let $W_1 = \{v_{5i+1}, v_{5i+4} \ : \ 0 \leq i \leq t-1\} \cup \{v_{n-2}\}$,
if $n = 5t + 4$, then let $W_1 = \{v_{5i+1}, v_{5i+4} \ : \ 0 \leq i \leq t-1\} \cup \{v_{n-3}, v_{n-1}\}$.

It is easy to check that in each of these cases we have $|W_1| = \lfloor \frac{2n+2}{5} \rfloor$. Also, by using Theorem 2.2 we can see that in each case, the set $W = W_0 \cup W_1$ is an adjacency resolving set for $B_{n,k}$ and hence,

$$adim(B_{n,k}) \leq |W| = |W_0| + |W_1| = (k-1) + \left\lfloor \frac{2n+2}{5} \right\rfloor .$$

□

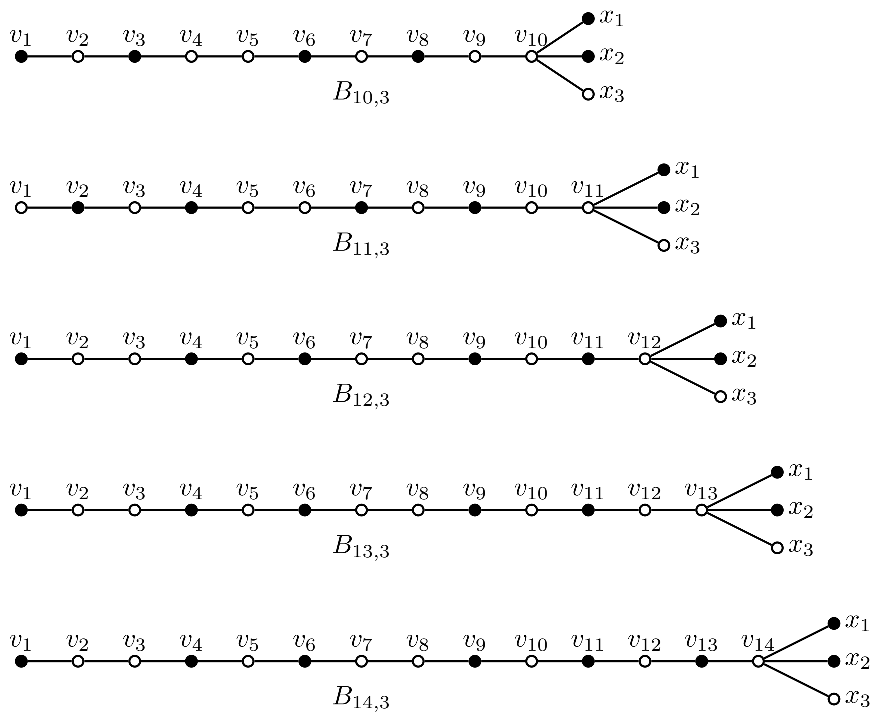

FIGURE 2. An adjacency basis for the broom graph $B_{n,3}$, $10 \leq n \leq 14$.

Theorem 3.2. *Let $B_{n,k}$ be a broom graph in which $n \geq 1$ and $k \geq 2$. Then*

$$adim(B_{n,k}) = k + \left\lfloor \frac{2n+2}{5} \right\rfloor - 1.$$

Proof. From Lemma 3.1 we obtain $adim(B_{n,k}) \leq k + \lfloor \frac{2n+2}{5} \rfloor - 1$. Hence, to complete the proof, it is sufficient to show that $adim(B_{n,k}) \geq (k-1) + \lfloor \frac{2n+2}{5} \rfloor$. Let B be an adjacency basis for $B_{n,k}$ and hence, $adim(B_{n,k}) = |B|$. Thus, it is sufficient to show that $|B| \geq (k-1) + \lfloor \frac{2n+2}{5} \rfloor$. Since $x_1, x_2, ..., x_k$ are twin vertices, we must have $|B \cap \{x_1, x_2, ..., x_k\}| \geq k-1$. Without loss of generality, assume that $\{x_1, x_2, ..., x_{k-1}\} \subseteq B$. We claim that $|B \cap \{v_1, v_2, ..., v_n\}| \geq \lfloor \frac{2n+2}{5} \rfloor$ and this completes the proof. Consider the following two cases.

Case 1. $v_n \in B$.
In this case, each vertex in $\{v_1, v_2, ..., v_n\}$ is resolved by vertices in $B \cap \{v_1, v_2, ..., v_n\}$ and hence, $B \cap \{v_1, v_2, ..., v_n\}$ is an adjacency resolving set for the induced n-vertex path on $\{v_1, v_2, ..., v_n\}$. This implies that

$$|B \cap \{v_1, v_2, ..., v_n\}| \geq adim(P_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor,$$

as desired.

Case 2. $v_n \notin B$.
Assume that $x_k \in B$. If $r_a(v_{n-1}|B) \neq (2, 2, ..., 2)$, then removing x_k from B and replacing it with v_n leads to an adjacency basis for $B_{n,k}$ which satisfies Case 1, and the proof is complete. If $r_a(v_{n-1}|B) = (2, 2, ..., 2)$, then removing x_k from B and replacing it with v_{n-1} leads to a basis for $B_{n,k}$ in which $v_n \notin B$ and $x_k \notin B$. Thus, here after we can assume that $v_n \notin B$ and $x_k \notin B$. Hence, $r_a(x_k|B) = (2, 2, ..., 2)$ and vertices $v_1, v_2, ..., v_{n-1}$ are resolved by $B \cap \{v_1, v_2, ..., v_{n-1}\}$, and except x_k no vertex in $B_{n,k}$ has the adjacency representation $(2, 2, ..., 2)$. This implies that

$$|B \cap \{v_1, v_2, ..., v_{n-1}\}| \geq adim(P_{n-1}) = \left\lfloor \frac{2(n-1)+2}{5} \right\rfloor = \left\lfloor \frac{2n}{5} \right\rfloor.$$

When $n \equiv r \pmod 5$ with $r \in \{0, 1, 3\}$, we have $\lfloor \frac{2n}{5} \rfloor = \lfloor \frac{2n+2}{5} \rfloor$ and hence,

$$|B \cap \{v_1, v_2, ..., v_n\}| \geq \lfloor \frac{2n+2}{5} \rfloor,$$

as desired. Suppose that $n \equiv 2$ or $4 \pmod 5$. Thus, $\lfloor \frac{2n+2}{5} \rfloor = \lfloor \frac{2n}{5} \rfloor + 1$. Since $n \equiv 2$ or $4 \pmod 5$ we see that $(n-1) \equiv 1$ or $3 \pmod 5$. Since $B \cap \{v_1, v_2, ..., v_{n-1}\}$ adjacently resolves $v_1, v_2, ..., v_{n-1}$, if $|B \cap \{v_1, v_2, ..., v_{n-1}\}| = \lfloor \frac{2n}{5} \rfloor$, then $B \cap \{v_1, v_2, ..., v_{n-1}\}$ is a basis for the (induced) path P_{n-1} and Theorem 2.5 implies that among $v_1, v_2, ..., v_{n-1}$ there exists a vertex with adjacency representation $(2, 2, ..., 2)$, which is a contradiction because $r_a(x_k|B) = (2, 2, ..., 2)$. Therefore,

$$|B \cap \{v_1, v_2, ..., v_n\}| \geq \left\lfloor \frac{2n}{5} \right\rfloor + 1 = \left\lfloor \frac{2n+2}{5} \right\rfloor,$$

as desired, and this completes the proof. □

Note that the star graph S_m is an m-vertex tree with $m-1$ leaves, $B_{1,k} = S_{k+1}$ and $B_{2,k} = S_{k+2}$. Hence, the following result is a confirmation of Theorem 3.2.

Corollary 3.3. *We have* $adim(S_{k+1}) = adim(B_{1,k}) = k-1$ *and* $adim(S_{k+2}) = adim(B_{2,k}) = k$.

Lemma 3.4. *If* $n \geq 1$ *and* $k \geq 2$, *then*

$$adim(B_{n,k,k}) \leq 2k + \left\lfloor \frac{2n-1}{5} \right\rfloor - 1.$$

Proof. As before, assume that

$$V(B_{n,k,k}) = \{v_1, v_2, ..., v_n\} \cup \{x_1, x_2, ..., x_k\} \cup \{x'_1, x'_2, ..., x'_k\},$$

in which $\{v_1, v_2, ..., v_n\}$ induces an n-vertex path, each pendant vertex x_i is adjacent to v_1 and, each pendant vertex x'_j is adjacent to v_n (see Figure 3). Note that $\{x_1, x_2, ..., x_k\}$ are twin vertices and similarly, $\{x'_1, x'_2, ..., x'_k\}$ are twins. At first, we want to show that $adim(B_{n,k,k}) \le 2k + \lfloor \frac{2n-1}{5} \rfloor - 1$. When $n \in \{1, 2, 3, 4\}$ let

$$W = \begin{cases} \{x_1, x_2, ..., x_k, x'_1, x'_2, ..., x'_{k-1}\} & \text{n=1} \\ \{v_1, x_1, x_2, ..., x_{k-1}, x'_1, x'_2, ..., x'_{k-1}\} & \text{n=2} \\ \{v_1, v_3, x_1, x_2, ..., x_{k-1}, x'_1, x'_2, ..., x'_{k-1}\} & \text{n=3} \\ \{v_1, v_3, x_1, x_2, ..., x_{k-1}, x'_1, x'_2, ..., x'_{k-1}\} & \text{n=4.} \end{cases}$$

It can be easily seen that when $n \in \{1, 2, 3, 4\}$, the relating set W is an adjacency resolving set for $B_{n,k,k}$ and hence,

$$adim(B_{n,k,k}) \le |W| = 2k - 1 + \left\lfloor \frac{2n-1}{5} \right\rfloor.$$

Now assume that $n \ge 5$. Let $X_0 = \{x_1, x_2, ..., x_{k-1}\}$ and $X'_0 = \{x'_1, x'_2, ..., x'_{k-1}\}$. Consider the following cases (see Figure 3 for more details in each case).

Case 1. $n \equiv 0 \pmod 5$.
Suppose that $n = 5t$ in which $t \ge 1$ and let $Y_0 = \{v_{5i+1}, v_{5i+3} \ : \ 0 \le i \le t-1\}$. It is straightforward to check that $W = X_0 \cup X'_0 \cup Y_0$ is an adjacency resolving set for $B_{n,k,k}$ (for instance see $B_{10,3,3}$ in Figure 3). Hence,

$$adim(B_{n,k,k}) \le |W| = |X_0| + |X'_0| + |Y_0| = 2(k-1) + 2t = (2k-1) + \left\lfloor \frac{2n-1}{5} \right\rfloor.$$

Case 2. $n \equiv 1$ or $2 \pmod 5$.
Thus $n = 5t + r$ in which $t \in \mathbb{N}$, $r \in \{1, 2\}$ and, $\lfloor \frac{2n-1}{5} \rfloor = 2t$. Let $Y_0 = \{v_{5i+1}, v_{5i+3} \ : \ 0 \le i \le t-1\} \cup \{v_{n-1}\}$. Then by a simple investigation we can see that $W = X_0 \cup X'_0 \cup Y_0$ is an adjacency resolving set for $B_{n,k,k}$. Thus,

$$adim(B_{n,k,k}) \le |W| = 2(k-1) + (2t+1) = (2k-1) + 2t = (2k-1) + \left\lfloor \frac{2n-1}{5} \right\rfloor.$$

Case 3. $n \equiv 3$ or $4 \pmod 5$.
Let $n = 5t + r$ in which $t \in \mathbb{N}$, $r \in \{3, 4\}$ and, $Y_0 = \{v_{5i+1}, v_{5i+3} \ : \ 0 \le i \le t\}$. It is not hard to check that $W = X_0 \cup X'_0 \cup Y_0$ is an adjacency resolving set

for $B_{n,k,k}$ and hence,

$$adim(B_{n,k,k}) \leq |W| = 2(k-1) + 2(t+1) = (2k-1) + \left\lfloor \frac{2n-1}{5} \right\rfloor.$$

Therefore, in all of cases we see that $adim(B_{n,k,k}) \leq 2k + \lfloor \frac{2n-1}{5} \rfloor - 1$. □

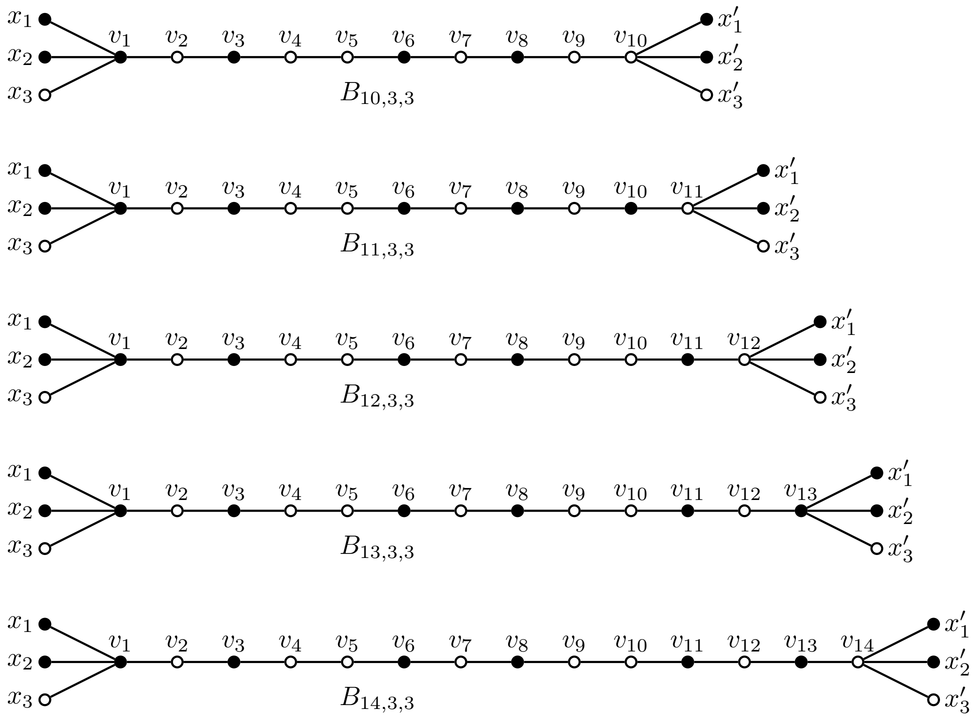

FIGURE 3. An adjacency basis for double broom $B_{n,3,3}$, $10 \leq n \leq 14$.

Theorem 3.5. *For the adjacency dimension of the double broom $B_{n,k,k}$ with $n \geq 1$ and $k \geq 2$ we have*

$$adim(B_{n,k,k}) = 2k + \left\lfloor \frac{2n-1}{5} \right\rfloor - 1.$$

Proof. By using Lemma 3.4 we see that $adim(B_{n,k,k}) \geq 2k + \lfloor \frac{2n-1}{5} \rfloor - 1$. Hence, to complete the proof, it is sufficient to show that $adim(B_{n,k,k}) \geq 2k + \lfloor \frac{2n-1}{5} \rfloor - 1$. Let B be an adjacency basis for $B_{n,k,k}$ and hence, $adim(B_{n,k,k}) = |B|$. We use the same notations applied in the proof of Lemma 3.4. Since $x_1, x_2, ..., x_k$ are twins and similarly, $x'_1, x'_2, ..., x'_k$ are twins, we must have

$$|B \cap \{x_1, x_2, ..., x_k\}| \geq k-1 \ , \ |B \cap \{x'_1, x'_2, ..., x'_k\}| \geq k-1.$$

Without loss of generality, assume that

$$\{x_1, x_2, ..., x_{k-1}\} \subseteq B \ , \ \{x'_1, x'_2, ..., x'_{k-1}\} \subseteq B.$$

Since $r_a(x_k|B) \neq r_a(x'_k|B)$, we obtain

$$B \cap \{v_1, v_n, x_k, x'_k\} \neq \emptyset. \tag{1}$$

Note that $k \geq 2$, v_1 is the unique vertex adjacent to $x_1 \in B$ and v_n is the unique vertex adjacent to $x'_1 \in B$. Especially, two vertices v_1 and v_n are resolved from other vertices. Regarding the relation (1), we consider two following cases.

Case 1. $B \cap \{v_1, v_n\} = \emptyset$.
In this case, relation (1) implies that $B \cap \{x_k, x'_k\} \neq \emptyset$. Without loss of generality, assume that $x_k \in B$. Since $B \cap \{v_1, v_n\} = \emptyset$ and $v_2, v_3, ..., v_{n-1}$ are adjacently resolved by each other, we have

$$|B \cap \{v_2, v_3, ..., v_{n-1}\}| \geq adim(P_{n-2}) = \left\lfloor \frac{2(n-2)+2}{5} \right\rfloor = \left\lfloor \frac{2n-2}{5} \right\rfloor.$$

Thus,

$$|B| \geq 2(k-1) + 1 + \left\lfloor \frac{2n-2}{5} \right\rfloor = 2k - 1 + \left\lfloor \frac{2n-2}{5} \right\rfloor.$$

If $n \equiv r \pmod 5$ with $r \in \{0, 1, 2, 4\}$, then the equality $\lfloor \frac{2n-2}{5} \rfloor = \lfloor \frac{2n-1}{5} \rfloor$ implies that $|B| \geq (2k-1) + \lfloor \frac{2n-1}{5} \rfloor$, which completes the proof. Thus, assume that $n \equiv 3 \pmod 5$ and hence, $(n-2) \equiv 1 \pmod 5$. If $|B \cap \{v_2, v_3, ..., v_{n-1}\}| = \lfloor \frac{2n-2}{5} \rfloor$, then $B \cap \{v_2, v_3, ..., v_{n-1}\}$ is a basis for the induced path on $n-2$ vertices $v_2, v_3, ..., v_{n-1}$ and Theorem 2.5 implies that there exists a vertex v_i, $2 \leq i \leq n-1$, whose adjacency representation is $(2, 2, ..., 2)$. Now if $B \cap \{v_n, x'_k\} = \emptyset$, then x'_k and v_i both have the same adjacency representation $(2, 2, ..., 2)$, a contradiction. Therefore, either $|B \cap \{v_2, v_3, ..., v_{n-1}\}| \geq \lfloor \frac{2n-2}{5} \rfloor + 1$ or $|B \cap \{v_n, x'_k\}| \geq 1$. Thus,

$$|B| \geq 2k - 1 + \left\lfloor \frac{2n-2}{5} \right\rfloor + 1 = 2k - 1 + \left\lfloor \frac{2n-1}{5} \right\rfloor,$$

as desired, which completes the proof.

Case 2. $B \cap \{v_1, v_n\} \neq \emptyset$.
Without loss of generality, assume that $v_1 \in B$. Since $k \geq 2$, v_n is the unique vertex adjacent to $x'_1 \in B$ and is resolved from the other vertices. Since $r_a(x_k|B) \neq r_a(v_2|B)$, we must have $|B \cap \{x_k, v_2, v_3\}| \geq 1$. If $B \cap \{x_k, v_2, v_3\} = \{v_2, v_3\}$, then we can remove v_2 from B and we get $B \setminus \{v_2\}$ as an adjacency resolving set which is smaller that the basis B, a contradiction. In a similar way, if $|B \cap \{x_k, v_2, v_3\}| \geq 2$, then by an easy check we can see that at least one member of $B \cap \{x_k, v_2, v_3\}$ is irredental and can be removed from B, which contradicts the minimality of B. Therefore, we have $|B \cap \{x_k, v_2, v_3\}| = 1$. Consider the following subcases.

Subcase i. $B \cap \{x_k, v_2, v_3\} = \{x_k\}$.
Then $|B \cap \{v_3, v_4, ..., v_{n-1}\}| \geq adim(P_{n-3})$ and

$$\begin{aligned} |B| &\geq 2(k-1)+1+1+adim(P_{n-3}) \\ &= 2k-1+1+\left\lfloor \frac{2(n-3)+2}{5} \right\rfloor \\ &= 2k-1+\left\lfloor \frac{2n+1}{5} \right\rfloor \\ &\geq 2k-1+\left\lfloor \frac{2n-1}{5} \right\rfloor. \end{aligned}$$

Subcase ii. $B \cap \{x_k, v_2, v_3\} = \{v_2\}$.
Then $|B \cap \{v_4, v_5, ..., v_{n-1}\}| \geq adim(P_{n-4})$, hence

$$\begin{aligned} |B| &\geq 2(k-1)+1+1+adim(P_{n-4}) \\ &= 2k-1+1+\left\lfloor \frac{2(n-4)+2}{5} \right\rfloor \\ &= 2k-1+\left\lfloor \frac{2n-1}{5} \right\rfloor. \end{aligned}$$

Subcase iii. $B \cap \{x_k, v_2, v_3\} = \{v_3\}$.
If $v_4 \in B$, then we can replace it with v_5. Now we have $|B \cap \{v_5, v_6, ..., v_{n-1}\}| \geq adim(P_{n-5})$ and

$$\begin{aligned} |B| &\geq 2(k-1)+1+1+adim(P_{n-5}) \\ &= 2k-1+1+\left\lfloor \frac{2(n-5)+2}{5} \right\rfloor \\ &= 2k-1+\left\lfloor \frac{2n-3}{5} \right\rfloor. \end{aligned}$$

If $n \equiv r \pmod 5$ with $r \in \{0,2,4\}$, then $\lfloor \frac{2n-3}{5} \rfloor = \lfloor \frac{2n-1}{5} \rfloor$ and the proof is complete. For $n \equiv 1$ or $3 \pmod 5$, we have $n-5 \equiv 1$ or $3 \pmod 5$ and $\lfloor \frac{2n-3}{5} \rfloor = \lfloor \frac{2n-1}{5} \rfloor - 1$. If $|B \cap \{v_5, v_6, ..., v_{n-1}\}| = adim(P_{n-5})$, then by Theorem 2.5 a vertex in $\{v_5, v_6, ..., v_{n-1}\}$ has the adjacency representation $(2, 2, ..., 2)$. If $B \cap \{v_n, x'_k\} = \emptyset$, then the adjacency representation of x'_k is also $(2, 2, ..., 2)$, a contradiction. Thus, $|B \cap \{v_5, v_6, ..., v_{n-1}\}| \geq 1 + adim(P_{n-5})$ or $|B \cap \{v_n, x'_k\}| \geq 1$. Therefore,

$$\begin{aligned} |B| &\geq 2(k-1)+1+1+\left(1+\left\lfloor \frac{2n-3}{5} \right\rfloor\right) \\ &= 2k-1+\left\lfloor \frac{2n-1}{5} \right\rfloor. \end{aligned}$$

This completes the proof. □

Note that Figure 3 provides some examples for Theorem 3.5 and two following results can be considered as a confirmation of it.

Corollary 3.6. *If $n = 1$, then $B_{n,k,k}$ is the star graph S_{2k+1}, and*

$$adim(B_{1,k,k}) = 2k - 1 = adim(S_{2k+1}).$$

Corollary 3.7. *If $n = 2$, then $B_{n,k,k}$ is the double star graph $S_{k,k}$ of order 2k+2, and*

$$adim(B_{2,k,k}) = 2k - 1 = adim(S_{k,k}).$$

In some literature, a double broom $B_{n,k,k'}$ is defined as a tree obtained from a path P_n by attaching k pendant edges to one end vertex of P_n and k' pendant edges to the other end vertex of P_n, where $k, k' \geq 2$. Since each pair of twin vertices has non-empty intersection with each adjacency resolving set, we can obtain the following result directly from Theorem 3.5 and its proof.

Corollary 3.8. *If $k, k' \geq 2$, then $adim(B_{n,k,k'}) = k + k' - 1 + \lfloor \frac{2n-1}{5} \rfloor$.*

4. Conclusion and further works

In this paper we determine the properties of adjacency resolving sets of paths. By using these properties, we determine the adjacency dimension of broom and double broom graphs. These properties can be applied for determining the adjacency dimension of some other path related graphs and some graph operations like subdivision.

5. Acknowledgments

The authors warmly thank the anonymous referees for reading this manuscript very carefully and providing numerous valuable corrections and suggestions which improve the quality of this paper.
The authors declare that they have no conflict of interest.

References

[1] Brigham, R.C., Chartrand, G., Dutton, R.D., & Zhang, P. (2003). Resolving domination in graphs. Math. Bohem., 128(1), 25-36. https://doi.org/10.21136/mb.2003.133935

[2] Buczkowski, P.S., Chartrand, G., Poisson, C., & Zhang, P. (2003). On k-dimensional graphs and their bases. Period. Math. Hung., 46(1), 9-15. https://doi.org/10.1023/A:1025745406160

[3] Caceres, J., Hernando, C., Mora, M., Pelayo, I.M., Puertas, M.L., Seara, C., & Wood, D.R. (2007). On the metric dimension of Cartesian product of graphs. SIAM J. Discrete Math., 21(2), 423-441. https://doi.org/10.1137/050641867

[4] Charon, I., Hudry, O., & Lobstein, A. (2003). Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard. Theoret. Comput. Sci., 290(3), 2109-2120. https://doi.org/10.1016/s0304-3975(02)00536-4

[5] Chartrand, G., Eroh, L., Johnson, M. A., & Oellermann, O. (2000). Resolvability in Graphs and the Metric Dimension of a Graph. Discrete Appl. Math., 105(3), 99-113. https://doi.org/10.1016/s0166-218x(00)00198-0

[6] Chartrand, G., Saenpholphat, V., & Zhang, P. (2003). The independent resolving number of a graph. Math. Bohem., 128(4), 379-393. https://doi.org/10.21136/mb.2003.134003

[7] Estrada-Moreno, A., Ramrez-Cruz, Y., & Rodriguez-Velazquez, J. A. (2016). On the adjacency dimension of graphs. Appl. Anal. Discrete Math., 10(1), 102-127. https://doi.org/10.2298/aadm151109022e

[8] Fernau, H., & Rodriguez-Velazquez, J. A. (2014). Notions of metric dimension of corona products: combinatorial and computational results, in: Computer science—theory and applications. Springer, Cham, Lect. Notes Comput. Sci., 84(1), 153-166. https://doi.org/10.1007/978-3-319-06686-8

[9] Fernau, H., & Rodriguez-Velazquez, J. A. (2018). On the (adjacency) metric dimension of corona and strong product graphs and their local variants: combinatorial and computational results. Discrete Appl. Math., 23(6), 183-202. https://doi.org/10.1016/j.dam.2017.11.019

[10] Harary, F. (1969). Graph Theory, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London.

[11] Harary, F., & Melter, R. A. (1976). On the Metric Dimension of a Graph. Ars Combin., 2(1), 191-195.

[12] Haynes, T.W., Henning, M.A., & Howard, J. (2006). Locating and total dominating sets in trees. Discrete Appl. Math., 154(8), 1293-1300. https://doi.org/10.1016/j.dam.2006.01.002

[13] Iswadi, H., Baskoro, E.T., & Simanjuntak, R. (2011). On the metric dimension of corona product of graphs. Far East J. Math. Sci., 52(2), 155-170. http://pphmj.com/journals/fjms.htm

[14] Jannesari, M. (2023). On lower bounds for the metric dimension of graphs. J. Mahani math. res., 12(1), 35-41. https://doi.org/10.22103/jmmrc.2022.19121.1211

[15] Jannesari, M. (2024). Metric dimension of lexicographic product of some known graphs. J. Mahani math. res., 13(1), 269-277. https://doi.org/10.22103/jmmr.2023.20814.1384

[16] Jannesari, M., & Omoomi, B. (2012). The metric dimension of the lexicographic product of graphs. Discrete Math., 312(22), 3349-3356. https://doi.org/10.1016/j.disc.2012.07.025

[17] Jannesari, M. (2022). Graphs with constant adjacency dimension. Discrete Math. Algorithms Appl., 14(4), 1-12. https://doi.org/10.1142/s1793830921501342

[18] Johnson, M. (1993). Structure–activity maps for visualizing the graph variables arising in drug design. J. Biopharm. Statist., 3(2), 203-236. https://doi.org/10.1080/10543409308835060

[19] Khuller, S., Raghavachari, B., & Rosenfeld, A. (1996). Landmarks in graphs. Discrete Appl. Math., 70(1), 217-229. https://doi.org/10.1016/0166-218x(95)00106-2

[20] Okamoto, F., Phinezy, B., & Zhang, P. (2010). The local metric dimension of a graph. Math. Bohem., 135(3), 239-255. https://doi.org/10.21136/mb.2010.140702

[21] Rodríguez-Velázquez, J. A., & Fernau, H. (2018). On the (adjacency) metric dimension of corona and strong product graphs and their local variants: combinatorial and computational results. Discrete Appl. Math., 236(1), 183-202. https://doi.org/10.1016/j.dam.2017.11.019

[22] Saenpholphat, V., & Zhang, P. (2004). Conditional resolvability in graphs: a survey. Int. J. Math. Math. Sci., 38(2), 1997-2017. https://doi.org/10.1155/s0161171204311403

[23] Sebö, A., & Tannier, E. (2004). On metric generators of graphs. Math. Oper. Res., 29(2), 383-393. https://doi.org/10.1287/moor.1030.0070

[24] Slater, P.J. (1975). Leaves of trees. Congr. Numer., 14(1), 549-559.

[25] Yero, I.G., Kuziak, D., & Rodriquez-Velazquez, J.A. (2011). On the metric dimension of corona product graphs. Comput. Math. Appl., 61(9), 2793-2798. https://doi.org/10.1016/j.camwa.2011.03.046

Elham Hardani
Orcid number: 0009-0003-9664-4985
Department of Mathematics, Faculty of Science,
Imam Khomeini International University
P.O. Box: 34149-16818, Qazvin, Iran
Email address: e.hardany@yahoo.com

Ali Behtoei
Orcid number: 0000-0003-2113-0210
Department of Mathematics, Faculty of Science
Imam Khomeini International University
P.O. Box: 34149-16818, Qazvin, Iran
Email address: a.behtoei@sci.ikiu.ac.ir