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QUASI z°-SUBMODULES OF REDUCED MULTIPLICATION MODULES

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Abstract. The purpose of this paper is to define and investigate the notion of quasi z°-submodules of modules over a commutative ring as an extension of z°-ideals of commutative rings. Also, we obtain some related results when M is a reduced multiplication R-module.

Keywords: Multiplication module, reduced module, z°-ideal, z°-submodule, quasi z°-submodule.
2020 MSC: 16N99, 13C13, 13C99.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity. An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [5].

A proper ideal I of R is called a *z-ideal* whenever any two elements of R are contained in the same set of maximal ideals and I contains one of them, then it also contains the other one [15]. For each $a \in R$, let $\mathfrak{P}_a$, be the intersection of all minimal prime ideals of R containing a. A proper ideal I of R is called a z°*-ideal* if for each $a \in I$ we have $\mathfrak{P}_a \subseteq I$ [4]. In fact, the concepts of z-ideals and z°-ideals play very important roles in the research on the rings of continuous real-valued functions (or, more generally reduced rings) and related subjects, for more information, we refer the reader to [3, 13, 17].

For a submodule N of an R-module M, let $\mathcal{M}(N)$ be the set of maximal submodules of M containing N and $Max(M)$ be the set of all maximal submodules of M. The intersection of all maximal submodules of M containing N is said to be the *Jacobson radical* of N and denote by $Rad_N(M)$ [6]. In case N is not contained in any maximal submodule, the Jacobson radical of N is defined to be M. We denote the Jacobson radical of zero submodule of M by $Rad_M(M)$. A proper submodule N of M is said to be a *z-submodule* if for every $x, y \in M$, $\mathcal{M}(x) = \mathcal{M}(y) \neq \emptyset$ and $x \in N$ imply $y \in N$ [9].

Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, either $m \in P$ or $r \in (P :_R M)$. In this case, $(P :_R M)$ is a prime ideal of R [7, 11]. An R-module M is said to be

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reduced if the intersection of all prime submodules of M is equal to zero [18]. The intersection of all prime submodules of M containing a submodule N of M is said to be the *prime radical* of N and denote by $rad_N M$. In case N is not contained in any prime submodule, the prime radical of N is defined to be M [16]. A prime submodule P of M is a *minimal prime submodule over* N if P is a minimal element of the set of all prime submodules of M that contain N. *A minimal prime submodule* of M means a minimal prime submodule over the 0 submodule of M. The set of all minimal prime submodules of M will be denoted by $Min^p(M)$. The intersection of all minimal prime submodules of M containing a submodule K of M is denote by $\mathfrak{P}_K$. In case K is not contained in any minimal prime submodule of M, $\mathfrak{P}_K$ is defined to be M. If N is a submodule of M, define $V(N) = \{P \in Min^p(M) : N \subseteq P\}$.

In [10], the notion of z°-submodules of an R-module M as an extension of z°-ideals was introduced and some of their properties when M is a reduced multiplication R-module are dealt with. A proper submodule N of an R-module M is said to be a z°*-submodule* of M if $\mathfrak{P}_x \subseteq N$ for all $x \in N$ [10]. In this paper, we introduce the notion of quasi z°-submodules of an R-module M as another generalization of z°-ideals. Also, we investigate some related results when M is a reduced multiplication R-module.

2. Reduced multiplication modules

Lemma 2.1. Let M be a reduced multiplication R-module and P be a minimal prime submodule of M. If $a \in (P :_R M)$, then

$$Ann_{R/Ann_R(M)}(a + Ann_R(M)) \not\subseteq (P :_R M)/Ann_R(M).$$

Proof. By [14, Proposition 1.5], we have $(P :_R M)$ is a minimal prime ideal of R over $Ann_R(M)$. As M is a reduced multiplication R-module, $R/Ann_R(M)$ is a reduced ring by [18]. Let $a + Ann_R(M) \in (P :_R M)/Ann_R(M)$. Since $R/Ann_R(M)$ is a reduced ring, $(a + Ann_R(M))^2 \neq 0$. Hence $a + Ann_R(M) \notin Ann_{R/Ann_R(M)}(a + Ann_R(M))$. Therefore, $Ann_{R/Ann_R(M)}(a + Ann_R(M)) \not\subseteq (P :_R M)/Ann_R(M)$. □

Remark 2.2. [10, Remark] Let M be a multiplication R-module and Ω be a subset of $Min^p(M)$. Set $\mathfrak{P}_\Omega = \cap\{P : P \in \Omega\}$. A subset Ω of $Min^p(M)$ is said to be closed if $\Omega = V(\mathfrak{P}_\Omega)$. With this notion of closed set, one can see that the space of minimal prime submodules of M becomes a topological space.

Theorem 2.3. *Let M be a faithful reduced multiplication R-module. Then for each $a \in R$, we have $V(Ann_R(a)M) = Min^p(M) \setminus V(aM)$. In particular, $V(Ann_R(a)M)$ and $V(aM)$ are disjoint open-and-closed sets of $Min^p(M)$.*

Proof. If $P \in V(aM)$, then $aM \subseteq P$, so $a \in (P :_R M)$. Now, since $Ann_R(M) = 0$, we have $Ann_R(a) \not\subseteq (P :_R M)$ by Lemma 2.1. It follows that $Ann_R(a)M \not\subseteq P$. Thus $V(Ann_R(a)M) \cap V(aM) = \emptyset$. On the other hand, if $P \in Min^p(M) \setminus V(aM)$, then for any $b \in Ann_R(a)$, we have $abM = 0 \subseteq P$. Since $a \notin$

$(P :_R M)$ and P is prime, $bM \subseteq P$. Therefore, $P \in V(Ann_R(a)M)$. Thus $V(Ann_R(a)M) = Min^p(M) \setminus V(aM)$. Both sets $V(Ann_R(a)M)$ and $V(aM)$ are closed, and since they are complementary, they are also open. □

Corollary 2.4. Let M be a faithful reduced multiplication R-module. Then $Min^p(M)$ is a Hausdorff space with a base of open-and-closed sets.

Proof. Let $P \neq Q \in Min^p(M)$. If $(P :_R M) \subseteq (Q :_R M)$, then $P = (P :_R M)M \subseteq (Q :_R M)M = Q$ because M is a multiplication R-module. This is a contradiction because $P, Q \in Min^p(M)$ and $P \neq Q$. So, we can assume that $a \in (P :_R M) \setminus (Q :_R M)$. Then $V(aM)$ and $V(Ann_R(a)M)$ are disjoint open sets containing P and $\acute{P}$, respectively. Hence $Min^p(M)$ is a Hausdorff space. In fact, the family $\{V(aM)\}$ is a base for the closed sets. Thus $V(Ann_R(a)M)$ is a base for the open sets. □

Theorem 2.5. *Let M be a reduced multiplication R-module. Then we have the following.*

(a) *If M is a faithful R-module, then $V(aM) = V((0 :_M Ann_R(aM)))$ for each $a \in R$.*
(b) *$(0 :_M Ann_R(IJM)) = (0 :_M Ann_R(IM)) \cap (0 :_M Ann_R(JM))$ for each ideals I, J of R.*

Proof. (a) Let M be a faithful R-module and $a \in R$. Then $Ann_R(aM) = Ann_R(a)$. As $aM \subseteq (0 :_M Ann_R(aM))$, we have $V((0 :_M Ann_R(aM))) \subseteq V(aM)$. Now, let P be a minimal prime submodule of M containing aM. Then, there exists $b \in Ann_R(aM) \setminus (P :_R M)$ by Lemma 2.1. Then, for any $y \in (0 :_M Ann_R(aM))$, we have $by = 0 \in P$. Hence $y \in P$. So, $(0 :_M Ann_R(aM)) \subseteq P$, as needed.

(b) Let I, J be ideals of R. As $Ann_R(IM) \subseteq Ann_R(IJM)$ and $Ann_R(JM) \subseteq Ann_R(IJM)$, we have

$$(0 :_M Ann_R(IJM)) \subseteq (0 :_M Ann_R(JM)) \cap (0 :_M Ann_R(IM)).$$

Now, suppose that $z \in (0 :_M Ann_R(JM)) \cap (0 :_M Ann_R(IM))$. Let $t \in Ann_R(IJM)$. Then $Jt \subseteq Ann_R(IM)$, so $tJz = 0$. Since M is a multiplication module, $tzR = AM$ for some ideal A of R. Hence, $A \subseteq Ann_R(JM)$, so $Az = 0$. Thus $t(Rz)^2 = 0$. It follows that $(Rtz)^2 = 0$. This means that $(Rtz :_R M)^2 M = 0$, so $(Rtz :_R M)^2 \subseteq Ann_R(M)$. It follows that $((Rtz :_R M) + Ann_R(M))^2 = 0 \in R/Ann_R(M)$. As M is a reduced multiplication R-module, $R/Ann_R(M)$ is a reduced ring by [18]. Therefore, $(Rtz :_R M) + Ann_R(M) = 0_{R/Ann_R(M)}$. This implies that $tz = 0$. Hence, $zAnn_R(IJM) = 0$. It follows that $z \in (0 :_M Ann_R(IJM))$. □

Corollary 2.6. Let M be a faithful reduced multiplication R-module. Then for each $a \in R$, $(0 :_M Ann_R(aM)) = \mathfrak{P}_{aM}$.

Proof. Since, by [10, Theorem 2.11], $(0 :_M I) = \mathfrak{P}_{(0:_M I)}$ for each ideal I of R. The result follows from Theorem 2.5 (a). □

Theorem 2.7. *Let M be a reduced multiplication R-module. Then the following are equivalent:*

(a) *For $a, b \in R$, $Ann_R(aM) = Ann_R(bM)$ and $aM \subseteq N$ imply that $bM \subseteq N$;*

(b) *For $a, b \in R$, $Ann_R(aM) \subseteq Ann_R(bM)$ and $aM \subseteq N$ imply that $bM \subseteq N$.*

Proof. $(a) \Rightarrow (b)$ Let for $a, b \in R$, $Ann_R(aM) \subseteq Ann_R(bM)$ and $aM \subseteq N$. Then $(0 :_M Ann_R(bM)) \subseteq (0 :_M Ann_R(aM))$. Hence $(0 :_M Ann_R(abM)) = (0 :_M Ann_R(bM))$ by Theorem 2.5 (b). It follows that $Ann_R(abM) = Ann_R(bM)$. Now, as $abM \subseteq N$ we have $bM \subseteq N$ by part (a).

$(b) \Rightarrow (a)$ This is clear. □

Theorem 2.8. *Let M be a faithful multiplication R-module. Then $\mathfrak{P}_I M \subseteq \mathfrak{P}_{IM}$ for each ideal I of R. The reverse inclusion holds when M is a finitely generated R-module.*

Proof. Let I be an ideal of R and X be a minimal prime submodule of M such that $IM \subseteq X$. Then $I \subseteq (X :_R M)$. Since, by [14, Proposition 1.5], $(X :_R M)$ is a minimal prime ideal of R, $\mathfrak{P}_I \subseteq (X :_R M)$. Thus $\mathfrak{P}_I M \subseteq (X :_R M)M \subseteq X$. Hence, $\mathfrak{P}_I M \subseteq \mathfrak{P}_{IM}$ for each ideal I of R. For the converse, let $\mathfrak{P}_I = \cap \mathfrak{p}_i$, where $\mathfrak{p}_i \in Min^p(R), I \subseteq \mathfrak{p}_i$, where $Min^p(R)$ is the set of all minimal prime ideals of R. As M is a faithful multiplication R-module, by using [8, Theorem 1.6],

$$IM \subseteq \mathfrak{P}_I M = (\bigcap \mathfrak{p}_i)M = \bigcap \mathfrak{p}_i M = \bigcap_{\mathfrak{p}_i M \neq M} \mathfrak{p}_i M.$$

This implies that $\mathfrak{P}_{IM} \subseteq \mathfrak{P}_I M$ since M is finitely generated, so by using [8, Page 762], $\mathfrak{p}_i M \neq M$ is a minimal prime submodule of M. □

3. *Quasi z°-submodules*

Definition 3.1. We say that a proper submodule N of an R-module M is a *quasi z°-submodule* of M if $\mathfrak{P}_{aM} \subseteq N$ for all $a \in (N :_R M)$.

Remark 3.2. Let M be an R-module. If N is a quasi z°-submodule of M, then for each $a \in (N :_R M)$ we have $\mathfrak{P}_{aM} \neq M$, i.e., aM contained in at least a minimal prime submodule of M. Clearly, every minimal prime submodule of M is a quasi z°-submodule of M. Also, the family of quasi z°-submodules of M is closed under intersection. Therefore, if $\mathfrak{P}_0 \neq M$, then $\mathfrak{P}_0$ is a quasi z°-submodule of M and it is contained in every quasi z°-submodule of M.

Example 3.3. Let K be a field and let $R = K[[x, y]]$, where x, y are indeterminates. Put $P = (x, y)$. Then $(Py :_R P) = (y)$. Put $P_1 = P/(y)$, $R_1 = R/(y)$, and $M = P/Py$. Then by [12, Example 2.7], $P_1 M$ is a minimal prime submodule of the R_1-module M and so, $P_1 M$ is a quasi z°-submodule of M as an R_1-module.

We give the following easy results without proofs.

Proposition 3.4. Let N be a submodule of a cyclic R-module M. Then N is a z°-submodule of M if and only if N is a quasi z°-submodule of M. In particular, an ideal I of the ring R is a z°-ideal if and only if I is a quasi z°-ideal of R.

Lemma 3.5. Let M be an R-module. A submodule N of M is a quasi z°-submodule if and only if $N = \sum_{a \in (N:_R M)} \mathfrak{P}_{aM}$.

Proposition 3.6. Let N be a proper submodule of an R-module M. Then N as an R-submodule is a quasi z°-submodule if and only if as an $R/Ann_R(M)$-submodule is a quasi z°-submodule.

Lemma 3.7. Let M be a faithful multiplication R-module. If N is a quasi z°-submodule M, then $(N :_R M)$ is a z°-ideal of R. The converse holds when M is a finitely generated R-module.

Proof. This follows from Theorem 2.8. □

Theorem 3.8. *Let M be a faithful multiplication R-module. Then we have the following.*

(a) *Let M be a finitely generated R-module and I be a z°-ideal of R. Then IM is a quasi z°-submodule of M.*

(b) *Let R be a reduced ring and N be a quasi z°-submodule of an R-module M. Then $(N :_R (K :_R M)M)$ is a z°-ideal of R for each submodule K of M. In particular, if $\mathfrak{P}_0 = 0$, then $Ann_R((K :_R M)M)$ is a z°-ideal of R for each submodule K of M.*

(c) *Let R be a reduced ring and N be a quasi z°-submodule of M. Then $(N :_R K)$ is a z°-ideal of R for each submodule K of M. In particular, if $\mathfrak{P}_0 = 0$, then $Ann_R(K)$ is a z°-ideal of R for each submodule K of M.*

Proof. (a) By [19, Theorem 10], $I = (IM :_R M)$. Now, the result follows from Lemma 3.7.

(b) As N is a quasi z°-submodule, $(N :_R M)$ is a z°-ideal of R by Lemma 3.7. Let K be a submodule of M. Then by [4, Examples of z°-ideals], $((N :_R M) :_R (K :_R M))$ is a z°-ideal of R. Now, $(N :_R (K :_R M)M) = ((N :_R M) :_R (K :_R M))$ implies that $(N :_R (K :_R M)M)$ is a z°-ideal of R. Now, the last assertion follows from the fact that $\mathfrak{P}_0$ is a quasi z°-submodule of M by Remark 3.2.

(c) As M is a multiplication R-module, $K = (K :_R M)M$. Now, the result follows from part (b) □

Let M be an R-module. The set of torsion elements of M with respect to R is the set $T_0(M) = \{m \in M : rm = 0 \text{ for some } 0 \neq r \in R\}$.

Theorem 3.9. *Let M be a faithful reduced multiplication R-module, $a, b \in M$, and $b \in Ann_R(aM)$. If $Ann(aM)M = \mathfrak{P}_{bM}$, then $(a + b)M \not\subseteq T_0(M)$. The* *converse holds when* $Ann_R(a)M = Ann_R(aM)M$ *is a quasi* z°*-submodule of* M*.*

Proof. First note that $T_0(M) = \cup_{P\in Min^p(M)} P$ by [10, Corollary 2.9]. Now, let $Ann(aM)M = \mathfrak{P}_{bM}$ and $(a+b)M$ belong to a minimal prime submodule P of M and seek a contradiction. If $a \in (P :_R M)$, then $b \in (P :_R M)$ implies that $\mathfrak{P}_{bM} \subseteq P$, i.e., $\mathfrak{P}_{bM} = Ann_R(aM)M \subseteq P$, which is a contradiction by Theorem 2.3. Now, if $a \notin (P :_R M)$, then we must have $b \notin (P :_R M)$, i.e., $Ann(aM)M = \mathfrak{P}_{bM} \not\subseteq P$, which is impossible by Theorem 2.3. Conversely, if $(a+b)M \not\subseteq T_0(M)$, we have to show that $Ann(aM)M \subseteq \mathfrak{P}_{bM}$. Let P be a minima1 prime submodule with $b \in (P :_R M)$. Then $a+b \notin (P :_R M)$ implies that $a \notin (P :_R M)$, i.e., $Ann(aM)M \subseteq P$. Hence $Ann(aM)M \subseteq \mathfrak{P}_{bM}$. The reverse inclusion follows from the fact that $Ann_R(aM)M$ is a quasi z°-submodule of M. □

Let M *be an* R*-module. The set of zero divisors of* R *on* M *is the set* $Zd_R(M) = \{r \in R : rm = 0 \ for\ some\ nonzero\ m \in M\}$*.*

Proposition 3.10. Let M be a faithful multiplication R-module. If N is a quasi z°-submodule of M, then $(N :_R M) \subseteq Zd_R(M)$.

Proof. By [1, Lemma 2.1], $Zd_R(R) = Zd_R(M)$. Now, the result follows from the fact that $(N :_R M)$ is a z°-ideal of R by Lemma 3.7. □

Theorem 3.11. *Let* M *be a faithful reduced multiplication* R*-module. Then the following are equivalent:*

(a) *N is a quasi z°-submodule of M;*
(b) *For each $a \in (N :_R M)$ and submodule K of M, $\mathfrak{P}_{aM} = \mathfrak{P}_K$ implies that $K \subseteq N$;*
(c) *For each $a \in (N :_R M)$ and submodule K of M, $V(aM) = V(K)$ implies that $K \subseteq N$;*
(d) *For each $a \in R$, we have $aM \subseteq N$ implies that $(0 :_M Ann_R(aM)) \subseteq N$;*
(e) *For $a, b \in R$, $Ann_R(aM) = Ann_R(bM)$ and $aM \subseteq N$ imply that $bM \subseteq N$;*
(f) *For $a, b \in R$, $Ann_R(aM) \subseteq Ann_R(bM)$ and $aM \subseteq N$ imply that $bM \subseteq N$.*

Proof. $(a) \Rightarrow (b)$ Let $a \in (N :_R M)$ and K be a submodule of M such that $\mathfrak{P}_{aM} = \mathfrak{P}_K$. By assumption, $\mathfrak{P}_{aM} \subseteq N$. Thus $K \subseteq \mathfrak{P}_K \subseteq N$.

$(b) \Rightarrow (c)$ Let $a \in (N :_R M)$ and K be a submodule of M such that $V(aM) = V(K)$. Then $\mathfrak{P}_{aM} = \mathfrak{P}_K$. Thus, by part (b), $K \subseteq N$.

$(c) \Rightarrow (d)$ Let $aM \subseteq N$. Then $V(aM) = V((0 :_M Ann_R(aM))$ by Theorem 2.5. Thus, by part (c), $(0 :_M Ann_R(aM)) \subseteq N$.

$(d) \Rightarrow (e)$ Let $a, b \in R$, $Ann_R(aM) = Ann_R(bM)$ and $aM \subseteq N$. Then $(0 :_M Ann_R(aM)) = (0 :_M Ann_R(bM))$. By part (d), $(0 :_M Ann_R(aM)) \subseteq N$. Thus $bM \subseteq (0 :_M Ann_R(bM)) \subseteq N$.

$(e) \Rightarrow (f)$ By Theorem 2.7.

$(f) \Rightarrow (a)$ Let $aM \subseteq N$. By Corollary 2.6, $(0 :_M Ann_R(aM)) = \mathfrak{P}_{aM}$. Let $x \in (0 :_M Ann_R(aM))$. Then $Ann_R(aM) \subseteq Ann_R(x)$. As M is a multiplication R-module, $Rx = JM$ for some ideal J or R. Let $b \in J$. Then $Ann_R(JM) \subseteq Ann_R(bM)$. Therefore, $Ann_R(aM) \subseteq Ann_R(bM)$. Thus by part (f), $bM \subseteq N$, so $Rx = JM \subseteq N$. This implies that $\mathfrak{P}_{aM} = (0 :_M Ann_R(aM)) \subseteq N$. □

An R-module M is said to be a co-multiplication module *if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [2].*

Corollary 3.12. Every proper submodule of a faithful reduced multiplication and co-multiplication R-module is a quasi z°-submodule.

Proof. As M is a co-multiplication R-module, for each $a \in M$ we have $aM = (0 :_M Ann_R(aM))$. Now, the result follows from Theorem 2.5 ($(d) \Rightarrow (a)$). □

Theorem 3.13. *Let M be a faithful reduced multiplication R-module and N be a quasi z°-submodule of M. Then every minimal prime submodule over N is a prime quasi z°-submodule of M.*

Proof. Let P be a minimal prime submodule over N. Assume that $Ann(aM) \subseteq Ann(bM)$, where $a \in (P :_R M)$ and $b \in R$. Since P/N is a minimal prime submodule of M/N, by Lemma 2.1 (b), there exists $c \in Ann_R(a(M/N)) \setminus (P/N :_R M/N)$. Thus $ca \in (N :_R M)$ and $c \notin (P :_R M)$. Now, we have $Ann(caM) \subseteq Ann(cbM)$. As N is a quasi z°-submodule of M, we get that $cb \in (N :_R M) \subseteq (P :_R M)$. Since $c \notin (P :_R M)$ and P is a prime submodule, $b \in (P :_R M)$, as needed. □

Corollary 3.14. If $f : M \to M/N$ is the natural epimorphism, where M is a faithful reduced multiplication R-module and N is a quasi z°-submodule of M, then every quasi z°-submodule of M/N contracts to a quasi z°-submodule of M.

Corollary 3.15. Let M be a faithful reduced multiplication R-module. Then we have the following.

(a) Every maximal quasi z°-submodule is a prime quasi z°-submodule.
(b) If P is a prime submodule of M, then either P is a quasi z°-submodule or contains a maximal quasi z°-submodule which is a prime quasi z°-submodule.

Theorem 3.16. *Let M be a faithful multiplication R-module. Then the following are equivalent:*

(a) *M is a reduced module, i.e., R is a reduced ring;*
(b) *The submodule 0 is a quasi z°-submodule of M.*

Proof. $(a) \Rightarrow (b)$ Let $a \in Ann_R(M)$. Then $(0 :_M Ann_R(aM)) = 0$. Thus the result follows from Theorem 3.11 $(d) \Rightarrow (a)$.

$(b) \Rightarrow (a)$ Let $a \in R$ such that $(Ra)^2 = 0$. It is clear that $\mathfrak{P}_{aM} = \mathfrak{P}_{(a^2M)}$. Thus $\mathfrak{P}_{aM} = \mathfrak{P}_{a^2M} = \mathfrak{P}_0$. Since the submodule 0 is a quasi z°-submodule, $aM = 0$. Now, as M is faithful, $a = 0$. □

4. ***Data Availability Statement***

Not applicable.

5. ***Aknowledgement***

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