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MINIMAL NON-ABELIAN GROUPS WITH AN AVERAGE CONDITION ON SUBGROUPS

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Abstract. For a finite group G, the average order $o(G)$ is defined to be the average of all order elements in G. We say that G satisfies the average condition if $o(H) \leq o(G)$ for all subgroups H of G. In [On a question of Jaikin-Zapirain about the average order elements of finite groups, Int. J. Group Theory, To appear] we proved that every abelian group satisfies the average condition. In this paper, we classify minimal non-abelian groups which satisfy the average condition.

Keywords: Minimal non-abelian groups, Group element orders, Sum of element orders, Average condition.
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1. Introduction

All groups in this paper are assumed to be finite. Our notation and terminology are standard and primarily derived from [5]. In particular, the order of a finite group G is shown by $|G|$. The cyclic group of order n, the quaternion group of order 8 and the elementary abelian p-group of order p^m are denoted by C_n, Q_8 and C_p^m, respectively. The symbol $G = H \rtimes K$ indicates that G is a split extension (semi-direct product) of a normal subgroup H of G by a complement K. The dihedral group of order $2n$ is shown by D_{2n} and defined as $D_{2n} = \langle a, b \mid a^n = b^2 = 1,\ a^b = a^{-1}\rangle \cong C_n \rtimes C_2$.

Let G be a finite group. For a non-empty subset X of G, let $\psi(X)$ be the sum of the orders of all elements of X, that is

$$\psi(X) = \sum_{x \in X} o(x),$$

where $o(x)$ denotes the order of the element x of G. Let $o(G)$ be the average order of elements of G, i.e.,

$$o(G) = \frac{\psi(G)}{|G|}.$$

In the last years there has been a growing interest in studying the properties of these functions and their relations with the structure of G. Jaikin-Zapirain mentioned in [11] that it would be very interesting to understand the relation

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between $o(G)$ and $o(N)$, where N is a normal subgroup of G. He determined a lower bound for the number of conjugacy classes of a finite p-group (nilpotent group). Also, he considered the average order for some powerful p-groups of exponent p^r and posed the following question:

Question 1. Let G be a finite (p-) group and N be a normal (abelian) subgroup of G. Is it true that $o(N)^{1/2} \leq o(G)$?

Ten years later, Khokhro, Moreto, and Zarrin provided a strong negative answer to this question [12]. They proved that "If c is a real number and $p \geq 3/c$ is a prime, then there exists a finite p-group with a normal abelian subgroup N such that $o(G) < o(N)^c$". In addition, they posed the following question:

Question 2. Fix a prime p. Does there exist a number $c = c(p) > 0$ depending on p such that $o(G) \geq o(N)^c$ for any finite p-group G and any (abelian) normal subgroup N of G?

In [12], it is conjectured that $o(G) < o(A_5)$, guarantees the solvability of G, which turned out to be true when M. Herzog, P. Longobardi and M. Maj proved it in [9]. In that paper, they also classified all finite groups with $o(G) \leq o(S_3)$. Later in [10], they proved that A_5 is characterizable by average order, meaning that $o(G) = o(A_5)$ implies $G \cong A_5$. Note that the same results were obtained by M. Tărnăuceanu [15] (his proofs were different from that of [9,10]). Meanwhile Tărnăuceanu in [14], classified all finite groups with $o(G) \leq o(A_4)$, where he stated that if $o(G) < o(A_4)$, G would be supersolvable, and $o(G) = o(A_4)$ leads to $G \cong A_4$. We leave the following question as an open problem.

Question 3. Let p be the odd prime divisor of the order of finite group G and $o(G) \leq o(D_{2p})$, then what is the structure of G?

Inspired by the above questions, in [16], we considered a strong condition and asked the following question:

Question 4. For which finite group G, $o(H) \leq o(G)$ for all (normal) subgroups H of G?

We say that a subgroup H of a group G has the *average property* if $o(H) \leq o(G)$. It is obvious that trivial subgroups of G have the average property. Investigating the average property of certain subgroups of a finite group is an interesting topic. Jaikin-Zapirain [11] proved that the center of a finite group G has the average property, that is, $o(Z(G)) \leq o(G)$, where $Z(G)$ is the center of G. H. Amiri and S. M. Jafarian Amiri [2] conjectured that $o(A_n) < o(S_n)$ for every positive integer $n \geq 4$. We say that G satisfies the *average condition* if every subgroup of G has the average property, that is, $o(H) \leq o(G)$ for all subgroups H of G. In [16], we proved the following result:

Theorem 1.1. *If G is a finite abelian group then G satisfies the average condition.*

Using Theorem 1.1, C_8, $C_2 \times C_4$ and $C_2 \times C_2 \times C_2$ satisfy the average condition. Also, if $G \cong Q_8$ then $o(H) \in \{1, 3/2, 11/4\}$ for all proper subgroup H of G and $o(G) = 27/4$. So, Q_8 satisfies the average condition. But D_8 does not satisfy the average condition, since $19/8 = o(D_8) < o(\langle a \rangle) = 11/4$. Therefore, every finite group of order 8, except D_8, satisfies the average condition.

A finite non-abelian group G is called minimal non-abelian if every proper subgroup of G is abelian. We note that Q_8 and D_8 are minimal non-abelian groups such that Q_8 satisfies the average condition but D_8 does not satisfy the average condition. In this paper, we classify minimal non-abelian groups which satisfy the average condition and prove the following theorem:

Theorem A. Let G be a finite minimal non-abelian group. Then, G satisfies the average condition if and only if one of the following holds:
(a) $G \ncong D_8$ is a p-group, where p is a prime,
(b) $G \cong C_3 \rtimes C_{2^\alpha}$, $\alpha \geq 2$,
(c) $G \cong C_q^\beta \rtimes C_{p^\alpha}$, where $p > q \geq 3$ are distinct primes, $\beta \geq 2$ and $\alpha \geq 1$.

2. **Preliminaries**

We recall the well-known classification of minimal non-abelian groups. Let $Z(G)$ denote the center of the group G.

Theorem 2.1. *[3, 13]A finite group G is minimal non-abelian if and only if G is isomorphic to one of the following groups:*

(a) $G = Q \rtimes \langle b \rangle \cong C_q^\beta \rtimes C_{p^\alpha}$*, where Q is an elementary abelian q-subgroup and a minimal normal subgroup of G,* $\langle b \rangle$ *is a non-normal cyclic Sylow p-subgroup, p and q are distinct primes and* $b^p \in Z(G)$*. Also, the action of* $\langle b \rangle$ *on Q is irreducible with kernel* $\langle b^p \rangle$*. More precisely,* $Q = \langle a_1, a_2, \ldots, a_\beta \rangle$*, where* ${a_i}^q = b^{p^\alpha} = 1$, $a_i^b = a_{i+1}$ *and* ${a_\beta}^b = a_1^{d_1} \cdots a_\beta^{d_\beta}$.

(b) The quaternion group of order 8.

(c) $M_{m,n,p} = \langle a, b \mid a^{p^m} = b^{p^n} = 1,\ a^b = a^{1+p^{m-1}} \rangle = \langle a \rangle \rtimes \langle b \rangle$ *of order* p^{m+n}*, where p is a prime number,* $m \geq 2$ *and* $n \geq 1$.

(d) $N_{m,n,p} = \langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1,\ [a, b] = c,\ [a, c] = [b, c] = 1 \rangle = (\langle a \rangle \times \langle c \rangle) \rtimes \langle b \rangle = (\langle b \rangle \times \langle c \rangle) \rtimes \langle a \rangle$ *of order* p^{m+n+1}*, where p is a prime and* $m \geq n \geq 1$.

Note that the function $o(G)$ is multiplicative. This is a direct consequence of the following lemma:

Lemma 2.2. *([1, Lemma 2.1])If G and H are finite groups then* $\psi(G \times H) \leq \psi(G)\psi(H)$*. Also,* $\psi(G \times H) = \psi(G)\psi(H)$ *if and only if* $\gcd(|G|, |H|) = 1$.

We need the following results on the sum of element orders cyclic and elementary abelian p-groups, respectively.

Lemma 2.3. *([8, Lemma 2.9])If P is a cyclic group of order p^n for some prime p then*

$$\psi(P) = \frac{p^{2n+1}+1}{p+1}.$$

Lemma 2.4. *([7, Proposition 8])Let a and n be positive integers and let p be a prime. Then,*

$$\psi(C_{p^a}^n) = \frac{p^{(n+1)(a+1)} - p^{a(n+1)+1} + p - 1}{p^{n+1}-1}.$$

Also, we use the following simple results:

Lemma 2.5. *([4, Lemma 2.3])Let $G = H_1 \times H_2 \times \cdots \times H_n$, where H_i's are p-groups, then $o\big((x_1, x_2, \ldots, x_n)\big) = \max\{o(x_1), o(x_2), \ldots, o(x_n)\}$ for any $(x_1, x_2, \ldots, x_n) \in G$.*

Lemma 2.6. *([5, Lemma 2.2])Let $a, b \in G$ and suppose $c = [a, b]$ commutes with both a and b. Then,*
(a) $\left[a^i, b^j\right] = c^{ij}$ for all i and j.
(b) $(ab)^i = a^i b^i c^{-i(i-1)/2}$ for all i.

The proof of the following lemma is straightforward.

Lemma 2.7. *Let $a, b \in G$ and suppose that r and n are positive integers. Then,*
(a) $(b^{-1}ab)^n = b^{-1}a^n b$.
(b) If $b^{-1}ab = a^r$ then $b^{-n}ab^n = a^{r^n}$.

We may restrict the study of the average condition on a group to its maximal subgroups using the following straightforward result:

Proposition 2.8. *([16, Proposition 2.2])Let G be a finite group. If every maximal subgroup M of G satisfies the average condition and $o(M) \leq o(G)$ then G satisfies the average condition.*

3. Proof of the main result

To prove that every minimal non-abelian p-group, except D_8, satisfies the average condition, we need the following two lemmas:

Lemma 3.1. *Let $G = M_{m,n,p} = \langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}}\rangle$ be a group of order p^{m+n}, where p is a prime number, $m \geq 2$ and $n \geq 1$. If $G \neq M_{2,1,2} \cong D_8$ then $o(a^i b^j) = \max\left\{o(a^i), o(b^j)\right\}$ for all $1 \leq i \leq p^m$ and $1 \leq j \leq p^n$.*

Proof. Using the assumption $b^{-1}ab = a^{1+p^{m-1}}$ and Lemma 2.7, we infer that

$$b^{-j}a^k b^j = (b^{-j}ab^j)^k = (a^{(1+p^{m-1})^j})^k = a^{k(1+p^{m-1})^j}$$

for all positive integers j and k. Since $(1+p^{m-1})^j \equiv 1+jp^{m-1} \pmod{p^m}$, we have $b^{-j}a^kb^j = a^{k(1+jp^{m-1})}$ and it follows that $a^kb^j = b^ja^ka^{kjp^{m-1}}$. We claim that

$$(a^kb^j)^t = (b^j)^t(a^k)^ta^{\frac{t(t+1)}{2}kjp^{m-1}}$$

for all positive integer t. We proceed by induction on t. If $t=1$ then the result is true. Assuming the result holds for t. Then,

$$\begin{aligned}(a^kb^j)^{t+1} &= (a^kb^j)^t(a^kb^j)\\ &= (b^j)^t(a^k)^ta^{\frac{t(t+1)}{2}kjp^{m-1}}(a^kb^j) \quad \text{(by induction hypothesis)}\\ &= (b^j)^t\left(a^{kt+k+\frac{t(t+1)}{2}kjp^{m-1}}b^j\right)\\ &= (b^j)^tb^ja^{kt+k+\frac{t(t+1)}{2}kjp^{m-1}}a^{(kt+k+\frac{t(t+1)}{2}kjp^{m-1})jp^{m-1}}\\ &= (b^j)^{t+1}(a^k)^{t+1}a^{\frac{t(t+1)}{2}kjp^{m-1}+k(t+1)jp^{m-1}+\frac{t(t+1)}{2}kj^2p^{2m-2}}\\ &= (b^j)^{t+1}(a^k)^{t+1}a^{kjp^{m-1}(t+1)(\frac{t}{2}+1)}\\ &= (b^j)^{t+1}(a^k)^{t+1}a^{\frac{(t+1)(t+2)}{2}kjp^{m-1}}.\end{aligned}$$

Therefore, our claim is true.

Now, consider a^ib^j, where $1 \le i \le p^m$ and $1 \le j \le p^n$. If $p \mid i$ or $p \mid j$ then $\langle a^i, b^j\rangle$ is a proper subgroup of G and thus is abelian. So, by Lemma 2.5, we have $o(a^ib^j) = \max\{o(a^i), o(b^j)\}$ and the result follows. Suppose that $p \nmid i$ and $p \nmid j$. Thus, $\langle a^i, b^j\rangle = G$. Suppose that $p^r = \max\{o(a^i), o(b^j)\} = \max\{p^m, p^n\}$. It is obvious that $r \ge 2$. By the claim, we have $(a^ib^j)^{p^r} = (b^j)^{p^r}(a^i)^{p^r}a^{p^r(p^r+1)ijp^{m-1}/2} = a^{p^r(p^r+1)ijp^{m-1}/2}$. If $p>2$ then $2 \mid p^r+1$, which implies that $p^m \mid p^r(p^r+1)ijp^{m-1}/2$. If $p=2$ then $2^m \mid 2^r(2^r+1)ij2^{m-1}/2$, since $r \ge 2$. Therefore, in any case $(a^ib^j)^{p^r} = 1$.

Now, we prove that r is the smallest positive integer such that $(a^ib^j)^{p^r} = 1$. Suppose that there exists a positive integer $s<r$ such that $(a^ib^j)^{p^s} = 1$. By the claim, we have $(b^j)^{p^s}(a^i)^{p^s}a^{p^s(p^s+1)ijp^{m-1}/2} = 1$. If $s=1$ and $p=2$ then $(b^j)^2(a^i)^2a^{3ij2^{m-1}} = 1$ and so, $(a^{i(1+3j2^{m-2})})^2 = (b^j)^{-2} \in \langle a^i\rangle \cap \langle b^j\rangle = 1$ which implies that $o(b^j) = 2$ and $(a^i)^{2(1+3j2^{m-2})} = 1$. If $m>2$ then $(2, 1+3j2^{m-2}) = 1$. It follows that $o(a) = 2$. This is a contradiction. Hence, $m=2$ and so, $G \cong D_8$. This contradicts our assumption. If $s=1$ and $p>2$ or $s \ge 2$ then $a^{p^s(p^s+1)ijp^{m-1}/2} = 1$ and so, $(b^j)^{p^s}(a^i)^{p^s} = 1$. Hence, $(a^i)^{p^s} = (b^j)^{-p^s} \in \langle a^i\rangle \cap \langle b^j\rangle = 1$, contradicting $s<r$. Therefore, $o(a^ib^j) = p^r$ which completes the proof. □

Lemma 3.2. *Let $G = N_{m,n,p} = \langle a,b,c \mid a^{p^m} = b^{p^n} = c^p = 1, [a,b] = c, [a,c] = [b,c] = 1\rangle$ be a group of order p^{m+n+1}, where p is a prime and $m \ge n \ge 1$. If $G \ne N_{1,1,2} \cong D_8$ then $o(c^ka^ib^j) = \max\{o(c^k), o(a^i), o(b^j)\}$ for all $1 \le i \le p^m$, $1 \le j \le p^n$ and $1 \le k \le p$.*

Proof. If $p \mid i$ or $p \mid j$ or $p \mid k$ then $\langle a^i, b^j, c^k \rangle$ is a proper subgroup of G and so, is abelian. Hence, $o(c^k a^i b^j) = \max\{o(c^k), o(a^i), o(b^j)\}$ by Lemma 2.5 and the result follows. Thus, suppose that $p \nmid i$, $p \nmid j$ and $p \nmid k$. Then, $\langle a^i, b^j, c^k \rangle = G$. Put $x = a^i, y = b^j$ and $z = c^{ij}$. By Lemma 2.6(a), $[x, y] = [a^i, b^j] = c^{ij} = z$. By assumption $[a, c] = [b, c] = 1$ which implies that $[x, z] = [y, z] = 1$. So, we may show that $o(cab) = \max\{o(c), o(a), o(b)\}$. Other cases are similar. Suppose that $p^t = \max\{o(c), o(a), o(b)\}$. It is obvious $t \geq 1$. Also, if $p = 2$ then $t \geq 2$. Therefore, by Lemma 2.6(b), we have $(cab)^{p^t} = c^{p^t} c^{p^t(p^t-1)/2} a^{p^t} b^{p^t} = c^{p^t(p^t+1)/2} = 1$.
Now, we prove that t is the smallest positive integer such that $(cab)^{p^t} = 1$. Suppose that there exists a positive integer $s < t$ such that $(cab)^{p^s} = 1$. By Lemma 2.6(b), $c^{-p^s(p^s-1)/2} a^{p^s} b^{p^s} = 1$. If $s = 1$ and $p = 2$ then $c^{-1}a^2b^2$ and so, $a^2 \in \langle a \rangle \cap (\langle b \rangle \times \langle c \rangle) = 1$. Therefore, $c^{-1}b^2 = 1$ and $b^2 = c \in \langle c \rangle \cap \langle b \rangle = 1$. Thus, $G \cong D_8$ which is a contradiction. If $s = 1$ and $p > 2$ or $s \geq 2$ then $c^{-p^s(p^s-1)/2} = 1$ and so, $(b)^{p^s}(a)^{p^s} = 1$. Hence, $(a)^{p^s} = (b)^{-p^s} \in \langle a \rangle \cap (\langle b \rangle \times \langle c \rangle) = 1$ which is a contradiction. Therefore, $o(cab) = p^t$, and the proof is complete. □

Now, we can prove that every minimal non-abelian p-group, except D_8, satisfies the average condition:

Proof of Theorem A in p-group case: Let $G \ncong D_8$ be a minimal non-abelian p-group, p a prime. By Theorem 2.1, we have three cases:

Case 1. $G \cong Q_8$. Up to isomorphism, G has two proper non-trivial subgroups C_2 and C_4. Since $o(C_2) = 3/2$, $o(C_4) = 11/4$ and $o(G) = 27/8$, we have $o(H) < o(G)$ for all subgroups H of G. Hence, G satisfies the average condition.

Case 2. $G = M_{m,n,p}$. By Lemma 2.5 and Lemma 3.1,

$$o(a^i b^j) = \max\{o(a^i), o(b^j)\} = o((a^i, b^j)),$$

for all $1 \leq i \leq p^m$ and $1 \leq j \leq p^n$. Hence,

$$\begin{aligned}
\psi(G) &= \sum_{j=1}^{p^n} \sum_{i=1}^{p^m} o(a^i b^j) \\
&= \sum_{j=1}^{p^n} \sum_{i=1}^{p^m} \max\{o(a^i), o(b^j)\} \\
&= \sum_{j=1}^{p^n} \sum_{i=1}^{p^m} o((a^i, b^j)) \\
&= \psi(C_{p^m} \times C_{p^n}).
\end{aligned}$$

Since $|G| = |C_{p^m} \times C_{p^n}|$, it follows that $o(G) = o(C_{p^m} \times C_{p^n})$. Since G is a minimal non-abelian p-group, every proper subgroup H of G is also a proper subgroup of $C_{p^m} \times C_{p^n}$. By Theorem 1.1, $o(H) < o(C_{p^m} \times C_{p^n})$ and so, $o(H) < o(G)$. Hence, G satisfies the average condition.

Case 3. $G = N_{m,n,p}$. By Lemma 2.5 and Lemma 3.2,

$$o(c^k a^i b^j) = \max\{o(c^k), o(a^i), o(b^j)\} = o\big((c^k, a^i, b^j)\big),$$

for all $1 \le i \le p^m$, $1 \le j \le p^n$ and $1 \le k \le p$. Hence,

$$\begin{aligned}\psi(G) &= \sum_{k=1}^{p}\sum_{j=1}^{p^n}\sum_{i=1}^{p^m} o(c^k a^i b^j)\\ &= \sum_{k=1}^{p}\sum_{j=1}^{p^n}\sum_{i=1}^{p^m} \max\{o(c^k), o(a^i), o(b^j)\}\\ &= \sum_{k=1}^{p}\sum_{j=1}^{p^n}\sum_{i=1}^{p^m} o\big((c^k, a^i, b^j)\big)\\ &= \psi(C_p \times C_{p^m} \times C_{p^n}).\end{aligned}$$

Since $|G| = |C_p \times C_{p^m} \times C_{p^n}|$, it follows that $o(G) = o(C_p \times C_{p^m} \times C_{p^n})$. Since G is a minimal non-abelian p-group, every proper subgroup H of G is also a proper subgroup of $C_p \times C_{p^m} \times C_{p^n}$. By Theorem 1.1, $o(H) < o(C_p \times C_{p^m} \times C_{p^n})$ and so, $o(H) < o(G)$. Thus, the proof is complete. □

Corollary 3.3. *Let $G \ncong D_8$ be a finite minimal non-abelian p-group. Then, $o(H) < o(G)$ for all maximal subgroups H of G.*

Consider the following example:

Example 3.4. *Suppose that $G = M_{2,1,3}$. Using GAP [6] we see that $o(G) =$ 6.926 and, up to isomorphism, $G_1 = C_9$ and $G_2 = C_3 \times C_3$ are maximal subgroups of G, and $o(G_1) = 6.78$ and $o(G_2) = 2.78$. So, G satisfies the average condition.*

Proof of Theorem A in non p-group case: Let G be a finite minimal non-abelian group of order $p^\alpha q^\beta$, where p and q are distinct primes. By Theorem 2.1(a), $G = Q \rtimes \langle b\rangle$, where $Q \cong C_q^\beta$ and $\langle b\rangle \cong C_{p^\alpha}$. Miller and Moreno [13, pp. 400] proved that if $\beta = 1$ and $p \mid q-1$ then there exists one and only one group G for every α. If $\beta = 1$ and $p \nmid q-1$ then there is no group G. In the case $\beta > 1$, we have $1 < \beta < p$, $\beta \mid p-1$, $p \mid q^\beta - 1$ and $p \nmid q^j - 1$ for all $1 \le j < \beta$ and for all α. They also showed that G has $q^\beta + 1$ maximal subgroups, q^β subgroups of order p^α and one subgroup $G_1 = C_q^\beta \times C_{p^{\alpha-1}}$ that contains all elements of G except those of order p^α. It follows that

$$\begin{aligned}\psi(G) &= \psi(G_1) + \psi(G\backslash G_1)\\ &= \psi(G_1) + p^\alpha|G\backslash G_1|\\ &= \psi(G_1) + p^\alpha(q^\beta p^{\alpha-1}(p-1))\\ &= \psi(G_1) + q^\beta p^{2\alpha-1}(p-1). \qquad (1)\end{aligned}$$

By Theorem 1.1, all maximal subgroups of G satisfy the average condition. Since maximal subgroups of order p^α are isomorphic to C_{p^α}, by Proposition 2.8, it suffices to study the average property on G_1 and $G_2 \cong C_{p^\alpha}$.

At first, we prove that $o(G_2) < o(G)$. By Lemma 2.3, $\psi(G_2) = (p^{2\alpha+1} + 1)/(p+1)$ and by (1), we have $\psi(G) = \psi(G_1) + q^\beta p^{2\alpha-1}(p-1)$, where $\psi(G_1) = (q^{\beta+1} - q + 1)(p^{2\alpha-1} + 1)/(p+1)$ by Lemmas 2.3, 2.4 and 2.2. Therefore, by some computations,

$$\begin{aligned} o(G) - o(G_2) &= \frac{\psi(G_1) + q^\beta p^{2\alpha-1}(p-1)}{q^\beta p^\alpha} - \frac{\psi(C_{p^\alpha})}{p^\alpha} \\ &= \frac{(q^{\beta+1} - q + 1)(p^{2\alpha-1} + 1) + q^\beta p^{2\alpha-1}(p^2 - 1) - q^\beta(p^{2\alpha+1} + 1)}{q^\beta p^\alpha (p+1)} \\ &= \frac{(q-1)(p^{2\alpha-1} + 1)(q^\beta - 1)}{q^\beta p^\alpha (p+1)} > 0. \end{aligned}$$

Therefore, G_2 has the average property. For the rest of the proof, by Proposition 2.8, it suffices to study the average property on G_1. We have

$$\begin{aligned} o(G) - o(G_1) &= \frac{\psi(G_1) + q^\beta p^{2\alpha-1}(p-1)}{q^\beta p^\alpha} - \frac{\psi(G_1)}{q^\beta p^{\alpha-1}} \\ &= \frac{q^\beta p^{2\alpha-1}(p-1) - (p-1)\psi(G_1)}{q^\beta p^\alpha} \\ &= (p-1)\frac{q^\beta p^{2\alpha-1}(p+1) - (q^{\beta+1} - q + 1)(p^{2\alpha-1} + 1)}{q^\beta p^\alpha (p+1)}. \qquad (2) \end{aligned}$$

(a) Suppose that $\beta = 1$. Then by (2),

$$o(G) - o(G_1) = (p-1)\frac{qp^{2\alpha-1}(p+1) - (q^2 - q + 1)(p^{2\alpha-1} + 1)}{qp^\alpha (p+1)}.$$

If $p = 2$ and $q = 3$ then $o(G) - o(G_1) = (4^\alpha - 7)/(9 \cdot 2^\alpha)$. Therefore, if $\alpha = 1$ then $o(G_1) > o(G)$ and so, G_1 does not have the average property and G does not satisfy the average condition. If $\alpha \geq 2$ then $o(G_1) < o(G)$. Hence, G_1 has the average property and G satisfies the average condition. If $p = 2$ and $q > 3$ then by (2) we have

$$o(G) - o(G_1) = \frac{3 \cdot 2^{2\alpha-1}q - (q^2 - q + 1)(2^{2\alpha-1} + 1)}{3 \cdot 2^\alpha q}$$

which is negative as $q \geq 5$. Therefore, G_1 does not have the average property. Hence, G does not satisfy the average condition.

To complete the proof in the case $\beta = 1$, we show that if $p > 2$ and $q \geq 3$ then G does not satisfy the average condition. In this case, p and q are odd primes and since p divides $q-1$, there exists a positive even integer k such that $q = kp + 1$. Thus $q \geq 2p + 1 > 2p > p + 2$. By (2), we have

$$o(G) - o(G_1) = (p-1)\frac{qp^{2\alpha-1}(p+1) - (q(q-1) + 1)(p^{2\alpha-1} + 1)}{qp^\alpha (p+1)}).$$

Since $q-1>p+1$,

$$(q(q-1)+1)(p^{2\alpha-1}+1) > (q(p+1)+1)(p^{2\alpha-1}+1) > q(p+1)p^{2\alpha-1}.$$

This implies that $o(G)-o(G_1)<0$. Therefore, G_1 does not have the average property and G does not satisfy the average condition.

(b) Suppose that $\beta \geq 2$. By (2),

$$o(G)-o(G_1) = (p-1)\frac{q^{\beta}p^{2\alpha-1}(p+1)-(q^{\beta+1}-q+1)(p^{2\alpha-1}+1)}{q^{\beta}p^{\alpha}(p+1)}.$$

If $p>q$ then $p-q+1>1$. It follows that $p^{2\alpha-1}(p-q+1)>q$ for all $\alpha \geq 1$. This is equivalent to $p^{2\alpha-1}(p+1)>q(p^{2\alpha-1}+1)$. Therefore, $q^{\beta}p^{2\alpha-1}(p+1) > q^{\beta+1}(p^{2\alpha-1}+1) > (q^{\beta+1}-(q-1))(p^{2\alpha-1}+1)$ and we infer that $o(G)-o(G_1)>0$. Hence, G_1 has the average property and G satisfies the average condition. If $p<q$ then, by Theorem 2.1, $\beta<p$. This implies that p and q are odd primes. It follows that $q-1 \geq p+1$. So, $(q^{\beta}(q-1)+1)(p^{2\alpha-1}+1) > (q-1)q^{\beta}p^{2\alpha-1} \geq (p+1)q^{\beta}p^{2\alpha-1}$ and $o(G)-o(G_1)<0$. Hence, G_1 does not have the average property and G does not satisfy the average condition. □

Consider the following example:

Example 3.5. *Suppose that $G = C_3 \rtimes C_4$. $G_1 = C_3 \times C_2$ and $G_2 = C_4$ are maximal subgroups of G, up to isomorphism. We obtain that $o(G) = 3.75$, $o(G_1) = 3.5$ and $o(G_2) = 2.75$ so, G satisfies the average condition.*

Corollary 3.6. *Let $G \cong C_q^{\beta} \rtimes C_{p^{\alpha}}$ be a finite minimal non-abelian group, where p and q are distinct primes and $G_1 \cong C_q^{\beta} \ltimes C_{p^{\alpha-1}}$ is a maximal subgroup of G. Hence, $\beta = 1$, $p=2$, $q=3$, $\alpha \geq 2$ or $\beta \geq 2$, $p>q$ and $\alpha \geq 1$ if and only if $o(G_1)<o(G)$.*

4. Conclusion

In this paper, we continue the study of average condition on minimal non-abelian groups. We say that a finite group satisfies the average condition if the average order of its subgroup is less than or equal to the average condition of the group. We classify finite minimal non-abelian groups which satisfies the average condition.

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