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ON FUNCTIONAL IDENTITIES INVOLVING n-DERIVATIONS IN RINGS

V. Varshney ✉, S. Ali, N. N. Rafiquee, and K. B. Wong

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Abstract. In this paper, we explore various properties associated with the traces of permuting n-derivations satisfying certain functional identities that operate on a Lie ideal within prime and semiprime rings. Additionally, we address and discuss correlated findings pertaining to left n-multipliers. Lastly, we enrich our results with examples that show the necessity of their assumptions.

Keywords: Derivation, Semiprime ring, Lie ideal, Derivation, Symmetric n-derivation, n-multiplier.
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1. Introduction & Preliminaries

In ring theory, functional identities play a crucial role in unraveling intricate relationships between various algebraic structures. One such area of exploration lies in understanding the behavior of permuting n-derivations and their interactions within rings. A ring $\mathfrak{S}$ is said to be prime if $\varepsilon\mathfrak{S}\varsigma = \{0\}$ implies that either $\varepsilon = 0$ or $\varsigma = 0$, for all $\varepsilon, \varsigma \in \mathfrak{S}$. Similarly, $\mathfrak{S}$ is called semiprime if $\varepsilon\mathfrak{S}\varepsilon = \{0\}$ implies $\varepsilon = 0$, where $\varepsilon \in \mathfrak{S}$. The notation $[\varepsilon, \varsigma]$ represents the commutator $\varepsilon\varsigma - \varsigma\varepsilon$, while $\varepsilon \circ \varsigma$ denotes the anti-commutator $\varepsilon\varsigma + \varsigma\varepsilon$, where ε and ς are any elements belonging to the ring $\mathfrak{S}$. A ring $\mathfrak{S}$ is considered n-torsion free if the condition $n\varepsilon = 0$ implies that $\varepsilon = 0$ for every element ε in $\mathfrak{S}$. Moreover, if $\mathfrak{S}$ is $n!$-torsion free, then it is also m-torsion free for every divisor m of $n!$. An additive subgroup U of $\mathfrak{S}$ is called a Lie ideal of $\mathfrak{S}$ if the commutator $[u, r]$ belongs to U for every u in U and r in $\mathfrak{S}$. Furthermore, U is termed as a square closed Lie ideal of $\mathfrak{S}$ if it satisfies the conditions of being a Lie ideal and if the square of every element u in U also belongs to U. A mapping $\mathcal{D} : \mathfrak{S} \to \mathfrak{S}$ is called a derivation if $\mathcal{D}$ is additive and satisfies the condition $\mathcal{D}(\varepsilon\varsigma) = \mathcal{D}(\varepsilon)\varsigma + \varepsilon\mathcal{D}(\varsigma)$ for all $\varepsilon, \varsigma \in \mathfrak{S}$. Following [8], an additive mapping $g : \mathfrak{S} \longrightarrow \mathfrak{S}$ is said to be a generalized derivation on $\mathfrak{S}$ if there exists a derivation $d : \mathfrak{S} \longrightarrow \mathfrak{S}$ such that $g(\varepsilon\varsigma) = g(\varepsilon)\varsigma + \varepsilon d(\varsigma)$ holds for all $\varepsilon, \varsigma \in \mathfrak{S}$. To expand the range of derivation, the concept of symmetric bi-derivations on rings was proposed

✉ vaishali.varshney@gla.ac.in, ORCID: 0000-0002-5787-2059
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by Maksa [14]. Vukman further investigated this concept extensively in his works [21] and [22]. A bi-additive map $\mathfrak{D} : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ is said to be a bi-derivation if $\mathfrak{D}(\varepsilon\varepsilon', \varsigma) = \mathfrak{D}(\varepsilon, \varsigma)\varepsilon' + \varepsilon\mathfrak{D}(\varepsilon', \varsigma)$ and $\mathfrak{D}(\varepsilon, \varsigma\varsigma') = \mathfrak{D}(\varepsilon, \varsigma)\varsigma' + \varsigma\mathfrak{D}(\varepsilon, \varsigma')$ hold for any $\varepsilon, \varepsilon', \varsigma, \varsigma' \in \mathfrak{S}$. The foregoing conditions are identical if $\mathfrak{D}$ is also a symmetric map meaning that $\mathfrak{D}(\varepsilon, \varsigma) = \mathfrak{D}(\varsigma, \varepsilon)$ for all $\varepsilon, \varsigma \in \mathfrak{S}$. When this condition holds, $\mathfrak{D}$ is a symmetric bi-derivation on $\mathfrak{S}$. Numerous researchers have investigated symmetric bi-derivations on rings (see [13], [20] and references therein). Argac and Yenigul [1] and Muthana [15] achieved analogous findings regarding Lie ideals within the ring $\mathfrak{S}$. In [18], Rehman and Ansari characterized the trace of symmetric bi-derivation and attained more general outcomes by examining various conditions on a subset of a ring $\mathfrak{S}$, specifically focusing on Lie ideals within $\mathfrak{S}$.

From the given framework of bi-derivation, Park [16] brought forth the idea of permuting n-derivation in the following manner:

Definition 1.1. Let $n \geq 2$ be a fixed integer, and $\mathfrak{S}^n = \mathfrak{S} \times \mathfrak{S} \times \cdots \times \mathfrak{S}$. A map $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be symmetric (permuting) if

$$\mathfrak{D}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \mathfrak{D}(\varepsilon_{\varkappa(1)}, \varepsilon_{\varkappa(2)}, \ldots, \varepsilon_{\varkappa(n)})$$

for all permutations $\varkappa(i) \in S_n$ and $\varepsilon_i \in \mathfrak{S}$, where $i = 1, 2, \ldots, n$.

Definition 1.2. Let $n \geq 2$ be a fixed integer. A map $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ is said to be a symmetric n-derivation if $\mathfrak{D}$ is symmetric and n-additive (i.e., additive in each argument) and

$$\mathfrak{D}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i\varepsilon_i', \ldots, \varepsilon_n) = \varepsilon_i\mathfrak{D}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i', \ldots, \varepsilon_n) + \mathfrak{D}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_i, \ldots, \varepsilon_n)\varepsilon_i'$$

holds for all $\varepsilon_i, \varepsilon_i' \in \mathfrak{S}$.

Of course, 1-derivation is a derivation and for $n = 2$, $\mathfrak{D}$ is referred to as a symmetric bi-derivation on rings.

The trace is a crucial aspect of a symmetric n-derivation. It plays a significant role when extending results established for derivations or bi-derivations to n-derivations. It is defined as follows:

Definition 1.3. Let $n \geq 2$ be a fixed integer and a map $\mathfrak{f} : \mathfrak{S} \to \mathfrak{S}$ defined by $\mathfrak{f}(\varepsilon) = \mathfrak{D}(\varepsilon, \varepsilon, \ldots, \varepsilon)$ for all $\varepsilon \in \mathfrak{S}$, is called the trace of $\mathfrak{D}$.

If $\mathfrak{D}$ is symmetric n-additive, then the trace $\mathfrak{f}$ of $\mathfrak{D}$ satisfies the relation

$$\mathfrak{f}(\varepsilon + \varsigma) = \mathfrak{f}(\varepsilon) + \mathfrak{f}(\varsigma) + \sum_{l=1}^{n-1} {}^nC_l \; \mathfrak{D}(\underbrace{\varepsilon, \ldots, \varepsilon}_{(n-l)-\text{times}}, \underbrace{\varsigma, \ldots, \varsigma}_{l-\text{times}})$$

for all $\varepsilon, \varsigma \in \mathfrak{S}$, where ${}^nC_l = \binom{n}{l}$.

Recently, Ashraf et al. [2] introduced the notion of permuting n-multipliers in the following manner:

Definition 1.4. A permuting n-additive map $\mathcal{F}:\mathfrak{S}^n\to\mathfrak{S}$ is called a permuting left n-multiplier (resp. permuting right n-multiplier) if

$$\mathcal{F}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_i\varepsilon_i',\ldots,\varepsilon_n)=\mathcal{F}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_i,\ldots,\varepsilon_n)\varepsilon_i'$$

$$(\text{resp. } \mathcal{F}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_i\varepsilon_i',\ldots,\varepsilon_n)=\varepsilon_i\mathcal{F}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_i',\ldots,\varepsilon_n))$$

holds for all $\varepsilon_i,\varepsilon_i'\in\mathfrak{S},\ i=1,2,\ldots,n$. If $\mathcal{F}$ is both a permuting left n-multiplier and a permuting right n-multiplier, it is referred to as a permuting n-multiplier.

Inspired by the idea of generalized derivation in rings, they also introduced the concept of symmetric generalized n-derivation in rings. Let $n\geq 1$ be a fixed positive integer. A symmetric n-additive map $\mathcal{G}:\mathfrak{S}^n\to\mathfrak{S}$ is known to be symmetric generalized n-derivation if there exists a symmetric n-derivation $\mathfrak{D}:\mathfrak{S}^n\to\mathfrak{S}$ such that $\mathcal{G}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_i\varepsilon_i',\ldots,\varepsilon_n)=\mathcal{G}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_i,\ldots,\varepsilon_n)\varepsilon_i'+\varepsilon_i\mathfrak{D}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_i',\ldots,\varepsilon_n)$ holds for all $\varepsilon_i,\varepsilon_i'\in\mathfrak{S}$.

The study of commuting and centralizing mappings on a prime ring was initiated by Posner [17]. In his work, the author demonstrated that if a prime ring $\mathfrak{S}$ possesses a nonzero centralizing derivation, then $\mathfrak{S}$ must be commutative. In 1993, Brešar [7] generalized the result of Posner for left ideals of a prime ring. Certainly, Brešar's proof establishes that in a prime ring $\mathfrak{S}$, if I is a nonzero left ideal, and there exist nonzero derivations f and g of $\mathfrak{S}$ satisfying $f(\varepsilon)\varepsilon-\varepsilon g(\varepsilon)\in\mathcal{Z}(\mathfrak{S})$ for all $\varepsilon\in I$, then $\mathfrak{S}$ is commutative. In [18], the authors meticulously explored the commutativity properties of prime rings under specific identities dictated by the trace of symmetric bi-derivations. In fact, they proved that in a prime ring $\mathfrak{S}$ with characteristic not equal to 2, if U is a square closed Lie ideal of $\mathfrak{S}$ and $B:\mathfrak{S}\times\mathfrak{S}\to\mathfrak{S}$ is a symmetric bi-derivation with trace f, then for any mapping $g:\mathfrak{S}\to\mathfrak{S}$ satisfying $f(\varepsilon)\varsigma-\varepsilon g(\varsigma)\in Z(\mathfrak{S})$ for all $\varepsilon,\varsigma\in U$, it must hold that either $U\subseteq Z(\mathfrak{S})$ or $f=0$. In the study conducted by Gölbaşi and Sögütcü, as outlined in [10], they examined the characteristics of a prime ring $\mathfrak{S}$ equipped with a generalized derivation (F,d) operating on a Lie ideal U of $\mathfrak{S}$. They proved that if $[F(u),u]\in\mathcal{Z}(\mathfrak{S})$, for all $u\in U$, then either $d=0$ or $U\subseteq\mathcal{Z}(\mathfrak{S})$. Very recently, in [19], they also conducted an investigation into the characteristics of semiprime rings acting on the Lie ideal of $\mathfrak{S}$, where they characterize certain functional identities in the presence of trace of symmetric bi-derivation on $\mathfrak{S}$. They demonstrated that if $\mathfrak{S}$ is a semiprime ring with U as a square-closed Lie ideal, and $D:\mathfrak{S}\times\mathfrak{S}\to\mathfrak{S}$ a symmetric bi-derivation with trace d, then $\mathfrak{S}$ contains a nonzero central ideal if the condition $d(\varepsilon)d(\varsigma)\pm\varepsilon\varsigma\in Z(\mathfrak{S})$ holds for all $\varepsilon,\varsigma\in U$. In [2], Ashraf et al. established that "for a fixed integer $n\geq 2$, in a non-commutative semiprime ring $\mathfrak{S}$ with $n!$-torsion free property, if there exists a permuting generalized n-derivation $\mathcal{T}$ with an associated n-derivation $\mathcal{D}$ such that the trace ζ of $\mathcal{T}$ centralizes on $\mathfrak{S}$, then $\mathcal{T}$ acts as a left n-multiplier on $\mathfrak{S}$." Numerous researchers have investigated different identities that involve traces of bi-derivations and n-derivations, yielding several intriguing outcomes(viz.; [2], [20], [21], [22]).

The motivation behind this research is to expand on earlier results about derivations and biderivations by exploring similar properties in permuting n-derivations. By imposing conditions on a subset of the ring $\mathfrak{S}$, with a particular focus on the Lie ideal, this study aims to reveal new insights into the behavior and structure of prime and semiprime rings. Understanding this research not only enriches the theoretical framework of ring theory but also contributes to the broader mathematical understanding of algebraic systems. In particular, we establish that "for a given fixed integer $n \geq 2$ and an $n!$-torsion free semiprime ring $\mathfrak{S}$, along with a noncentral square closed Lie ideal U of $\mathfrak{S}$, if $\mathfrak{S}$ possesses two nonzero symmetric n-derivations $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $f : \mathfrak{S} \to \mathfrak{S}$ and $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$, satisfying $f(\varepsilon)\varsigma \pm \varepsilon g(\varsigma) \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$, then $\mathfrak{S}$ has a nonzero central ideal" (Theorem 2.1). Furthermore, we provide a characterization of the traces of q-iterations of n-derivations in prime rings. Specifically, we establish that "for a fixed integer $n \geq 2$, in a $n!$-torsion free prime ring $\mathfrak{S}$ with a noncentral square closed Lie ideal U, and for any integer $q \geq 1$, admitting q-iterations of n-derivations $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_q : \mathfrak{S}^n \to \mathfrak{S}$ such that the product of the traces of $\mathfrak{D}_1, \mathfrak{D}_2, \ldots, \mathfrak{D}_q$ respectively is zero on a nonzero Lie ideal of $\mathfrak{S}$, then either $\mathfrak{D}_1$ is identically zero or the remaining $\mathfrak{D}_i$'s act as n-multipliers on U".

We now provide some previously known results that play a crucial role in deriving our main findings in the subsequent section.

Lemma 1.5. *[16, Lemma 2.2] Let n be a fixed positive integer and $\mathfrak{S}$ a $n!$-torsion free ring. Suppose that $a_1, a_2, \ldots, a_n \in \mathfrak{S}$ satisfy $\lambda a_1 + \lambda^2 a_2 + \cdots + \lambda^n a_n = 0$ (or $\in \mathcal{Z}(\mathfrak{S})$) for $\lambda = 1, 2, \ldots, n$. Then $a_i = 0$ (or $\in \mathcal{Z}(\mathfrak{S})$) for $i = 1, 2, \ldots, n$.*

Lemma 1.6. *[9, Lemma 2(b)] If $\mathfrak{S}$ is a semiprime ring, then the center of a nonzero ideal of $\mathfrak{S}$ is contained in the center of $\mathfrak{S}$.*

Lemma 1.7. *[5, Theorem 3] Let $\mathfrak{S}$ be a 2-torsion free semiprime ring and U a noncentral Lie ideal of $\mathfrak{S}$ such that $u^2 \in U$ for all $u \in U$. Then there exist a nonzero ideal I of $\mathfrak{S}$ such that $I \subseteq U$.*

Lemma 1.8. *[6, Lemma 4] If $U \nsubseteq \mathcal{Z}(\mathfrak{S})$ is a Lie ideal of a prime ring $\mathfrak{S}$ and if $aUb = (0)$, then $a = 0$ or $b = 0$.*

Lemma 1.9. *[12, Corollary 2.1] Let $\mathfrak{S}$ be a 2-torsion free semiprime ring, U a Lie ideal of $\mathfrak{S}$ such that $U \nsubseteq Z(\mathfrak{S})$ and $a, b \in U$.*

(*i*) *If $aUa = \{0\}$, then $a = 0$.*
(*ii*) *If $aU = \{0\}$ ($Ua = \{0\}$), then $a = 0$.*
(*iii*) *If U is a square closed Lie ideal and $aUb = \{0\}$, then $ab = 0$ and $ba = 0$.*

Lemma 1.10. *[11, Lemma 1] Let $\mathfrak{S}$ be a semiprime, 2-torsion free ring and let U be a Lie ideal of $\mathfrak{S}$. Suppose that $[U,U] \subseteq Z(\mathfrak{S})$, then $U \subseteq Z(\mathfrak{S})$.*

2. Main results

In [18], Rehman and Ansari investigated the trace of symmetric bi-derivations. They achieved broader results by analyzing different conditions on a subset of the ring $\mathfrak{S}$, with a particular emphasis on Lie ideals within $\mathfrak{S}$. Specifically, they proved that in a prime ring $\mathfrak{S}$ with characteristic not equal to 2, if U is a square closed Lie ideal of $\mathfrak{S}$ and $B : \mathfrak{S}\times\mathfrak{S} \to \mathfrak{S}$ is a symmetric bi-derivation with trace f, then for any map $g : \mathfrak{S} \to \mathfrak{S}$ satisfying $f(\varepsilon)\varsigma - \varsigma g(\varepsilon) \in Z(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$ implies that either $U \subseteq Z(\mathfrak{S})$ or $f = 0$. This idea inspires our investigation into similar criteria for symmetric n-derivations and the corresponding ring behavior. In fact, our first main result is the following:

Theorem 2.1. *For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring, U a noncentral square closed Lie ideal of $\mathfrak{S}$ and $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ & $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ be two nonzero symmetric n-derivations on $\mathfrak{S}$ with traces $f : \mathfrak{S} \to \mathfrak{S}$ and $g : \mathfrak{S} \to \mathfrak{S}$ respectively satisfying $f(\varepsilon)\varsigma \pm \varepsilon g(\varsigma) \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$, then $\mathfrak{S}$ contains a nonzero central ideal.*

Proof. Given that

(1) $$f(\varepsilon)\varsigma \pm \varepsilon g(\varsigma) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

Replacing ς by $\varsigma + m\tau$ for $\tau \in U$ and $1 \leq m \leq n-1$, we get

$$f(\varepsilon)(\varsigma + m\tau) \pm \varepsilon g(\varsigma + m\tau) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Solving further, we see that

$$f(\varepsilon)\varsigma + f(\varepsilon)m\tau \pm \varepsilon g(\varsigma) \pm \varepsilon g(m\tau) \pm \varepsilon \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varsigma, \ldots, \varsigma}_{(n-l)-\text{times}}, \underbrace{m\tau, \ldots, m\tau}_{l-\text{times}}) \in \mathcal{Z}(\mathfrak{S})$$

for all $\varepsilon, \varsigma, \tau \in U$. Employing the given condition, we find that

$$\varepsilon \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varsigma, \ldots, \varsigma}_{(n-l)-\text{times}}, \underbrace{m\tau, \ldots, m\tau}_{l-\text{times}}) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U$$

which implies that,

$$m\binom{n}{1}\varepsilon h_1(\varsigma;\tau) + m^2\binom{n}{2}\varepsilon h_2(\varsigma;\tau) + \cdots + m^{n-1}\binom{n}{n-1}\varepsilon h_{n-1}(\varsigma;\tau) \in \mathcal{Z}(\mathfrak{S}),$$

where $h_l(\varsigma;\tau)$ represents the term in which τ appears l- times.
Applying Lemma 1.5 results in

$$n\varepsilon\mathcal{G}(\varsigma, \ldots, \varsigma, \tau) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Using the torsion restriction, we get

$$\varepsilon\mathcal{G}(\varsigma, \ldots, \varsigma, \tau) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Replacing τ by ς, we find that

$$\varepsilon g(\varsigma) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

Hence, by the hypothesis, we see that

$$f(\varepsilon)\varsigma \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

Commuting with r where $r \in \mathfrak{S}$ results in

$$[f(\varepsilon)\varsigma, r] = 0 \text{ for all } \varepsilon, \varsigma \in U,\ r \in \mathfrak{S}$$

$$\text{or, } f(\varepsilon)[\varsigma, r] + [f(\varepsilon), r]\varsigma = 0 \text{ for all } \varepsilon, \varsigma \in U,\ r \in \mathfrak{S}. \tag{2}$$

Replacing ς by $\varsigma\tau$ where $\tau \in U$ in (2) and using (2), we get

$$f(\varepsilon)\varsigma[\tau, r] = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U,\ r \in \mathfrak{S}.$$

Substituting r with $f(\varepsilon)$ in the preceding equation yields

$$f(\varepsilon)\varsigma[\tau, f(\varepsilon)] = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U. \tag{3}$$

Multiplying by τ from left, we get

$$\tau f(\varepsilon)\varsigma[\tau, f(\varepsilon)] = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U. \tag{4}$$

Taking $\tau\varsigma$ in place of ς in (3), we see that

$$f(\varepsilon)\tau\varsigma[\tau, f(\varepsilon)] = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U. \tag{5}$$

Subtracting (5) from (4), we get

$$[\tau, f(\varepsilon)]\varsigma[\tau, f(\varepsilon)] = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U$$

i.e.,

$$[\tau, f(\varepsilon)]U[\tau, f(\varepsilon)] = 0 \text{ for all } \varepsilon, \tau \in U,\ r \in \mathfrak{S}.$$

By Lemma 1.9, the last expression gives

$$[\tau, f(\varepsilon)] = 0 \text{ for all } \varepsilon, \tau \in U. \tag{6}$$

Invoking Lemma 1.7, there exists a nonzero ideal I of $\mathfrak{S}$ such that $I \subseteq U$. Hence, by (6) and using Lemma 1.6, we have

$$f(\varepsilon) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon \in I. \tag{7}$$

Now again replacing ε by $\varepsilon + m\varsigma_1$ for $\varsigma_1 \in I$ and $1 \leq m \leq n-1$ in (7) and using (7), we obtain

$$\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varepsilon, \ldots, \varepsilon}_{(n-l)-\text{times}}, \underbrace{m\varsigma_1, \ldots, m\varsigma_1}_{l-\text{times}}) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma_1 \in I.$$

Again using Lemma 1.5 and taking into account that $\mathfrak{S}$ is $n!$-torsion free, we get

$$\mathfrak{D}(\varsigma_1, \varepsilon, \ldots, \varepsilon) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma_1 \in I. \tag{8}$$

Replace ε by $\varepsilon + m\varsigma_2$ for $\varsigma_2 \in I$ and $1 \leq m \leq n-1$ in (8) to get

$$\mathfrak{D}(\varsigma_1, \varepsilon + m\varsigma_2, \dots, \varepsilon + m\varsigma_2) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma_1, \varsigma_2 \in I.$$

After further solving and applying torsion restriction, we obtain

$$\mathfrak{D}(\varsigma_1, \varsigma_2, \varepsilon, \dots, \varepsilon) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varsigma_1, \varsigma_2, \varepsilon \in I.$$

Continuing in the same manner, we get

$$\mathfrak{D}(\varsigma_1, \varsigma_2, \varsigma_3, \dots, \varsigma_n) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varsigma_1, \varsigma_2, \dots, \varsigma_n \in I. \tag{9}$$

On commuting with r, we get

$$[\mathfrak{D}(\varsigma_1, \varsigma_2, \varsigma_3, \dots, \varsigma_n), r] = 0 \text{ for all } \varsigma_1, \varsigma_2, \dots, \varsigma_n \in I, r \in \mathfrak{S}.$$

Substituting ς_1 with ς_1^2 in the last equation and applying the torsion restriction of $\mathfrak{S}$, we reach the following

$$[\varsigma_1, r]\mathfrak{D}(\varsigma_1, \varsigma_2, \dots, \varsigma_n) = 0 \text{ for all } \varsigma_1, \varsigma_2, \dots, \varsigma_n \in I, r \in \mathfrak{S}.$$

Now taking r to be rr' where $r' \in \mathfrak{S}$, we get

$$[\varsigma_1, r]r'\mathfrak{D}(\varsigma_1, \varsigma_2, \dots, \varsigma_n) = 0 \text{ for all } r, r' \in \mathfrak{S},$$

i.e.,

$$[\varsigma_1, r]\mathfrak{S}\mathfrak{D}(\varsigma_1, \varsigma_2, \dots, \varsigma_n) = \{0\} \text{ for all } \varsigma_1, \varsigma_2, \dots, \varsigma_n \in I, r \in \mathfrak{S}. \tag{10}$$

As $\mathfrak{S}$ is a semiprime ring, it necessarily includes a family of prime ideals whose intersection is zero. Let $\Pi = \{P_j \mid j \in \Lambda\}$ be the family of all prime ideals such that $\bigcap_{j\in\Lambda} P_j = \{0\}$. Let P be a typical member of Π. From (10), we can infer that for a fixed $\varsigma_1 \in I$, either

(*i*) $[\varsigma_1, r] \in P$ or

(*ii*) $\mathfrak{D}(\varsigma_1, \varsigma_2, \dots, \varsigma_n) \in P$ for all $\varsigma_1, \varsigma_2, \dots, \varsigma_n \in I, r \in \mathfrak{S}$.

Let us set $\mathscr{L} = \{\varsigma_1 \in I \mid [\varsigma_1, \mathfrak{S}] \subseteq P\}$ and $\mathscr{K} = \{\varsigma_1 \in I \mid \mathfrak{D}(\varsigma_1, \varsigma_2, \dots, \varsigma_n) \in P$ for all $\varsigma_2, \dots, \varsigma_n \in I\}$. Both $\mathscr{L}$ and $\mathscr{K}$ are additive subgroups of I such that $I = \mathscr{L} \cup \mathscr{K}$. However, a group cannot be expressed as the union of two of its proper subgroups. Hence, either $I = \mathscr{L}$ or $I = \mathscr{K}$. Let us suppose that $I \neq \mathscr{L}$. Then, we have $I = \mathscr{K}$, i.e., $\mathfrak{D}(\varsigma_1, \varsigma_2, \dots, \varsigma_n) \in P$ for all $\varsigma_1, \varsigma_2, \dots, \varsigma_n \in I$. Replace ς_1 by $\varsigma_1 r_1$, i.e., $\mathfrak{D}(\varsigma_1 r_1, \varsigma_2, \dots, \varsigma_n) \in P$ for any $r_1 \in \mathfrak{S}$. On solving, we get $\varsigma_1\mathfrak{D}(r_1, \varsigma_2, \dots, \varsigma_n) \in P$. Using the primeness of P, we get either $\varsigma_1 \in P$ or $\mathfrak{D}(r_1, \varsigma_2, \dots, \varsigma_n) \in P$ for all $\varsigma_1, \varsigma_2, \dots, \varsigma_n \in I,\ r_1 \in \mathfrak{S}$. But $\varsigma_1 \in P$ implies that $[\varsigma_1, \mathfrak{S}] \subseteq P$, which leads to a contradiction. Thus, we have $\mathfrak{D}(r_1, \varsigma_2, \dots, \varsigma_n) \in P$ for all $\varsigma_2, \dots, \varsigma_n \in I,\ r_1 \in \mathfrak{S}$. Again replace ς_2 by $\varsigma_2 r_2$ and adhering to the same method, we get $\mathfrak{D}(r_1, r_2, \varsigma_3, \dots, \varsigma_n) \in P$ for all $\varsigma_3, \dots, \varsigma_n \in I,\ r_1, r_2 \in \mathfrak{S}$. Carrying on in a similar vein, we reach

$$\mathfrak{D}(\mathfrak{S}, \mathfrak{S}, \dots, \mathfrak{S}) \subseteq P \text{ for any } P \in \Pi.$$

As P was chosen arbitrarily from Π. Therefore,

$$\mathfrak{D}(\mathfrak{S}, \mathfrak{S}, \dots, \mathfrak{S}) \subseteq \bigcap_{j\in\Lambda} P_j = \{0\}$$

and consequently, $\mathfrak{D}(\mathfrak{S},\mathfrak{S},\ldots,\mathfrak{S})=\{0\}$. As a consequence, we find ourselves with a contradiction. Therefore, $I=\mathscr{L}$, i.e., $[\varsigma_1,\mathfrak{S}]\subseteq P$ for all $\varsigma_1\in I$ or $[I,\mathfrak{S}]\subseteq\bigcap_{j\in\Lambda}P_j=\{0\}$. That is, $[I,\mathfrak{S}]=\{0\}$. Therefore, I is a nonzero central ideal of $\mathfrak{S}$. Hence $\mathfrak{S}$ has a nonzero central ideal. □

The following corollary is the immediate consequence of the above result.

Corollary 2.2. *[20, Theorem 1] Let $\mathfrak{S}$ be a 2-torsion free semiprime ring and U be a square closed Lie ideal of $\mathfrak{S}$. Suppose $\mathfrak{D}:\mathfrak{S}\times\mathfrak{S}\to\mathfrak{S}$, $\mathcal{G}:\mathfrak{S}\times\mathfrak{S}\to\mathfrak{S}$ two symmetric bi-derivations where d is the trace of $\mathfrak{D}$ and g is the trace of $\mathfrak{D}$ where $U\mathfrak{D}(U,U)\neq(0)$. If $d(\varepsilon)\varsigma\pm\varepsilon g(\varsigma)\in\mathscr{Z}(\mathfrak{S})$, for all $\varepsilon,\varsigma\in U$, then $\mathfrak{S}$ contains a nonzero central ideal.*

Proof. In the above theorem, if we put $n=2$, we get the required result. □

Lemma 2.3. *For a fixed integer $n\geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring, U a noncentral square closed Lie ideal of $\mathfrak{S}$. Suppose that $\mathfrak{S}$ admits two nonzero symmetric n-derivations $\mathfrak{D}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $f:\mathfrak{S}\to\mathfrak{S}$ and $\mathcal{G}:\mathfrak{S}^n\to\mathfrak{S}$ with trace $g:\mathfrak{S}\to\mathfrak{S}$ satisfying $f([\varepsilon,\varsigma])=[f(\varepsilon),\varsigma]+[f(\varsigma),\varepsilon]$* for all *$\varepsilon,\varsigma\in U$, then $\mathfrak{S}$ contains a nonzero central ideal.*

Proof. We have been given that

$$[f(\varepsilon),\varsigma]=\pm\,\varepsilon\circ g(\varsigma)\text{ for all }\varepsilon,\varsigma\in U.$$

Now replace ς by $\varsigma+m\tau$ for $\tau\in U$ and $1\leq m\leq n-1$, we get

$$[f(\varepsilon),\varsigma]+[f(\varepsilon),m\tau]=\pm\,\varepsilon\circ g(\varsigma)\pm\varepsilon\circ g(m\tau)\pm\varepsilon\circ\sum_{l=1}^{n-1}{}^{n}C_l\mathcal{G}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}},\underbrace{m\tau,\ldots,m\tau}_{l-\text{times}})$$

for all $\varepsilon,\varsigma,\tau\in U$. By using hypothesis, we get

$$\varepsilon\circ\sum_{l=1}^{n-1}{}^{n}C_l\mathcal{G}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}},\underbrace{m\tau,\ldots,m\tau}_{l-\text{times}})=0\text{ for all }\varepsilon,\varsigma,\tau\in U.$$

The application of Lemma 1.5 yields that

$$n\{\varepsilon\circ\mathcal{G}(\varsigma,\ldots,\varsigma,\tau)\}=0\text{ for all }\varepsilon,\varsigma,\tau\in U.$$

Using the torsion free restriction in $\mathfrak{S}$, we find that

$$\varepsilon\circ\mathcal{G}(\varsigma,\ldots,\varsigma,\tau)=0\text{ for all }\varepsilon,\varsigma,\tau\in U.$$

After replacing τ by ς, we get

$$\varepsilon\circ g(\varsigma)=0\text{ for all }\varepsilon,\varsigma\in U.$$

Again using the hypothesis, we get

$$[f(\varepsilon),\varsigma]=0\text{ for all }\varepsilon,\varsigma\in U$$

which is the same as (6). Thus, continuing in the same manner as our previous steps, we can infer that $\mathfrak{S}$ contains a nonzero central ideal. □

Lemma 2.4. *For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring, U a noncentral square closed Lie ideal of $\mathfrak{S}$. Suppose that $\mathfrak{S}$ admits two nonzero symmetric n-derivations $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $f : \mathfrak{S} \to \mathfrak{S}$ and $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ satisfying $[f(\varepsilon), \varsigma] = \pm\ \varepsilon \circ g(\varsigma)$* for all *$\varepsilon, \varsigma \in U$, then $\mathfrak{S}$ contains a nonzero central ideal.*

Proof. It is given that

$$f([\varepsilon,\varsigma]) = [f(\varepsilon),\varsigma] + [f(\varsigma),\varepsilon] \text{ for all } \varepsilon,\varsigma \in U.$$

On replacing ς by $\varsigma + m\tau$ for $\tau \in U$ and $1 \leq m \leq n-1$, we get

$$f([\varepsilon,\varsigma] + [\varepsilon, m\tau]) = [f(\varepsilon),\varsigma] + [f(\varepsilon), m\tau] + \\ [f(\varsigma) + f(m\tau) + \sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}}, \underbrace{m\tau,\ldots,m\tau}_{l-\text{times}}), \varepsilon]$$

for all $\varepsilon,\varsigma,\tau \in U$. On simplifying, we get

$$f([\varepsilon,\varsigma]) + f([\varepsilon,m\tau]) + \sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{[\varepsilon,\varsigma],\ldots,[\varepsilon,\varsigma]}_{(n-l)-\text{times}}, \underbrace{[\varepsilon,m\tau],\ldots,[\varepsilon,m\tau]}_{l-\text{times}})$$

$$= [f(\varepsilon),\varsigma] + [f(\varepsilon),m\tau] + [f(\varsigma),\varepsilon] + [f(m\tau),\varepsilon] + [\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}}, \underbrace{m\tau,\ldots,m\tau}_{l-\text{times}}), \varepsilon]$$

for all $\varepsilon,\varsigma,\tau \in U$. Using the hypothesis, we get

$$\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{[\varepsilon,\varsigma],\ldots,[\varepsilon,\varsigma]}_{(n-l)-\text{times}}, \underbrace{[\varepsilon,m\tau],\ldots,[\varepsilon,m\tau]}_{l-\text{times}}) = [\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}}, \underbrace{m\tau,\ldots,m\tau}_{l-\text{times}}), \varepsilon]$$

for all $\varepsilon,\varsigma,\tau \in U$. Taking account of the Lemma 1.5 and torsion free restriction in $\mathfrak{S}$, we get

$$\mathfrak{D}([\varepsilon,\varsigma],\ldots,[\varepsilon,\varsigma],[\varepsilon,\tau]) = [\mathfrak{D}(\varsigma,\ldots,\varsigma,\tau),\varepsilon] \text{ for all } \varepsilon,\varsigma,\tau \in U.$$

Replacing τ by ς, we get

$$f([\varepsilon,\varsigma]) = [f(\varsigma),\varepsilon] \text{ for all } \varepsilon,\varsigma \in U.$$

Utilizing the hypothesis once more, we derive

$$[f(\varepsilon),\varsigma] = 0 \text{ for all } \varepsilon,\varsigma \in U$$

which is equivalent to (6). Consequently, the conclusion is reached by employing the same line of reasoning as outlined in Theorem 2.1.

□

Lemma 2.5. *For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring, U a noncentral square closed Lie ideal of $\mathfrak{S}$. Suppose that $\mathfrak{S}$ admits two nonzero symmetric n-derivations $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $f : \mathfrak{S} \to \mathfrak{S}$ and*

$\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ satisfying $f(\varepsilon) \circ \varsigma = \pm\ \varepsilon \circ g(\varsigma)$ for all *$\varepsilon, \varsigma \in U$, then $\mathfrak{S}$ contains a nonzero central ideal.*

Proof. Given that

$$f(\varepsilon) \circ \varsigma = \pm\ \varepsilon \circ g(\varsigma) \text{ for all } \varepsilon, \varsigma \in U.$$

On replacing ς by $\varsigma + m\tau$ for $\tau \in U$ and $1 \leq m \leq n-1$, we get

$$f(\varepsilon) \circ (\varsigma + m\tau) = \pm\ \varepsilon \circ g(\varsigma + m\tau) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

On simplifying, we get

$$f(\varepsilon) \circ \varsigma + f(\varepsilon) \circ m\tau = \pm\varepsilon \circ g(\varsigma) \pm \varepsilon \circ g(m\tau) \pm \varepsilon \circ \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varsigma, \ldots, \varsigma}_{(n-l)-\text{times}}, \underbrace{m\tau, \ldots, m\tau}_{l-\text{times}})$$

for all $\varepsilon, \varsigma, \tau \in U$. On using the given condition, we find that

$$\varepsilon \circ \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varsigma, \ldots, \varsigma}_{(n-l)-\text{times}}, \underbrace{m\tau, \ldots, m\tau}_{l-\text{times}}) = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Application of Lemma 1.5 gives

$$n(\varepsilon \circ \mathcal{G}(\varsigma, \ldots, \varsigma, \tau)) = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Since $\mathfrak{S}$ is $n!$-torsion free, we have

$$\varepsilon \circ \mathcal{G}(\varsigma, \ldots, \varsigma, \tau) = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

On replacing τ by ς, we get

$$\varepsilon \circ g(\varsigma) = 0 \text{ for all } \varepsilon, \varsigma \in U.$$

Using the hypothesis one more time, we see that

$$f(\varepsilon) \circ \varsigma = 0 \text{ for all } \varepsilon, \varsigma \in U.$$

Replacing ς by $\varsigma\tau$ where $\tau \in U$, we find that

$$\varsigma[\tau, f(\varepsilon)] = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Replacing ς by $[\tau, f(\varepsilon)]\varsigma$ in the above equation, we have

$$[\tau, f(\varepsilon)]\varsigma[\tau, f(\varepsilon)] = 0 \text{ for all } \varepsilon, \varsigma, \tau \in U$$

or

$$[\tau, f(\varepsilon)]U[\tau, f(\varepsilon)] = \{0\} \text{ for all } \varepsilon, \tau \in U.$$

Using Lemma 1.9, we get

$$[\tau, f(\varepsilon)] = 0 \text{ for all } \varepsilon, \tau \in U$$

which is the same as (6). Therefore, following a similar approach, we can deduce that $\mathfrak{S}$ possesses a nontrivial central ideal. □

In 2001, Ashraf and Rehman [3] demonstrated that if a prime ring $\mathfrak{S}$ with a nonzero ideal I possesses a derivation δ satisfying either of the conditions: $\delta(\varepsilon\varsigma) \pm \varepsilon\varsigma \in Z(\mathfrak{S})$ or $\delta(\varepsilon\varsigma) \pm \varsigma\varepsilon \in Z(\mathfrak{S})$ for all $\varepsilon, \varsigma \in I$, then $\mathfrak{S}$ must be commutative. Additionally, Ashraf et al. [4] broadened these results to generalized derivations (F, d) associated with a nonzero derivation d of $\mathfrak{S}$. Motivated by the above mentioned result, we prove the similar result for the traces of permuting n-derivations on semiprime rings acting on the Lie ideal of the ring.

Theorem 2.6. *For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring and U be a square closed Lie ideal of $\mathfrak{S}$. Suppose that $\mathfrak{S}$ admits a symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $f : \mathfrak{S} \to \mathfrak{S}$ such that any one of the following conditions hold:*

(*i*) $f(\varepsilon\varsigma) \pm \varepsilon\varsigma \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$,
(*ii*) $f(\varepsilon\varsigma) \pm \varsigma\varepsilon \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$,
(*iii*) $f(\varepsilon\varsigma) \pm [\varepsilon, \varsigma] \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$,
(*iv*) $f(\varepsilon\varsigma) \pm \varepsilon \circ \varsigma \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$.

Then $U \subseteq \mathcal{Z}(\mathfrak{S})$.

Proof. (*i*) Given that

$$f(\varepsilon\varsigma) \pm \varepsilon\varsigma \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

Replace ς by $\varsigma + m\tau$ for $\tau \in U$ and $1 \leq m \leq n-1$, we get

$$f(\varepsilon(\varsigma + m\tau)) \pm \varepsilon(\varsigma + m\tau) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

That is,

$$f(\varepsilon\varsigma) + f(\varepsilon m\tau) + \sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varepsilon\varsigma, \ldots, \varepsilon\varsigma}_{(n-l)-\text{times}}, \underbrace{\varepsilon m\tau, \ldots, \varepsilon m\tau}_{l-\text{times}}) \pm \varepsilon\varsigma \pm \varepsilon m\tau \in \mathcal{Z}(\mathfrak{S})$$

for all $\varepsilon, \varsigma, \tau \in U$. On using the given condition, we see that

$$\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varepsilon\varsigma, \ldots, \varepsilon\varsigma}_{(n-l)-\text{times}}, \underbrace{m\varepsilon\tau, \ldots, m\varepsilon\tau}_{l-\text{times}}) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Utilizing Lemma 1.5 and taking into account the characteristic of $\mathfrak{S}$ to be $n!$-torsion free, we can derive at

$$\mathfrak{D}(\varepsilon\varsigma, \ldots, \varepsilon\varsigma, \varepsilon\tau) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Replace τ by ς to get

$$f(\varepsilon\varsigma) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

Again using the hypothesis, we get

$$\varepsilon\varsigma \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U. \tag{11}$$

Also,

$$\varsigma\varepsilon \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

This implies that $[\varepsilon, \varsigma] \in \mathscr{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$. By Lemma 1.10, we have the required conclusion.

(ii) Apply analogous reasoning as used in part (i) to obtain the desired outcome.

(iii) Given that

$$f(\varepsilon\varsigma) \pm [\varepsilon, \varsigma] \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

Replace ς by $\varsigma + m\tau$ for $\tau \in U$ and $1 \leq m \leq n-1$, we get

$$f(\varepsilon(\varsigma + m\tau)) \ \pm \ [\varepsilon, \varsigma + m\tau] \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

That is,

$$f(\varepsilon\varsigma)+f(\varepsilon m\tau)+\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varepsilon\varsigma, \ldots, \varepsilon\varsigma}_{(n-l)-\text{times}}, \underbrace{\varepsilon m\tau, \ldots, \varepsilon m\tau}_{l-\text{times}}) \pm [\varepsilon, \varsigma] \pm [\varepsilon, m\tau] \in \mathscr{Z}(\mathfrak{S})$$

for all $\varepsilon, \varsigma, \tau \in U$. Using the hypothesis, we see that

$$\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varepsilon\varsigma, \ldots, \varepsilon\varsigma}_{(n-l)-\text{times}}, \underbrace{m\varepsilon\tau, \ldots, m\varepsilon\tau}_{l-\text{times}}) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Invoking Lemma 1.5 and using torsion free restriction of $\mathfrak{S}$, we get

$$\mathfrak{D}(\varepsilon\varsigma, \ldots, \varepsilon\varsigma, \varepsilon\tau) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Replace τ by ς, we obtain

$$f(\varepsilon\varsigma) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

On using the hypothesis, we see that

$$[\varepsilon, \varsigma] \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

That is, $[U, U] \subseteq \mathscr{Z}(\mathfrak{S})$. Hence by Lemma 1.10, $U \subseteq \mathscr{Z}(\mathfrak{S})$.

(iv) On the contrary, suppose that $U \nsubseteq \mathscr{Z}(\mathfrak{S})$. Now, we have been given that

$$f(\varepsilon\varsigma) \pm \varepsilon \circ \varsigma \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

Taking $\varsigma + m\tau$ in the place of ς for $\tau \in U$ and $1 \leq m \leq n-1$, we get

$$f(\varepsilon(\varsigma + m\tau)) \pm \varepsilon \circ (\varsigma + m\tau) \in \mathscr{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

That is,

$$f(\varepsilon\varsigma)+f(\varepsilon m\tau)+\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varepsilon\varsigma, \ldots, \varepsilon\varsigma}_{(n-l)-\text{times}}, \underbrace{\varepsilon m\tau, \ldots, \varepsilon m\tau}_{l-\text{times}}) \pm \varepsilon \circ \varsigma \pm \varepsilon \circ m\tau \in \mathscr{Z}(\mathfrak{S})$$

for all $\varepsilon, \varsigma, \tau \in U$. By utilizing the provided condition, we observe that

$$\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varepsilon\varsigma, \ldots, \varepsilon\varsigma}_{(n-l)-\text{times}}, \underbrace{m\varepsilon\tau, \ldots, m\varepsilon\tau}_{l-\text{times}}) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Now, using Lemma 1.5 and the fact that $\mathfrak{S}$ is $n!$-torsion free, we obtain

$$\mathfrak{D}(\varepsilon\varsigma, \ldots, \varepsilon\varsigma, \varepsilon\tau) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Replace τ by ς, we get

$$f(\varepsilon\varsigma) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

Utilizing the given hypothesis, it is evident that

$$\varepsilon \circ \varsigma \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U. \tag{12}$$

Again replace ε by $\varsigma\varepsilon$, we have $\varsigma(\varepsilon \circ \varsigma) \in \mathcal{Z}(\mathfrak{S})$ which imply $[\varsigma(\varepsilon \circ \varsigma), \tau] = 0$ for all $\varepsilon, \varsigma, \tau \in U$. On solving, we get $[\varsigma, \tau](\varepsilon \circ \varsigma) = 0$ for all $\varepsilon, \varsigma, \tau \in U$. Again replace ε by $\varepsilon\tau$, we have $[\varsigma, \tau]\varepsilon[\tau, \varsigma] = 0$ for all $\varepsilon, \varsigma, \tau \in U$. By Lemma 1.9, we have $[\tau, \varsigma] = 0$ for all $\varsigma, \tau \in U$. Again using Lemma 1.10, we get $U \subseteq \mathcal{Z}(\mathfrak{S})$, which is a contradiction. □

Theorem 2.7. *For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a $n!$-torsion free semiprime ring and U be a nonzero ideal of $\mathfrak{S}$. Suppose that $\mathfrak{S}$ admits two nonzero symmetric n-derivations $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ and $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with $f : \mathfrak{S} \to \mathfrak{S}$ and $g : \mathfrak{S} \to \mathfrak{S}$ as traces of $\mathfrak{D}$ and $\mathcal{G}$ respectively satisfying any one of the following conditions:*

(*i*) $g(\varepsilon\varsigma) + f(\varepsilon)f(\varsigma) \pm \varepsilon\varsigma \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$,
(*ii*) $g(\varepsilon\varsigma) + f(\varepsilon)f(\varsigma) \pm \varsigma\varepsilon \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$,
(*iii*) $g([\varepsilon, \varsigma]) + [f(\varepsilon), f(\varsigma)] \pm [\varepsilon, \varsigma] \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$,
(*iv*) $g(\varepsilon \circ \varsigma) + f(\varepsilon) \circ f(\varsigma) \pm \varepsilon \circ \varsigma \in \mathcal{Z}(\mathfrak{S})$ for all $\varepsilon, \varsigma \in U$.

Then $U \subseteq \mathcal{Z}(\mathfrak{S})$.

Proof. (*i*) Given that

$$g(\varepsilon\varsigma) + f(\varepsilon)f(\varsigma) \pm \varepsilon\varsigma \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma \in U.$$

Replacing ς by $\varsigma + m\tau$ for $\tau \in U$ and $1 \leq m \leq n-1$, we arrive at

$$g(\varepsilon\varsigma) + g(\varepsilon m\tau) + \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varepsilon\varsigma, \ldots, \varepsilon\varsigma}_{(n-l)-\text{times}}, \underbrace{\varepsilon m\tau, \ldots, \varepsilon m\tau}_{l-\text{times}}) +$$
$$f(\varepsilon)\Bigg(f(\varsigma) + f(m\tau) + \sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varsigma, \ldots, \varsigma}_{(n-l)-\text{times}}, \underbrace{m\tau, \ldots, m\tau}_{l-\text{times}})\Bigg)$$
$$\pm\ \varepsilon\varsigma\ \pm\ \varepsilon m\tau \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon, \varsigma, \tau \in U.$$

Using the hypothesis, we get

$$\sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varepsilon\varsigma,\ldots,\varepsilon\varsigma}_{(n-l)-\text{times}},\underbrace{\varepsilon m\tau,\ldots,\varepsilon m\tau}_{l-\text{times}}) + f(\varepsilon)\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}},\underbrace{m\tau,\ldots,m\tau}_{l-\text{times}}) \in \mathcal{Z}(\mathfrak{S})$$

for all $\varepsilon,\varsigma,\tau \in U$. Using Lemma 1.5, we see that

$$n\mathcal{G}(\varepsilon\varsigma,\ldots,\varepsilon\varsigma,\varepsilon\tau) + nf(\varepsilon)\mathfrak{D}(\varsigma,\ldots,\varsigma,\tau) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma,\tau \in U.$$

Since $\mathfrak{S}$ is $n!$-torsion free, we get

$$\mathcal{G}(\varepsilon\varsigma,\ldots,\varepsilon\varsigma,\varepsilon\tau) + f(\varepsilon)\mathfrak{D}(\varsigma,\ldots,\varsigma,\tau) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma,\tau \in U.$$

Writing ς in place of τ, we get

$$g(\varepsilon\varsigma) + f(\varepsilon)f(\varsigma) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma \in U.$$

Using the hypothesis, we obtain that

$$\varepsilon\varsigma \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma \in U.$$

Utilizing the same reasoning following (11), we arrive at the desired conclusion.

(ii) Using identical procedures as in (i), we ascertain that U is contained within the center of $\mathfrak{S}$.

(iii) Given that

$$g([\varepsilon,\varsigma]) + [f(\varepsilon),f(\varsigma)] \pm [\varepsilon,\varsigma] \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma \in U.$$

Replacing ς by $\varsigma + m\tau$ for $\tau \in U$ and $1 \leq m \leq n-1$, we conclude that

$$\begin{aligned} g([\varepsilon,\varsigma]) + g([\varepsilon,m\tau]) + \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{[\varepsilon,\varsigma],\ldots,[\varepsilon,\varsigma]}_{(n-l)-\text{times}},\underbrace{[\varepsilon,m\tau],\ldots,[\varepsilon,m\tau]}_{l-\text{times}}) + \\ [f(\varepsilon),f(\varsigma)] + [f(\varepsilon),f(m\tau)] + [f(\varepsilon),\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}},\underbrace{m\tau,\ldots,m\tau}_{l-\text{times}})] \\ \pm [\varepsilon,\varsigma] \pm [\varepsilon,m\tau] \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma,\tau \in U. \end{aligned}$$

On using hypothesis, we get

$$\begin{aligned} \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{[\varepsilon,\varsigma],\ldots,[\varepsilon,\varsigma]}_{(n-l)-\text{times}},\underbrace{[\varepsilon,m\tau],\ldots,[\varepsilon,m\tau]}_{l-\text{times}}) + \\ \Big[f(\varepsilon),\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}},\underbrace{m\tau,\ldots,m\tau}_{l-\text{times}})\Big] \in \mathcal{Z}(\mathfrak{S}) \end{aligned}$$

for all $\varepsilon, \varsigma, \tau \in U$. Using Lemma 1.5 and the fact that $\mathfrak{S}$ is $n!-$torsion free, we have

$$\mathcal{G}([\varepsilon,\varsigma],\ldots,[\varepsilon,\varsigma],[\varepsilon,\tau]) + [f(\varepsilon),\mathfrak{D}(\varsigma,\ldots,\varsigma,\tau)] \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma,\tau \in U.$$

Writing ς in place of τ, we obtain

$$g([\varepsilon,\varsigma]) + [f(\varepsilon), f(\varsigma)] \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma \in U.$$

Using the hypothesis, we obtain that

$$[\varepsilon,\varsigma] \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma \in U.$$

By Lemma 1.10, we conclude that $U \subseteq Z(\mathfrak{S})$.

(iv) Given that

$$g(\varepsilon \circ \varsigma) + f(\varepsilon) \circ f(\varsigma) \pm \varepsilon \circ \varsigma \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma \in U.$$

Replacing ς by $\varsigma + m\tau$ for $\tau \in U$ and $1 \leq m \leq n-1$, we arrive at

$$g(\varepsilon\circ\varsigma) + g(\varepsilon\circ m\tau) + \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varepsilon\circ\varsigma,\ldots,\varepsilon\circ\varsigma}_{(n-l)-\text{times}},\underbrace{\varepsilon\circ m\tau,\ldots,\varepsilon\circ m\tau}_{l-\text{times}}) +$$
$$f(\varepsilon)\circ f(\varsigma) + f(\varepsilon)\circ f(m\tau) + f(\varepsilon)\circ \sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}},\underbrace{m\tau,\ldots,m\tau}_{l-\text{times}})$$
$$\pm\, \varepsilon\circ\varsigma \pm \varepsilon\circ m\tau \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma,\tau \in U.$$

On using hypothesis, we get

$$\sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varepsilon\circ\varsigma,\ldots,\varepsilon\circ\varsigma}_{(n-l)-\text{times}},\underbrace{\varepsilon\circ m\tau,\ldots,\varepsilon\circ m\tau}_{l-\text{times}}) +$$
$$f(\varepsilon)\circ \sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}},\underbrace{m\tau,\ldots,m\tau}_{l-\text{times}}) \in \mathcal{Z}(\mathfrak{S})$$

for all $\varepsilon, \varsigma, \tau \in U$. Using Lemma 1.5 and using the fact that $\mathfrak{S}$ is $n!-$torsion free, we get

$$\mathcal{G}(\varepsilon\circ\varsigma,\ldots,\varepsilon\circ\varsigma,\varepsilon\circ\tau) + f(\varepsilon)\circ\mathfrak{D}(\varsigma,\ldots,\varsigma,\tau) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma,\tau \in U.$$

Write ς in place of τ to get

$$g(\varepsilon\circ\varsigma) + f(\varepsilon)\circ f(\varsigma) \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma \in U.$$

Using the hypothesis, we obtain that

$$\varepsilon\circ\varsigma \in \mathcal{Z}(\mathfrak{S}) \text{ for all } \varepsilon,\varsigma \in U$$

which is the same as (12). Hence, proceeding in the same way, we get our result. The proof is complete. □

The following outcome extends Vukman's finding [22] to Lie ideals. Certainly, Vukman demonstrated that if $\mathfrak{S}$ is a prime ring with characteristic other than two and three, and there exist symmetric bi-derivations $\mathfrak{D}_1 : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ and $\mathfrak{D}_2 : \mathfrak{S} \times \mathfrak{S} \to \mathfrak{S}$ such that $f_1(a)f_2(a) = 0$ for all $a \in \mathfrak{S}$, where f_1 and f_2 are the traces of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ respectively, then either $\mathfrak{D}_1 = 0$ or $\mathfrak{D}_2 = 0$. We generalize this theorem for the traces of q-iterations of n-derivations acting on the Lie ideal of $\mathfrak{S}$.

Theorem 2.8. *Let $\mathfrak{S}$ be a $n!$-torsion free prime ring, U a noncentral square closed Lie ideal of $\mathfrak{S}$ and $q \geq 1$, be a fixed integer. Consider $\mathfrak{D}_1, \mathfrak{D}_2, \dots, \mathfrak{D}_q : \mathfrak{S}^n \to \mathfrak{S}$ to be n-derivations on $\mathfrak{S}$ such that $f_1(\varepsilon_1)f_2(\varepsilon_2)\cdots f_q(\varepsilon_q) = 0$ for all $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_q \in U$ where $f_i's$ are traces of $\mathfrak{D}_i's$ respectively, $1 \leq i \leq q$. Then one of the following holds:*

(*i*) $f_1(\varepsilon_1) = 0$ for all $\varepsilon_1 \in U$,

(*ii*) *All $\mathfrak{D}_p$ act as left n-multipliers on U for $p = 2, 3, \dots, q$.*

Proof. We will utilize the technique of induction to establish our result. When substituting $q = 1$ into our hypothesis, it becomes evident that $f_1(\varepsilon_1) = 0$ for all $\varepsilon_1 \in U$. Now, taking into account the case when $q = 2$, according to the hypothesis, we obtain

$$f_1(\varepsilon_1)f_2(\varepsilon_2) = 0 \text{ for all } \varepsilon_1, \varepsilon_2 \in U. \tag{13}$$

Replacing ε_2 by $\varepsilon_2 + m\varsigma_2$ for $\varsigma_2 \in U$ and $1 \leq m \leq n-1$, we get

$$f_1(\varepsilon_1)f_2(\varepsilon_2 + m\varsigma_2) = 0 \text{ for all } \varepsilon_1, \varepsilon_2, \varsigma_2 \in U.$$

On simplifying, we get

$$f_1(\varepsilon_1)f_2(\varepsilon_2) + f_1(\varepsilon_1)f_2(m\varsigma_2) + f_1(\varepsilon_1)\sum_{l=1}^{n-1} {}^nC_l \mathfrak{D}_2(\underbrace{\varepsilon_2, \dots, \varepsilon_2}_{(n-l)-\text{times}}, \underbrace{m\varsigma_2, \dots, m\varsigma_2}_{l-\text{times}}) = 0 \tag{14}$$

for all $\varepsilon_1, \varepsilon_2, \varsigma_2 \in U$. Compare (13) and (14) and use Lemma 1.5 to get

$$nf_1(\varepsilon_1)\mathfrak{D}_2(\varepsilon_2, \dots, \varepsilon_2, \varsigma_2) = 0 \text{ for all } \varepsilon_1, \varepsilon_2, \varsigma_2 \in U.$$

Since $\mathfrak{S}$ is $n!$-torsion free, we obtain

$$f_1(\varepsilon_1)\mathfrak{D}_2(\varepsilon_2, \dots, \varepsilon_2, \varsigma_2) = 0 \text{ for all } \varepsilon_1, \varepsilon_2, \varsigma_2 \in U. \tag{15}$$

Replacing ς_2 by $\varsigma_2\varsigma_2'$ in (15), we obtain

$$f_1(\varepsilon_1)\varsigma_2\mathfrak{D}_2(\varepsilon_2, \dots, \varepsilon_2, \varsigma_2') = 0 \text{ for all } \varepsilon_1, \varepsilon_1, \varsigma_2, \varsigma_2' \in U$$

i.e.,

$$f_1(\varepsilon_1)U\mathfrak{D}_2(\varepsilon_2, \dots, \varepsilon_2, \varsigma_2') = \{0\} \text{ for all } \varepsilon_1, \varepsilon_2, \varsigma_2' \in U.$$

Using Lemma 1.8, we have either $f_1(\varepsilon_1) = 0$ or $\mathfrak{D}_2(\varepsilon_2, \dots, \varepsilon_2, \varsigma_2') = 0$ for all $\varepsilon_2, \varsigma_2' \in U$. Consider the later case $\mathfrak{D}_2(\varepsilon_2, \dots, \varepsilon_2, \varsigma_2') = 0$ for all $\varepsilon_2, \varsigma_2' \in U$. A straightforward modification shows that $\mathfrak{D}_2(\varepsilon_2, \dots, \varepsilon_2, \varsigma_2''\varsigma_2') = \mathfrak{D}_2(\varepsilon_2, \dots, \varepsilon_2, \varsigma_2'')\varsigma_2'$

for all $\varsigma_2', \varsigma_2'' \in U$. Hence, $\mathfrak{D}_2$ acts as a left n-multiplier on U as desired.
Next assuming the statement holds true for $n = q-1$ and we now aim to establish its validity for $n = q$. Using hypothesis,

$$f_1(\varepsilon_1)f_2(\varepsilon_2)\cdots f_q(\varepsilon_q) = 0 \text{ for all } \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q \in U. \tag{16}$$

Replacing ε_q by $\varepsilon_q + m\varsigma_q$ for $\varsigma_q \in U$ and $1 \leq m \leq n-1$ in (16) and taking account of Lemma 1.5, we get

$$nf_1(\varepsilon_1)f_2(\varepsilon_2)\cdots f_{q-1}(\varepsilon_{q-1})\mathfrak{D}_q(\varepsilon_q, \ldots, \varepsilon_q, \varsigma_q) = 0$$

for all $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q, \varsigma_q \in U$. Since $\mathfrak{S}$ is $n!$-torsion free, we see that

$$f_1(\varepsilon_1)f_2(\varepsilon_2)\cdots f_{q-1}(\varepsilon_{q-1})\mathfrak{D}_q(\varepsilon_q, \ldots, \varepsilon_q, \varsigma_q) = 0. \tag{17}$$

Substituting $\varsigma_q u$ for ς_q in (17) and using (17), we arrive at

$$f_1(\varepsilon_1)f_2(\varepsilon_2)\cdots f_{q-1}(\varepsilon_{q-1})\varsigma_q\mathfrak{D}_q(\varepsilon_q, \ldots, \varepsilon_q, u) = 0$$

i.e.,

$$f_1(\varepsilon_1)f_2(\varepsilon_2)\cdots f_{q-1}(\varepsilon_{q-1})U\mathfrak{D}_q(\varepsilon_q, \ldots, \varepsilon_q, u) = \{0\}$$

for all $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q, u \in U$. Again taking account of Lemma 1.8, we have that either $f_1(\varepsilon_1)f_2(\varepsilon_2)\cdots f_{q-1}(\varepsilon_{q-1}) = 0$ or $\mathfrak{D}_q(\varepsilon_q, \ldots, \varepsilon_q, u) = 0$ for all $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_q$, $u \in U$. If $f_1(\varepsilon_1)f_2(\varepsilon_2)\cdots f_{q-1}(\varepsilon_{q-1}) = 0$, then we are done by the former case. If $\mathfrak{D}_q(\varepsilon_q, \ldots, \varepsilon_q, u) = 0$ for all $\varepsilon_q, u \in U$, then we can easily compute that $\mathfrak{D}_q(\varepsilon_q, \ldots, \varepsilon_q, \varsigma_{q-1}u) = \mathfrak{D}_q(\varepsilon_q, \ldots, \varepsilon_q, \varsigma_{q-1})u$ for all $\varepsilon_q, \varsigma_{q-1}, u \in U$. Hence $\mathfrak{D}_q$ acts as a left n-multiplier on U as desired. The theorem's proof is completed with this conclusion. □

The subsequent example illustrates that the requirement of semiprimeness for $\mathfrak{S}$ in Theorems 2.1, 2.6, and 2.7 and Lemma 2.3 - 2.5 is indispensable and cannot be overlooked.

Example 2.9. *Consider the ring* $\mathfrak{S} = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z} \right\}$. *Consider* $U = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \mid b \in \mathbb{Z} \right\}$ *be a Lie ideal of* $\mathfrak{S}$. *Denote* $A_i = \begin{bmatrix} a_i & b_i \\ 0 & 0 \end{bmatrix} \in \mathfrak{S}$, $a_i, b_i \in \mathbb{Z}$, $1 \leq i \leq n$, *and let us define* $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ *by* $\mathfrak{D}(A_1, A_2, \ldots, A_n) = \begin{bmatrix} 0 & a_1a_2\cdots a_n \\ 0 & 0 \end{bmatrix}$ *with trace* $f : \mathfrak{S} \to \mathfrak{S}$ *define by* $f\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a^n \\ 0 & 0 \end{bmatrix}$. *One can easily see that* $\mathfrak{D}$ *is a symmetric* n*-derivation such that all the conditions in Theorems 2.1, 2.6, 2.7 and Lemma 2.3 - 2.5 are satisfied. However,* U *is non central and* $\mathfrak{S}$ *does not contain any nonzero central ideal.*

3. Conclusion and Future Work

In this paper, we investigated various characteristics related to the traces of permuting n-derivations that satisfy specific functional identities within the context of Lie ideals in prime and semiprime rings. Our exploration led to the establishment of several results, particularly focusing on the traces of permuting n-derivations. We demonstrated how these traces behave under certain algebraic conditions and their implications in the structure of the rings studied. In addition to prove certain results involving the traces of permuting n-derivations, the last theorem is related to the permuting n-multipliers. Nevertheless, there are various interesting open problems related to our work. In this final section, we will propose a direction for future further research. In view of the above mentioned work, the following problems remain unanswered.

Problem 3.1. For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a (semi)-prime ring of suitable torsion restriction and P be a prime ideal of $\mathfrak{S}$. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying $g(\varepsilon)\varsigma \pm \varepsilon g(\varsigma) \in \mathscr{Z}(\mathfrak{S})$ or $g(\varepsilon)\varsigma \pm \varepsilon g(\varsigma) \in P$. Then, what can we say about the structure of $\mathfrak{S}$, g and $\mathscr{d}$?

Problem 3.2. For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a (semi)-prime ring of suitable torsion restriction and P be a prime ideal of $\mathfrak{S}$. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying $g(\varepsilon\varsigma) \pm [\varepsilon, \varsigma] \in \mathscr{Z}(\mathfrak{S})$ or $g(\varepsilon\varsigma) \pm [\varepsilon, \varsigma] \in P$. Then, what can we say about the structure of $\mathfrak{S}$, g and $\mathscr{d}$?

Problem 3.3. For a fixed integer $n \geq 2$, let $\mathfrak{S}$ be a (semi)-prime ring of suitable torsion restriction and P be a prime ideal of $\mathfrak{S}$. If $\mathfrak{S}$ admits a nonzero symmetric generalized n-derivation $\mathcal{G} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $g : \mathfrak{S} \to \mathfrak{S}$ associated with symmetric n-derivation $\mathfrak{D} : \mathfrak{S}^n \to \mathfrak{S}$ with trace $\mathscr{d} : \mathfrak{S} \to \mathfrak{S}$ satisfying $g(\varepsilon\varsigma) \pm \varepsilon \circ \varsigma \in \mathscr{Z}(\mathfrak{S})$ or $g(\varepsilon\varsigma) \pm \varepsilon \circ \varsigma \in P$. Then, what can we say about the structure of $\mathfrak{S}$, g and $\mathscr{d}$?

4. Author Contributions

All authors made equal contributions.

5. Data Availability Statement

No data were used to support the findings of this study.

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7. Conflict of interest

All authors made equal contributions.

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Vaishali Varshney
Orcid number: 0000-0002-5787-2059
Institute of Applied Sciences & Humanities,
GLA University, Mathura-281406
Mathura-281406, India
Email address: vaishalivarshney7625@gmail.com, vaishali.varshney@gla.ac.in

Shakir Ali[1,2]
Orcid number: 0000-0001-5162-7522
[1]Department of Mathematics
Aligarh Muslim University
Aligarh, India
and
[2]Institute of Mathematical Sciences
Faculty of Science, Universiti Malaya
50603, Kuala Lumpur, Malaysia
Email address: shakir.ali.mm@amu.ac.in,drshakir1971@gmail.com

Naira Noor Rafiquee
Orcid number: 0000-0003-1964-8870
Department of Mathematics
Aligarh Muslim University
Aligarh, India
Email address: rafiqinaira@gmail.com

Kok Bin Wong
Orcid number: 0000-0001-5640-4432
Institute of Mathematical Sciences
Faculty of Science, Universiti Malaya
50603, Kuala Lumpur, Malaysia
Email address: kbwong@um.edu.my