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# INDEPENDENCE NUMBER IN GRAPHS AND ITS UPPER BOUNDS

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Abstract. In this paper, we use the double counting method to find some upper bounds for the independence number of a simple graph in terms of its order, size and maximum degree. Moreover, we determine extremal graphs attaining equality in upper bounds. In addition, some lower bounds for the energy of graphs in terms of their size and maximum degree and the number of odd cycle, are determined.

*Keywords*: Independence number; Maximum degree; Edge; Energy.
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## 1. Introduction

Let $G$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The number of adjacent vertices with the vertex $v_i$ is called its *degree* and it is denoted by $d(v_i)$; the maximum number between vertex degrees of $G$ is called its *maximum degree* which we denote by $\Delta = \Delta(G)$. A vertex with degree one is called a *pendant* vertex and a vertex of degree zero is called an *isolated* vertex. A graph $G$ is called *connected* if for all distinct vertices $x$ and $y$, there exists at least one path from $x$ to $y$; otherwise, it is said that $G$ is *disconnected*. Any maximal connected subgraph of $G$ is called a *connected component*; and, the number of these components is denoted by $c(G)$.

The *adjacency matrix* of $G$, denoted by $A = A(G)$ is a square matrix of order $n$ whose diagonal entries are zero and the $ij^{\text{th}}$ entry with $i \neq j$ equals the number of edges between $v_i$ and $v_j$. Since $G$ is a simple graph, $A$ is a symmetric matrix with entries 0 and 1. So, all of its eigenvalues are real numbers. In algebraic graph theory, the set of these eigenvalues are sorted non-increasingly and it is said to be the *spectrum* of $G$, denoted by $Spec(G)$. Also, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $A$, then the *energy* of $G$ is considered as:

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

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It may be worthy to mention that Gutman in 1978 [3] was the first researcher that introduced the notion of the energy of graphs; however, the authors in [5] have studied many properties of the energy of graphs.

An induced subgraph of $G$ on $X \subseteq V$, denoted by $G[X]$ is a subgraph whose vertex set is $X$ and for any two distinct vertices $x, y \in X$, $xy$ is an edge of $G[X]$ if and only if it is an edge of $G$. A subset $S \subseteq V$ of vertices of $G$ is called an *independent* set if the induced subgraph on $S$ contains no edge. In addition, the *independence number* of $G$ is the size of the largest independent set of $G$ and here, we denote by $\alpha = \alpha(G)$. If the vertex set of $G$ can be partitioned in to two disjoint independent sets, then we say that $G$ is a *bipartite* graph. If moreover, every vertex of one part is joined to every vertex of the other part, it is said that $G$ is a *complete bipartite graph*. It is well-known that bipartite graphs are graphs with no odd cycles. Here, for any graph $G$, we denote the number of odd cycles in $G$ by $c_o(G)$. So, it is clear that $G$ is a bipartite graph if and only if $c_o(G) = 0$. In this article, by $k$-connected graph, we mean $k$-vertex-connected graph. For more details about basic notations and definitions, see [2, 7].

The independence number is one of the most applicable parameters in graph theory. This concept is used in other fields of science such as Computer science, Biology, Engineering, Chemistry, Network Security, Coding theory, Operation research and Social sciences. Understanding the properties and applications of the independence number is very important for solving diverse real-world problems such as optimizing resource allocation, analyzing social networks and etc. Moreover, there is a relation between the independence number of a graph with its other parameters, say the clique number, the chromatic number, the vertex cover number, the domination number. Therefore, these reasons motivates any graph theorist to study the independence number of graphs. On the other hand, we know that the problem of determining the independence number of a general graph is NP-hard. Hence finding upper bounds for the independence number can be very important and useful.

In this article, we present some upper bounds for $\alpha$ and investigate when the equalities hold. Moreover, we prove some results about the energy of graphs.

## 2. Bounds for the independence number and the energy of graphs

First we start with the following known upper bound for independence number $\alpha$ of graphs. This result has been proved for $k$-independence number in [4] for graphs with diameter at least $k + 1$. Here, we will use the double counting technique to prove it. Also, a condition is found under which the equality holds.

**Theorem 2.1.** *If $G$ is a connected $n$-vertex graph ($n \geq 2$) with independence number and maximum degree, $\alpha$ and $\Delta$, respectively, then*

$$\alpha \leq \frac{n\Delta}{\Delta + 1}. \tag{1}$$

*Also, the equality holds if and only if* $\alpha = \Delta = n - 1$.

*Proof.* Let $A$ be an independent set of $G$ with $|A| = \alpha$ and let $H$ be a subgraph of $G$, induced on $V \setminus A$. Put

$$F = \{e = xy| \ e \text{ is an edge of G}, x \in V(H) \text{ and } y \in A\}.$$

Since $A$ is an independent set, any edge have an end in $H$. By hypothesis, $G$ contains no isolated verex and so,

$$|F| \geq |A| = \alpha \tag{2}$$

Moreover, one can easily check that

$$|F| \leq \quad (n - \alpha)\Delta. \tag{3}$$

Combining (2) with (3), we have

$$\alpha \leq \frac{n\Delta}{\Delta + 1}. \tag{4}$$

Also, it is clear that $\alpha = \frac{n\Delta}{\Delta+1}$ if and only if the following three statements hold:

(a) The equality in (3) yields that any vertex of $H$ has maximum degree;
(b) There is no edge between vertices of $H$;
(c) From the equality in (2), we deduce that any vertex, belonging to $A$, is adjacent to one (no more) vertex of the subgraph $H$; and so, $\alpha = |F|$.

Now, the connectivity of $G$ implies that $G$ satisfies all of the above three conditions if and only if $G \cong \beta K_{1,\alpha}$, for some $\beta$. Therefore, $\alpha = \frac{n\Delta}{\Delta+1}$ only when $\alpha = \Delta = n - 1$. □

From the proof of Theorem 2.1, one can conclude that equality $\alpha = \frac{n\Delta}{\Delta+1}$ holds only for star graphs. Next, we need a new notation. By $ca(G)$, we mean the minimum number of the connected components of the subgraphs $G \setminus A$, when $A$ changes on the maximum independent subsets of $G$.

**Example 2.2.** *Let $G$ be a refinement of a star of order $n$ (a graph whose one of its vertices is adjacent to any other of vertices). Then $\Delta = n - 1$ and from Theorem 2.1, we deduce that $1 \leq \alpha(G) \leq n - 1$. The lower bound is attained if and only if $G$ is a complete graph. Also, $\alpha(G) = n - 1$ if and only if $G$ is a star graph. About complete $r$-partite graph $K_{n_1,n_2,\ldots,n_r}$ with $n_1 \geq \cdots \geq n_r$, we have $\alpha \leq n\frac{n_1}{n_1+1}$; in particular, for the complete bipartite graph $K_{n_1,n_2}$ with $n_1 \geq n_2$, we have $\alpha = n_1 \leq \frac{nn_1}{n_1+1}$ and the equality holds if and only if $n_2 = 1$.*

**Theorem 2.3.** *For any $n$-vertex graph $G$ with $n \geq 3$, we have the following upper bound for the independence number,*

$$\alpha \leq \frac{n(\Delta - 2) + 2ca(G)}{\Delta - 1}.$$

*Proof.* Choose any maximum independent set $A$ of $G$ and let $H = G \setminus A$. Let $F$ be the set of edges between $V(H)$ and $A$. Now, we apply the double counting principle, for the set $F$. Since $A$ is an independent set and $G$ is a connected graph,

$$|F| \geq |A| = \alpha \tag{5}$$

On the other hand,

$$\begin{aligned} |F| = \sum_{v \notin A} (d_G(v) - d_H(v)) &= \sum_{v \notin A} d_G(v) - \sum_{v \notin A} d_H(v) \\ &\leq (n-\alpha)\Delta - 2|E(H)| \\ &\leq (n-\alpha)\Delta - 2(n-\alpha-c(H)) \\ &= (n-\alpha)(\Delta-2) + 2c(H). \end{aligned} \tag{6}$$

From the above two nonequalities, we conclude that

$$\alpha \leq \frac{n(\Delta-2)+2c(H)}{\Delta-1}. \tag{7}$$

Now, let $A$ be a maximum independent set such that $c(H) = ca(G)$. Then we have

$$\alpha \leq \frac{n(\Delta-2)+2ca(G)}{\Delta-1}, \tag{8}$$

as desired. □

Next, we state an immediate consequence of the previous theorem.

**Corollary 2.4.** *Let $G$ be a connected graph which has no isolated vertex. Also, assume that $G$ contains a maximum independent set $A$ such that the induced subgraph on $V \setminus A$ is a connected graph. Then $\alpha \leq \frac{n(\Delta-2)+2}{\Delta-1}$.*

*Proof.* The assertion follows from the hypothesis (the existence of $A$) and the Nonequality (8). □

*Remark* 2.5. It is clear that $\Delta \leq n-1$. Adding $(n\Delta^2 + \Delta - 2n\Delta)$ to both sides of this inequality and rearranging all phraces, we have $n\Delta(\Delta-2)+2\Delta \leq n(\Delta-1)^2 + \Delta - 1$ and so,

$$\frac{n(\Delta-2)+2}{\Delta-1} \leq \frac{n(\Delta-1)+1}{\Delta}.$$

So, we deduce that the upper bound for the independence number in Corollary 2.4 is better than that of Brog's upper (see [1]).

Recall that a graph $G$ is called a split graph if its vertex set can be partitioned into a clique and an independent set. It is clear that the complement of a split graph is a split graph, too.

**Example 2.6.** *If $G$ is a split graph whose vertex set has been partitioned to the clique $C$ and the independent set $A$. Then any maximum independent set of $G$ is of the form $A \cup C'$ with $C' \subset C$ and it is clear that $G - (A \cup C')$ is a connected graph. So, by Corollary 2.4, $\alpha \leq \frac{n(\Delta-2)+2}{\Delta-1}$. In particular, if $G$ is the join of a complete graph $K_r$ and a null graph $\overline{K_s}$, then every vertex of $C$ is adjacent to any vertex of $A$ and hence $\alpha = s$ which is not greater than $\frac{n(\Delta-2)+2}{\Delta-1} = \frac{(r+s)(r+s-3)+2}{r+s-2}$.*

**Theorem 2.7.** *Let $G$ be a connected graph with $m$ edges and it contains no isolated vertex. Then we have*

$$\alpha \leq n + \Delta - \frac{1+\sqrt{(2\Delta-1)^2+8m}}{2}.$$

*In addition, the equality holds if and only if $G$ is a star graph.*

*Proof.* Let $A$ be a maximum independent set of $G$ and $H = G \setminus A$. We count the number of edges of $H$ by using double counting. It is clear that $H$ is a subgraph of $K_{n-\alpha}$ and so,

$$|E(H)| \leq \frac{(n-\alpha)(n-\alpha-1)}{2} \tag{9}$$

On the other hand, since the induced subgraph $G[A]$ is an empty graph and $G$ contains no vertex of degree zero, so

$$|E(H)| = m - |F|, \tag{10}$$

where $F$ is the set of edges between $A$ and $V(H)$. If we use a proof similar to that of Theorem 2.1, we have

$$\begin{aligned} |F| = \sum_{v\in V(H)} (d_G(v) - d_H(v)) &= \sum_{v\in V(H)} d_G(v) - \sum_{v\in V(H)} d_H(v) \\ &= \sum_{v\in V(H)} d_G(v) - 2|E(H)| \\ &\leq (n-\alpha)\Delta. \qquad (11) \end{aligned}$$

Now, we use the above two inequalities to deduce that:

$$|E(H)| \geq m - (n-\alpha)\Delta. \tag{12}$$

Then we use (9) with (12), to obtain the following quadratic inequality:

$$\alpha^2 - (2n+2\Delta-1)\alpha + n^2 + (2\Delta-1)n - 2m \geq 0. \tag{13}$$

Therefore, one of the following inequalities holds:

$$\alpha \geq \frac{2n+2\Delta-1+\sqrt{(2\Delta-1)^2+8m}}{2};$$

$$\alpha \leq \frac{2n+2\Delta-1-\sqrt{(2\Delta-1)^2+8m}}{2}.$$

Since $\alpha \leq n-1$, the first inequality does not hold. So,

$$\alpha \leq n+\Delta-\frac{1+\sqrt{(2\Delta-1)^2+8m}}{2}. \tag{14}$$

Also, the equality in (14) holds if and only if all of the inequalities before it turn into equalities, this yields the following two staements:

(a) $H \cong K_{n-\alpha}$, by (9);
(b) By (11), $H$ is an empty graph and every one of its vertices is adjacent with $\Delta$ vertices in $A$.

Hence, the equality holds if and only if $G$ is a star graph. □

From Theorems 2.1 and 2.7, one can easily deduce the following consequence.

**Corollary 2.8.** *For any connected graph $G$ with $n$ vertices, $m$ edges, maximum degree $\Delta$, we have*

$$\alpha(G) \leq \min\left\{\left\lfloor\frac{n\Delta}{\Delta+1}\right\rfloor, \left\lfloor n+\Delta-\frac{1+\sqrt{(2\Delta-1)^2+8m}}{2}\right\rfloor\right\}.$$

Now, some bounds for the energy of graphs are presented. Before this, we recall the following theorem from Wang and Ma [6] which gives a lower bound for the energy.

**Theorem 2.9.** ( [6, Theorem 4.2]) *For any graph $G$, we have $\mathcal{E}(G) \geq 2\alpha - 2c_o(G)$. If moreover, $G$ is a bipartite, then $\mathcal{E}(G) \geq 2\alpha$.*

Using Theorems 2.7 and 2.9, we have the following proposition which determines another lower bound for $\mathcal{E}(G)$.

**Proposition 2.10.** *For any graph $G$, the following lower bound for energy holds:*

$$\mathcal{E}(G) \geq \sqrt{(2\Delta-1)^2+8m}-2(\Delta+c_o(G))+1.$$

*If moreover, $G$ contains no odd cycle, then*

$$\mathcal{E}(G) \geq \sqrt{(2\Delta-1)^2+8m}-(2\Delta-1).$$

Finally, by using Theorem 2.3 and Corollary 2.10, we determine a new lower bound for the energy of disconnected graphs.

**Corollary 2.11.** *If $G$ is a graph which may be disconnecred, then*

$$\mathcal{E}(G) \geq 2(n-\alpha-c_o(G)) \geq \sqrt{(2\Delta-1)^2+8m}-2(\Delta+c_o(G))+1.$$

## 3. Conclusion

The independence number of a graph is one of the most important and applicable parameters in graph theory and it has many applications in diverse fields of mathematics, biology and engineering. It has a crucial role in solving real-world problems such as social networks, marketing, optimizing resource allocation, resource allocation, protein folding. But since the problem of finding independence number of a graph is NP-hard, we are looking for bounds for it. In this paper, some upper bounds in terms of the order, the size and the maximum degree are studied. also, lower bounds for the energy of graphs are presented. In future work, we try to find bounds for the special families of graphs and then we will use them to prove useful bounds for the energy of graphs.

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