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THE BACH-FLAT AND CONFORMALLY EINSTEIN EQUATIONS FOR SIKLOS SPACETIMES

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Abstract. Within the large class of Siklos spacetimes, we completely classify Bach-flat metrics, which turn out to be related to a bi-harmonicity property of the defining function. Using this classification, we tackle the conformally Einstein property and several classes of conformally Einstein Siklos metrics are then determined, including all the homogeneous examples.

Keywords: Siklos spacetimes, Bach-flat metrics, conformally Einstein metrics.
2020 MSC: 53C20, 53C50, 53C44.

1. Introduction

A pseudo-Riemannian manifold (M, g) is said to be *(locally) conformally Einstein* if every point $p \in M$ admits an open neighborhood U and a positive smooth function φ defined on U, such that $(U, \tilde{g} = \varphi^{-2} g)$ is Einstein. As proved in [3], this function φ satisfies the partial differential equation

$$(n-2)\text{Hes}_\varphi + \varphi\varrho = \frac{1}{n}\{(n-2)\Delta\varphi + \varphi\tau\}\, g, \tag{1}$$

where $n = \dim M$, $\text{Hes}_\varphi = \nabla d\varphi$ is the Hessian of φ and ϱ and τ denote the Ricci tensor and the scalar curvature of g, respectively. To determine whether a given pseudo-Riemannian metric is conformally Einstein, is a fundamental problem in conformal geometry.

Integrating the conformally Einstein equation (1) is generally complicated. As the conformally Einstein condition is trivial in dimension two and equivalent to conformal flatness in dimension three, proper solutions may appear starting from dimension four, in particular for spacetimes.

Let W denote the Weyl tensor of (M^n, g). The *Bach tensor* $\mathcal{B}$ of (M^n, g) is defined as

$$\mathcal{B} = \text{div}_1\text{div}_4 W + \frac{n-3}{n-2} W[\varrho], \tag{2}$$

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where, with respect to a pseudo-orthonormal basis $\{e_i\}$, with $\varepsilon_i = g(e_i, e_i) = \pm 1$, tensor $W[\varrho]$ is the *Ricci contraction* of W:

$$W[\varrho](X,Y) = \sum_{i,j} \varepsilon_i \varepsilon_j W(e_i, X, Y, e_j)\varrho(e_i, e_j).$$

As such, $\mathcal{B}$ is completely determined by local components

$$\mathcal{B}_{ij} = \nabla^k \nabla^l W_{kijl} + \frac{1}{2}\varrho^{kl} W_{kijl}. \tag{3}$$

Conformally Einstein manifolds are necessarily Bach-flat. Determining the Bach-flat examples within a given class of metrics, is both a needed step toward the classification of conformally Einstein metrics, and an interesting problem in itself, which has been recently investigated by several authors, with particular attention toward homogeneous examples. In dimension four, Bach-flat metrics are precisely the critical points of the conformally invariant functional $\mathcal{W} = \int_M \|W\|^2 dV_g$. Moreover, Bach-flatness is a necessary condition for being half conformally flat (i.e., self-dual of anti-self-dual). Some recent results concerning Bach-flat and conformally Einstein manifolds may be found in [1], [7], [8] and references therein.

In this paper we completely classify Bach-flat examples and determine several classes of conformally Einstein metrics, within the large class of Siklos spacetimes [14]. Siklos spacetimes are solutions of Einstein's field equations with an Einstein-Maxwell source. They are of Petrov type N with cosmological constant $\Lambda < 0$ and always admit a null non-twisting Killing field. Following [14], [13], in global coordinates $(x_1, x_2, x_3, x_4) = (v, u, x, y)$ Siklos metrics have the general form

$$g = -\frac{3}{\Lambda x_3^2}\left(2dx_1 dx_2 + H dx_2^2 + dx_3^2 + dx_4^2\right), \tag{4}$$

where the defining smooth function $H = H(x_2, x_3, x_4)$ is arbitrary. These metrics have been intensively studied. They have been interpreted as exact gravitational waves propagating in the anti-de Sitter universe [13], and coincide with the subclass $(IV)_0$ of Kundt spacetimes [12]. Vacuum polarization in Siklos spacetimes was investigated in [11]. Conformally flat [5] and some well-known homogeneous [4] Siklos metrics have been proved to satisfy the Ricci soliton equation.

The paper is organized in the following way. In Section 2 we shall give some basic information on the curvature of Siklos metrics. In Section 3 we shall determine all Bach-flat Siklos metrics, which turn out to be completely characterized by a biharmonicity property of the defining function H. In fact, setting $\Delta_{34} = \frac{\partial^2}{\partial x_3 \partial x_3} + \frac{\partial^2}{\partial x_4 \partial x_4}$, we prove that a Siklos metrics, as described in (4), is Bach-flat if and only if $\Delta_{34}^2 H = 0$. In Section 4 we turn our attention to conformally Einstein examples, which are completely classified and described for several large subclasses, including all classes of homogeneous Siklos metrics. Calculations have been checked using the software *Maple 16*©.

2. On the curvature of Siklos metrics

The Levi-Civita connection and curvature of an arbitrary Siklos metric g have been described in [13] and [4]. Here we shall report below some information which we will use to determine Bach-flat and conformally Einstein examples.

Consider an arbitrary Siklos metric g, as described in (4) with respect to a system of global coordinates (x_1, x_2, x_3, x_4). Throughout the paper, we shall use the notation $\partial_i = \frac{\partial}{\partial x_i}$ for all indices i. The Levi-Civita connection ∇ of g is completely determined by the following possibly non-vanishing components:

$$(1)\quad \begin{array}{ll} \nabla_1 2 = \frac{1}{x_3} 3, & \nabla_1 3 = -\frac{1}{x_3} 1, \\ \nabla_2 2 = \frac{1}{2}(2H)1 + \frac{1}{2x_3}(2H - x_3 3H)3 - \frac{1}{2}(4H)4, & \\ \nabla_2 3 = \frac{1}{2}(3H)1 - \frac{1}{x_3} 2, & \nabla_2 4 = \frac{1}{2}(4H)1, \\ \nabla_3 3 = -\frac{1}{x_3} 3, & \nabla_3 4 = -\frac{1}{x_3} 4, \\ \nabla_4 4 = \frac{1}{x_3} 3. & \end{array}$$

A straightforward calculation yields that the possibly non-vanishing components of the Riemann-Christoffel curvature tensor R of g are then given by

$$(2)\quad \begin{array}{lll} R_{1212} = -\frac{3}{\Lambda x_3^4}, & R_{1323} = \frac{3}{\Lambda x_3^4}, & R_{1424} = \frac{3}{\Lambda x_3^4}, \\ R_{2323} = \frac{3\left(2H - x_3(3H) + x_3^2(\partial^2_{33}H)\right)}{2\Lambda x_3^4}, & R_{2324} = \frac{3\left(\partial^2_{34}H\right)}{2\Lambda x_3^2}, & \\ R_{2424} = \frac{3\left(2H - x_3(3H) + x_3^2(\partial^2_{44}H)\right)}{2\Lambda x_3^4}, & R_{3434} = -\frac{3}{\Lambda x_3^4} & \end{array}$$

and the *Ricci tensor* of g, defined by $\varrho(X, Y) = \mathrm{tr}(Z \mapsto R(Z, X)Y)$, is completely described by the matrix

$$(3)\quad \varrho = (\varrho_{ij}) = \begin{pmatrix} 0 & -3x_3^{-2} & 0 & 0 \\ -3x_3^{-2} & -\frac{6H - 2x_3(3H) + x_3^2\left(\partial^2_{33}H + \partial^2_{44}H\right)}{2x_3^2} & 0 & 0 \\ 0 & 0 & -3x_3^{-2} & 0 \\ 0 & 0 & 0 & -3x_3^{-2} \end{pmatrix},$$

where $\varrho_{ij} = \varrho(_i, _j)$. The *scalar curvature* of a Siklos metric is given by $\tau = \mathrm{tr}\varrho = 4\Lambda$. We report in the following proposition the classification of Einstein Siklos metrics.

Proposition 2.1 ([14], [13], [4]). *For an arbitrary Siklos metric g, as described in* (4)*, the following conditions are equivalent:*

(i) *g is Einstein. More precisely, $\varrho = \Lambda g$;*
(ii) *g is Ricci-parallel (that is, $\nabla\varrho = 0$);*

(iii) *the defining function $H = H(x_2, x_3, x_4)$ satisfies the PDE*

$$\frac{2}{x_3}(3H) - \partial^2_{33}H - \partial^2_{44}H = 0. \tag{4}$$

Whenever H does not satisfy (4)*, a Siklos spacetime (being not Ricci-parallel) is not locally symmetric, and its Ricci operator is of Segre type* $[(11, 2)]$*, having an eigenvalue of multiplicity four associated with a three-dimensional eigenspace.*

It is worth noting that if the Siklos metric adheres to the vacuum Einstein field equations with a cosmological constant, the equation outlined in part (iii) of the aforementioned proposition must be duly established.

Next, the *Weyl conformal curvature tensor field* W of a pseudo-Riemannian manifold (M^n, g) is defined by

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}(QX \wedge Y + X \wedge QY)Z + \frac{\tau}{(n-1)(n-2)}(X \wedge Y)Z, \tag{5}$$

where $(X \wedge Y)(Z) = g(Y,Z)X - g(X,Z)Y$, R is the $(1,3)-$curvature tensor field and Q is the Ricci operator. Applying (4), (2) and (3) we determine the possibly non-vanishing components W_{ijkh} of the Weyl tensor of type $(0,4)$ with respect to $\{\partial_i\}$, which we shall use in the next sections:

$$W_{2323} = -W_{2424} = \tfrac{3}{4\Lambda x_3^2}\left(\partial^2_{33}H - \partial^2_{44}H\right), \qquad W_{2324} = \tfrac{3}{2\Lambda x_3^2}\partial^2_{34}H. \tag{6}$$

As it is well-known, in dimension greater than three, a pseudo-Riemannian manifold (M, g) is conformally flat if and only if $W = 0$. Hence, for Siklos metrics by (6) one derives the following result.

Proposition 2.2 ([4], [5])**.** *A Siklos metric g, as described in* (4)*, is locally conformally flat if and only if the defining function $H = H(x_2, x_3, x_4)$ satisfies the system of PDEs*

$$\partial^2_{33}H - \partial^2_{44}H = \partial^2_{34}H = 0, \tag{7}$$

that is, when H is explicitly given by

$$H(x_2, x_3, x_4) = \frac{1}{2}T(x_2)\left(x_3^2 + x_4^2\right) + L(x_2)\,x_3 + M(x_2)\,x_4 + N(x_2), \tag{8}$$

where T, L, M, N are arbitrary smooth functions.

3. Bach-flat Siklos spacetimes

We already recalled in the Introduction that conformally Einstein manifolds are necessarily Bach-flat. For this reason, we shall now determine the Bach-flat examples within the class of Siklos spacetimes. Setting $\Delta_{34} = \partial^2_{33} + \partial^2_{44}$, we prove the following.

Theorem 3.1. *An arbitrary Siklos metric g, as described in* (4)*, is Bach-flat if and only if its defining function H is biharmonic with respect to* Δ_{34}*, that is, H satisfies the PDE*

$$\Delta^2_{34} H = \partial^4_{3333} H + \partial^4_{4444} H + 2\partial^4_{3344} H = 0. \tag{1}$$

Proof. We first use (3) and (6) in order to compute the components of the Ricci contraction of W. A straightforward calculation yields that all components of $W[\varrho]$ with respect to $\{\partial_i\}$ vanish identically. We then proceed applying the definition (2) of the Bach tensor $\mathcal{B}$ and we obtain its components with respect to the basis $\{\partial_i\}$. We find that up to symmetries, the only possibly non-vanishing component of the Bach tensor is given by

$$\mathcal{B}_{22} = -\tfrac{1}{12}\Lambda x_3^2 \left(\partial^4_{3333} H + \partial^4_{4444} H + 2\partial^4_{3344} H\right),$$

that is, by the above definition of Δ_{34},

$$\mathcal{B}_{22} = -\tfrac{1}{12}\Lambda x_3^2 \Delta^2_{34} H.$$

Since $\Lambda < 0$, from the above expression, we conclude that a Siklos metric is Bach-flat if and only if the PDE (1) holds. □

Corollary 3.2. *In the special case where a Siklos metric g is of the form* (4) *for a defining function* $H = H(x_3, x_4)$*, g is Bach-flat if and only if H is biharmonic (in the classical sense).*

Remark 3.3. *Observe that according to the fact that a conformally flat metric is necessarily Bach-flat, any (smooth) solution of* (7) *also satisfies* (1)*. In fact, substituting from* (7)*, one gets*

$$\begin{aligned}\Delta^2_{34} H &= \partial^4_{3333} H + \partial^4_{4444} H + 2\partial^4_{3344} H \\ &= \partial^2_{33}(\partial^2_{44} H) + \partial^2_{44}(\partial^2_{33} H) + 2\partial^2_{33}\partial^2_{44} H \\ &= 4\partial^2_{33}\partial^2_{44} H \\ &= 4\partial^2_{34}\left(\partial^2_{34} H\right) = 0.\end{aligned}$$

The theory of biharmonic and pluriharmonic functions is well-established and has a long tradition. As proved in the classical reference [2], the general solution of the pluriharmonic equation

$$\Delta^n u = 0,$$

where $u = u(x_1, ..., x_n)$ and $\Delta = \partial^2_{11} + ... + \partial^2_{nn}$ denotes the Laplace operator, is given by

$$u_n = p v_{n-1} + w_{n-1},$$

where $\Delta v_{n-1} = \Delta w_{n-1} = 0$ and p is a degree 1 polynomial in the variables $x_1, ..., x_n$ (observe that the Laplace operator is denoted by Δ^2 in [2], while here we denote it by the more usual notation Δ). Adapting this general result to our context, where we have a smooth function $H = H(x_2, x_3, x_4)$ which satisfies $\Delta^2_{34} H = 0$, by Theorem 3.1 we get the following.

Corollary 3.4. *An arbitrary Siklos metric g, as described in* (4)*, is Bach-flat if and only if its defining function H is of the form*

$$H(x_2, x_3, x_4) = A(x_2)U(x_3, x_4) + B(x_2)V(x_3, x_4) + W(x_3, x_4), \tag{2}$$

where A, B, C are arbitrary smooth functions and U, V, W are harmonic functions in the variables x_3, x_4, that is, they are solutions of $\Delta_{34}u = 0$.

4. Conformally Einstein Siklos spacetimes

We recall that the *Cotton tensor* $\mathcal{C}$ of an n-dimensional pseudo-Riemannian manifold (M, g) is completely determined in local coordinates by

$$\mathcal{C}_{ijk} = (\nabla_i Sc)_{jk} - (\nabla_j Sc)_{ik},$$

where $Sc = \varrho - \frac{\tau}{2(n-1)} g$ is the *Schouten tensor*. The following key result holds.

Proposition 4.1. *[7,9,10] Let (M, g) be a four-dimensional pseudo-Riemannian manifold such that $\tilde{g} = e^{\sigma} g$ is Einstein. Then*

(1) $\mathcal{C} - W(\cdot, \cdot, \cdot, \nabla\sigma) = 0$*, and*

(2) $\mathcal{B} = 0$.

Remark 4.2. *As reported in [7], conditions (1) and (2) listed in the above Proposition 4.1 are also* sufficient *for a four-dimensional pseudo-Riemannian manifold (M, g) to be conformally Einstein when the manifold is* weakly generic*, that is, its Weyl tensor W defines an injective map from TM to $\otimes^3 TM$. However, it is easily seen that excluding the trivial conformally flat case, this does not occur for general Siklos metrics.*

To determine the kernel of $W : TM \to \otimes^3 TM$, we express in coordinates the condition $W(X, \cdot, \cdot, \cdot) = 0$, where $X = X_i \partial_i$ is an arbitrary vector field on (M, g), and we get

$$\begin{cases} W(X, \cdot, \cdot, \cdot)_{223} = -\frac{3}{4\lambda x_3^2}(X_3(33^2H - 44^2H) + 2X_4 34^2 H) = 0, \\ W(X, \cdot, \cdot, \cdot)_{224} = -\frac{3}{4\lambda x_3^2}(X_4(44^2H - 33^2H) + 2X_3 34^2 H) = 0, \\ W(X, \cdot, \cdot, \cdot)_{332} = -W(X, \cdot, \cdot, \cdot)_{442} = \frac{3}{4\lambda x_3^2} X_2(44^2H - 33^2H) = 0, \\ W(X, \cdot, \cdot, \cdot)_{324} = \frac{3}{2\lambda x_3^2} X_2 34^2 H = 0. \end{cases}$$

Excluding the conformally flat solutions, equations (7) *do not hold and so, the above system necessarily yields $X_2 = X_3 = X_4 = 0$, but X_1 is arbitrary. Therefore, the metric is not weakly generic.*

For an arbitrary Siklos metric, as described in (4), equation $\mathcal{C}-W(\cdot,\cdot,\cdot,\nabla\sigma)=0$ in coordinates (x_1,x_2,x_3,x_4) yields the following system of independent PDE:

(1)
$$\begin{cases} (44^2H-33^2H)1\sigma=0,\\ (34^2H)1\sigma=0,\\ 2(34^2H)3\sigma-\big((33^2H)-(44^2H)\big)\,4\sigma=\frac{4}{x_3}34^2H-4444^3H-4433^3H,\\ \big((33^2H)-(44^2H)\big)\,3\sigma+2(34^2H)4\sigma=\frac{2}{x_3}(33^2H-44^2H)-2344^3H-2333^3H. \end{cases}$$

Again by the assumption that g is not conformally flat, the first two equations in (1) imply that $1\sigma=0$, so that σ only depends on (x_2,x_3,x_4), and
$$4(34^2H)^2+\big(33^2H-44^2H\big)^2\neq 0,$$
whence the last two equations of (1) determine 3σ, 4σ as

$$(2)\qquad \begin{cases} 3\sigma=\dfrac{\det\begin{vmatrix}\frac{4}{x_3}34^2H-44(\Delta_{34}H) & -(33^2H-44^2H)\\ \frac{2}{x_3}(33^2H-44^2H)-23(\Delta_{34}H) & 2(34^2H)\end{vmatrix}}{4(34^2H)^2+(33^2H-44^2H)^2},\\[2ex] 4\sigma=\dfrac{\det\begin{vmatrix}234^2H & \frac{4}{x_3}34^2H-4(\Delta_{34}H)\\ (33^2H)-(44^2H) & \frac{2}{x_3}(33^2H-44^2H)-23(\Delta_{34}H)\end{vmatrix}}{4(34^2H)^2+(33^2H-44^2H)^2}, \end{cases}$$

where we took into account the definition of the Laplace operator Δ_{34}. Therefore, we have the following.

Proposition 4.3. *Let g denote an arbitrary Siklos metric, as described in* (4) *in terms of an arbitrary defining smooth function $H(x_2,x_3,x_4)$. If a conformal metric $\tilde g=e^\sigma g$ is Einstein, then $\sigma=\sigma(x_2,x_3,x_4)$ satisfies the system of PDE* (2)*.*

It is worthwhile to observe that σ is a completely arbitrary function of variable x_2. By Corollary 3.4, the defining function H for a Bach-flat Siklos metric has the explicit form (2). The full integration of system (2) for H of the form (2) is very difficult. Moreover, the above restrictions only give necessary conditions for conformal Einstein metrics. We shall now obtain the explicit classification and description of the conformally Einstein cases within several relevant subclasses of Siklos metrics.

4.1. **Siklos metrics with a harmonic defining function.** We consider here conformally Einstein Siklos metrics defined by a smooth function $H(x_2,x_3,x_4)$ such that $\Delta_{34}H=0$. Condition (1) for Bach-flatness is then automatically satisfied. With regard to (2), it reduces to
$$\begin{cases} 3\sigma=\frac{2}{x_3},\\ 4\sigma=0, \end{cases}$$

so that by integration we get

$$\sigma = 2\ln(x_3) + f(x_2), \tag{3}$$

for an arbitrary smooth function f. Both conditions listed in Proposition 4.1 are now satisfied. However, as we already observed, they are only necessary conditions for a metric in order to be conformally Einstein. In fact, by (3) we get that a smooth function φ such that $\tilde{g} = \varphi^{-2}g = e^{\sigma}g$, is necessarily of the form

$$\varphi = \frac{h(x_2)}{x_3},$$

for some smooth function h. For such function φ, equation (1) reduces to $h''(x_2) = 0$. Therefore, by integrating we prove the following.

Theorem 4.4. *Siklos metrics g, described by* (4) *for a harmonic defining function H, are conformally Einstein. In fact,*

$$\tilde{g} = \frac{x_3^2}{(d_1x_2 + d_2)^2}g$$

is Einstein (more precisely, Ricci-flat).

4.2. **Siklos metrics for a defining function obtained by separation of variables.** We now consider the case where the defining function H of a Siklos metric satisfies the additional condition $\partial^2_{34}H = 0$, whence H is of the special form

$$H(x_2, x_3, x_4) = F(x_2, x_3) + G(x_2, x_4), \tag{4}$$

for some smooth functions F, G. As observed in [6], the family of Siklos metrics described by condition (4) is very large, and it includes conformally flat Siklos metrics as a very special case, because a function H as given by (8) is clearly of the form (4). With regard to Bach-flatness, when H is of the form (4), the corresponding Siklos metric is Bach-flat if and only if $3333^4F(x_2, x_3) - 4444^4G(x_2, x_4) = 0$. Integrating system (2) for H of the form (4), we prove the following.

Proposition 4.5. *When H is described by* (4)*, the corresponding Siklos metric g satisfies both conditions listed in Proposition* 4.1 *if and only if*

$$3333^4F + 4444^4G = 0$$

and

$$\sigma = -2\ln\left(\tfrac{K(x_2)}{x_3}\left(33^2F(x_2, x_3) - 44^2G(x_2, x_4)\right)\right), \tag{5}$$

where K is an arbitrary smooth function.

We can now obtain the explicit classification of conformally Einstein Siklos metrics defined by H of the form (4).

Theorem 4.6. *A Siklos metric g, corresponding to a defining function H given by (4) for some smooth functions F, G, is conformally Einstein if and only if* (6)

$$\begin{cases} F(x_2,x_3) = \frac{c_1}{6K(x_2)}x_3^3 + \frac{1}{2}\alpha(x_2)x_3^2 + \beta(x_2)x_3 + \gamma(x_2), \\ G(x_2,x_4) = \frac{1}{6c_2K(x_2)}\big(12K(x_2)K''(x_2)x_4(\alpha(x_2)-\mu(x_2)) \\ \qquad +12x_4K(x_2)^2(\alpha''(x_2)-\mu''(x_2)) + 24K(x_2)K'(x_2)x_4(\alpha'(x_2)-\mu'(x_2)) \\ \qquad +3x_4K(x_2)^2(\mu(x_2)^2-\alpha(x_2)^2) \\ \qquad +K(x_2)(3c_2\mu(x_2)x_4^2 + 6c_2\nu(x_2) + 6c_1x_4\beta(x_2)) + c_2^2x_4^3\big), \end{cases}$$

where c_1, c_2 are real constants and $K, \alpha, \beta, \gamma, \lambda, \mu, \nu$ are smooth functions. In this case, $\tilde{g} = \varphi^{-2}g$ is an Einstein metric, satisfying the Einstein equation $\tilde{\varrho} = \lambda\tilde{g}$ with $\lambda = \Lambda(c_1^2 + c_2^2)$, where

$$\varphi(x_2,x_3,x_4) = \pm\frac{1}{x_3}\big(c_1x_3 - c_2x_4 + K(x_2)\,(\alpha(x_2)-\mu(x_2))\,\big).$$

Proof. Since σ is of the form (5) and $e^\sigma = \varphi^{-2}$, we have

$$\varphi(x_2,x_3,x_4) = \pm\frac{K(x_2)\big(33^2F(x_2,x_3) - 44^2G(x_2,x_4)\big)}{x_3}.$$

We then apply (1) to such function φ. We get the following system of PDEs:

$$\begin{cases} 2(K(x_2)333^3F) = 0, \\ 2(K(x_2)444^3G) = 0, \\ K(x_2)\big(33333^4F + 4444^4G\big) = 0, \\ K(x_2)\big(3333^4F + 34444^4G\big) = 0, \\ \frac{1}{2}K(x_2)(F+G)\big(4444^4G - 3333^4F\big) + 2K(x_2)\big(2233^4F - 2244^4G\big) \\ \qquad +4K'(x_2)\big(233^3F - 244^3G\big) + K(x_2)\big(3F333^3F - 4G444^3G\big) \\ \qquad -\frac{1}{2}\big(K(x_2)(33^2F + 44^2G) - 4K''(x_2)\big)\big(33^2F - 44^2G\big) = 0, \end{cases}$$

where, for the sake of brevity, we put $F = F(x_2,x_3)$ and $G = G(x_2,x_4)$. We then integrate the above PDEs one by one, starting from the simpler ones, taking into account $\Delta_{34}^2H = 3333^4F + 4444^4G = 0$ and $K(x_2) \neq 0$, and we find the solution described in the statement. □

5. Conformally Einstein homogeneous Siklos metrics

A pseudo-Riemannian manifold (M, g) is considered homogeneous when the group of isometries exhibits transitive behavior on M. In other words, for any two points p and q within M, there exists an isometry within (M, g) that maps p to q. Consequently, (M, g) can be conceptualized as the coset space G/H, equipped with a metric that remains invariant under G-transformations. Homogeneous spaces hold significant importance in the field of geometry.

All Siklos metrics admit at least one Killing vector field, namely, the vector field ∂/∂_{x_1} in global coordinates (x_1, x_2, x_3, x_4). Siklos [14, p.262] also completely classified all subclasses of Siklos metrics admitting some additional Killing vector fields. The results of the previous sections can be applied to these subclasses in order to determine Bach-flat and conformally Einstein examples within each of them. Here we obtain the classification of Bach-flat and conformally Einstein cases for all Siklos metrics that admit at least four Killing vector fields, that is, for all *homogeneous Siklos metrics.*

Siklos metrics in the following subclasses I),...,V) all admit four linearly independent Killing vector fields, except for the ones defined by $H = \pm x_3^{-2}$ (a special case of V)), which admit five of them.

For each of subclasses I),...,V), we start writing down the special form of the defining function H for which (at least) three additional Killing vector fields occur, following the notation we used in this paper for the global coordinates, and determine the Bach-flat examples. We then proceed computing the necessary condition (1) of Proposition 4.1 and integrating equation (1), so determining all conformally Einstein examples for that class. Following [14], $A(x_i)$ will denote an arbitrary smooth function of variables x_i, while we shall use $A_\alpha(x_i)$ to denote a homogeneous function of degree α of the specified variables.

I) $H = A_{-2}(x_3, x_4)$. Since in this case, H is a homogeneous function of degree -2 in the variables x_3 and x_4, more explicitly we have

$$H = a_{33}x_3^{-2} + a_{34}x_3^{-1}x_4^{-1} + a_{44}x_4^{-2},$$

for some real constants a_{33}, a_{34}, a_{44}. A standard calculation then yields

$$\Delta_{34}^2 H = 120(a_{33}x_3^{-6} + a_{44}x_4^{-6}) + 24a_{34}(x_3^{-5}x_4^{-1} + x_3^{-1}x_4^{-5}) + 8a_{34}x_3^{-3}x_4^{-3}.$$

Therefore, $\Delta_{34}^2 H = 0$ if and only if $a_{33} = a_{34} = a_{34} = 0$, that is, only for $H = 0$. In this case, the Siklos metric is isometric to the anti-de Sitter space [14], which, being Einstein, we shall always exclude from our actual investigation. Consequently, excluding the trivial case $H = 0$, by Theorem 3.1 we conclude that the metric is *never Bach-flat* (in particular, *never conformally Einstein*).

Remark 5.1. *The same argument also shows that the wider class of Siklos metrics*

$$H = a_{33}(x_2)x_3^{-2} + a_{33}(x_2)x_3^{-1}x_4^{-1} + a_{44}(x_2)x_4^{-2},$$

for arbitrary smooth functions a_{33}, a_{34}, a_{44}*, does not include (nontrivial) Bach-flat examples.*

II) $H = A(x_3)$. Clearly, in this case, $\Delta_{34}^2 H = A^{(iv)}(x_3) = 0$, whose general integral is given by

$$H = \frac{r_1}{6}x_3^3 + \frac{r_2}{2}x_3^2 + r_3x_3 + r_4,$$

for some real constants $r_1, \ldots, r_4$. We then check when the corresponding Siklos metric is conformally Einstein. Excluding the conformally flat case (corresponding to $r_1 = r_2 = 0$) and the Einstein case ($r_2 = r_3 = 0$), by Equation

(1) we find that the necessary condition $\mathcal{C} - W(\cdot,\cdot,\cdot,\nabla\sigma) = 0$ yields

$$1\sigma = 4\sigma = 2r_2 - x_3(r_1x_3 + r_2)3\sigma = 0.$$

By integration we then get

$$\sigma = -2\ln\left(K(x_2)\frac{r_1x_3 + r_2}{x_3}\right),$$

for an arbitrary smooth function $K(x_2)$. Now, the equation (1) is satisfied if and only if

(1) $$r_1K'(x_2) = 4K''(x_2)(r_1x_3 + r_2) + K(x_2)(2r_1r_3 - r_2^2) = 0.$$

As $r_1K'(x_2) = 0$, we consider the following two cases:

First case: $r_1 \neq 0$**.** Then, $K(x_2) = c_1$ is a real constant, and the above equation (1) yields $c_1(2r_1r_3 - r_2^2) = 0$ whence, excluding the trivial case $c_1 = 0$, we conclude that $2r_1r_3 - r_2^2 = 0$.

Second case: $r_1 = 0$**.** Then, the above equation (1) reduces to

$$r_2(4K'' - r_2K) = 0.$$

Observe that since $r_1 = 0$, necessarily $r_2 \neq 0$, otherwise the metric is conformally flat. So, $4K'' - r_2K = 0$, which we can completely integrate according to whether $r_2 < 0$ or $r_2 > 0$.

Thus, we proved the following.

Theorem 5.2. *Homogeneous Siklos spacetimes determined by a defining function $H = A(x_3)$:*

- *are Bach-flat if and only if $H = \frac{r_1}{6}x_3^3 + \frac{r_2}{2}x_3^2 + r_3x_3 + r_4$, for some real constants $r_1, \dots, r_4$.*
- *are (proper) conformally Einstein if and only if H is of the above form and one of the following cases occurs:*
  (a) $r_3 = \frac{r_2^2}{2r_1} \neq 0$. *In this case, $\tilde{g} = \varphi^{-2}g$ is an Einstein metric, satisfying $\tilde{\varrho} = \lambda\tilde{g}$ with $\lambda = (r_1c_1)^2\Lambda$, where*
  $$\varphi = c_1\left(r_1 + \frac{r_2}{x_3}\right),$$
  for some real constant c_1.
  (b) $r_1 = 0 < r_2$. *In this case, $\tilde{g} = \varphi^{-2}g$ is a Ricci-flat metric, where*
  $$\varphi = \frac{r_2}{x_3}\left(c_1\cosh\big(\frac{\sqrt{r_2}}{2}x_2\big) + c_2\sinh\big(\frac{\sqrt{r_2}}{2}x_2\big)\right),$$
  for some real constants c_1, c_2.
  (c) $r_1 = 0 > r_2$. *In this case, $\tilde{g} = \varphi^{-2}g$ is a Ricci-flat metric, where*
  $$\varphi = \frac{r_2}{x_3}\left(c_1\cos\big(\frac{\sqrt{-r_2}}{2}x_2\big) + c_2\sin\big(\frac{\sqrt{-r_2}}{2}x_2\big)\right),$$
  for some real constants c_1, c_2.

III) $H = A(x_2)x_3^2$. It is easily seen that $\Delta^2_{34}H = 0$. Therefore, all homogeneous Siklos metrics within this class are Bach-flat. Apart from the trivial case $A(x_2) = 0$, these spaces are neither Einstein nor conformally flat. With regard to conformally Einstein examples, we first compute the necessary condition $\mathcal{C} - W(\cdot, \cdot, \cdot, \nabla\sigma) = 0$ listed in Proposition 4.1. Rewriting system (1) for $H = A(x_2)x_3^2 \neq 0$, we get

$$1\sigma = 4\sigma = 2 - x_3 3\sigma = 0,$$

which by integration gives at once

$$\sigma = -2\ln\left(\frac{K(x_2)}{x_3}\right),$$

for an arbitrary function $K(x_2)$. A direct calculation then yields that equation (1) is now equivalent to the following differential equation

$$A(x_2)K(x_2) - 2K''(x_2) = 0. \tag{2}$$

A general result about ODE ensures that equation (2) admits a solution for any given smooth datum $A(x_2)$. Therefore, we proved the following result.

Theorem 5.3. *All homogeneous Siklos spacetimes determined by a defining function $H = A(x_2)x_3^2$ are conformally Einstein (in particular, Bach-flat). In fact, $\tilde{g} = \varphi^{-2}g = e^{\sigma}g$ is a Ricci-flat metric, where $\varphi = \frac{K(x_2)}{x_3}$ and K is determined as a solution of* (2).

IV) $H = x_2^{2\beta-2}A(x_2^{\beta}x_3)$. In this case, a standard calculation yields

$$\Delta^2_{34}H = x_2^{6\beta-2}A^{(iv)}(x_3x_2^{\beta})$$

so that by integration we get that $\Delta^2_{34}H = 0$ if and only if

$$H = x_2^{2\beta-2}\left(\frac{r_1}{6}(x_2^{\beta}x_3)^3 + \frac{r_2}{2}(x_2^{\beta}x_3)^2 + r_3(x_2^{\beta}x_3) + r_4\right),$$

for some real constants $r_1, \ldots, r_4$.

By Proposition 2.2 it easily follows that these metrics are conformally flat when $r_1 = r_2 = 0$ and are Einstein when $r_2 = r_3 = 0$, so we shall exclude these trivial solutions. We have that for such defining function H, system (2) becomes

$$1\sigma = 4\sigma = 2r_2x_2^{4\beta-2} - x_3 3\sigma(r_1x_2^{5\beta-2}x_3 + r_2x_2^{4\beta-2}) = 0,$$

which by integration gives

$$\sigma = -2\ln\left(\frac{K(x_2)(r_1x_2^{\beta}x_3 + r_2)}{x_3}\right),$$

where $K(x_2)$ is an arbitrary smooth function. The conformally Einstein equation (1) is now equivalent to

$$\left\{\begin{array}{l} K(x_2)\left(x_2^{4\beta}(2r_1r_3-r_2^2)+4r_1\beta x_2^{\beta}x_3(\beta-1)\right) \\ \quad +8r_1\beta x_2^{\beta+1}x_3K'(x_2)+4x_2^2K''(x_2)(r_2+r_1x_2^{\beta}x_3)=0, \\ 2r_1x_2^{\beta-1}(\beta K(x_2)+x_2K'(x_2))=0. \end{array}\right. \tag{3}$$

From the second equation in (3) we have the following two cases:

First case: $r_1 \neq 0$**.** Then, $\beta K(x_2)+x_2K'(x_2)=0$ and by integration we get $K(x_2)=k_1 x_2{}^{-\beta}$, for some real constant $k_1 \neq 0$. We substitute this into the first equation of (3) and we get

$$x_2^{3\beta}(2r_1r_3-r_2^2)+4x_2^{-\beta}r_2\beta(\beta+1)=0,$$

for all values of x_2. If $\beta=0$ the above equation reduces to $2r_1r_3-r_2^2=0$, that is, $r_3=\frac{r_2^2}{2r_1}$. For $\beta\neq 0$ it yields

$$2r_1r_3-r_2^2=r_2\beta(\beta+1)=0$$

whence, excluding the conformally flat and Einstein cases, $\beta=-1$ and again $2r_1r_3-r_2^2=0$.

Second case: $r_1=0$**.** Since $r_2\neq 0$ in order to exclude the Einstein case, the first equation in (3) yields

$$r_2x_2^{4\beta}K(x_2)-4x_2^2K''(x_2)=0, \tag{4}$$

and a standard result about ODE ensures that equation (4) admits a solution. Therefore, we proved the following.

Theorem 5.4. *Homogeneous Siklos spacetimes determined by a defining function $H=x_2^{2\beta-2}A(x_2^{\beta}x_3)$:*

- *are Bach-flat if and only if*

$$H=x_2^{2\beta-2}\left(\frac{r_1}{6}(x_2^{\beta}x_3)^3+\frac{r_2}{2}(x_2^{\beta}x_3)^2+r_3(x_2^{\beta}x_3)+r_4\right),$$

  for some real constants $r_1,\ldots,r_4$.
- *are (proper) conformally Einstein if and only if H is of the above form and one of the following cases occurs:*
  (a) $\beta\in\{0,-1\}$ *and* $r_3=\frac{r_2^2}{2r_1}$. *In the first case, $\tilde g=\varphi^{-2}g=e^{\sigma}g$ is an Einstein metric, satisfying $\tilde\varrho=\lambda\tilde g$ with $\lambda=(k_1r_1)^2\Lambda$, where*

$$\varphi=\frac{k_1x_2^{-\beta}(r_1x_2^{\beta}x_3+r_2)}{x_3},$$

  for some real constant k_1.

(b) $r_1 = 0$. *In this case,* $\tilde{g} = \varphi^{-2} g = e^{\sigma} g$ *is Ricci-flat, where*

$$\varphi = \frac{K(x_2)(r_1 x_2^{\beta} x_3 + r_2)}{x_3}$$

and $K(x_2)$ *is determined, depending on the sign of* $r_2 \neq 0$*, as a solution of equation* (4)*.*

V) $H = \pm x_3^{\alpha}$. It is easy to check that in this case,

$$\Delta_{34}^2 H = \pm\alpha(\alpha-1)(\alpha-2)(\alpha-3)x_3^{\alpha-4}.$$

Therefore, $\Delta_{34}^2 H = 0$ if and only if $\alpha \in \{0,1,2,3\}$. Excluding the Einstein ($\alpha = 0,3$) and conformally flat solutions ($\alpha = 0,1$), the only non-trivial case is $\alpha = 2$. But then, $H = \pm x_3^2$, which is the special case of type III) obtained setting $A(x_2) = \pm 1$. Observe that equation 2 now reduces to

$$\pm K(x_2) - 2K''(x_2) = 0.$$

Consequently, upon solving the aforementioned second-order ordinary differential equation, we have the following conclusion.

Theorem 5.5. *Homogeneous Siklos spacetimes determined by a defining function* $H = \pm x_3^{\alpha}$*:*

- *are Bach-flat if and only if* $\alpha \in \{0,1,2,3\}$*.*
- *For* $\alpha = 0,3$ *(respectively,* $\alpha = 0,1$*) these homogeneous spaces are Einstein (respectively, conformally flat). For* $\alpha = 2$ *they are conformally Einstein. In this case,* $\tilde{g} = \varphi^{-2} g = e^{\sigma} g$ *is an Einstein metric, satisfying* $\tilde{\varrho} = \lambda \tilde{g}$ *with* $\lambda = ...$*, where*

$$\sigma = -2\ln\left(\frac{K(x_2)}{x_3}\right) \text{ and } K(x_2) = \begin{cases} c_1\cosh(\frac{x_2}{\sqrt{2}}) + c_2\sinh(\frac{x_2}{\sqrt{2}}) & \text{if } H = x_3^2, \\ c_1\cos(\frac{x_2}{\sqrt{2}}) + c_2\sin(\frac{x_2}{\sqrt{2}}) & \text{if } H = -x_3^2, \end{cases}$$

for some real constants c_1, c_2*.*

We summarize the complete classification of Bach-flat and conformally Einstein homogeneous Siklos metrics in the following Table I. For each class of homogeneous Siklos spacetimes, we list the type with the defining function H and the cases where H gives rise to Bach flat metrics and proper (that is, neither Einstein nor conformally flat) conformally Einstein metrics. The checkmark "✓" in the column "Bach-flat" (respectively, "Conformally Einstein") means that the property holds for all homogeneous Siklos metrics with such defining function H. The checkmark ""means that the corresponding property does not hold for homogeneous Siklos metrics with such defining function H. It may be observed that very different behaviours occur in the different cases.

Defining H	Bach-flat	Conformally Einstein
I): $\frac{k_1}{x_3^2}+\frac{k_2}{x_3x_4}+\frac{k_3}{x_4^2}$		
II): $A(x_3)$	$\frac{r_1}{6}x_3^3+\frac{r_2}{2}x_3^2+r_3x_3+r_4$	$r_1(2r_1r_3-r_2^2)=0$
III): $A(x_2)x_3^2$	✓	✓
IV): $x_2^{2\beta-2}\,A(x_3x_2^\beta)$	$x_2^{2\beta-2}\big(\frac{r_1}{6}(x_2^\beta x_3)^3$ $+\frac{r_2}{2}(x_2^\beta x_3)^2+r_3(x_2^\beta x_3)+r_4\big)$	either $r_1=0$ or $\beta\in\{0,-1\}, r_3=\frac{r_2^2}{2r_1}$
V): $\pm x_3^\alpha$	$\alpha\in\{0,1,2,3\}$	$\alpha=2$

Table I: Homogeneous Bach-flat and (proper) conformally Einstein Siklos metrics

Appendix: conformally symmetric Siklos metrics

As we showed in the previous section, the study of conformally Einstein Siklos metrics leads to several remarkable examples. The same is not true for conformal symmetry. We briefly recall that a pseudo-Riemannian manifold (M,g) is *(locally) conformally symmetric* if $\nabla W=0$, W denoting, as before, the Weyl curvature tensor. *Proper* conformally symmetric manifolds are the ones that are neither conformally flat nor locally symmetric. For Siklos metrics we have the following result.

Proposition 5.6. *A Siklos metric is locally conformally symmetric if and only if it is conformally flat.*

Proof. Starting from equations (1) and (6), we computed the components of ∇W with respect to coordinates (x_1,x_2,x_3,x_4). Up to symmetries, the possibly non-vanishing components are the following:

$$
\begin{aligned}
&(\nabla_2W)_{1224}=(\nabla_2W)_{2334}=\tfrac{3}{2\Lambda x_3^3}34^2H,\\
&(\nabla_2W)_{2324}=\tfrac{3}{2\Lambda x_3^2}234^3H,\\
&(\nabla_2W)_{2424}=-(\nabla_2W)_{2323}=\tfrac{3}{4\Lambda x_3^2}\big(244^3H-233^3H\big),\\
&(\nabla_2W)_{2434}=-(\nabla_2W)_{1223}=\tfrac{3}{4\Lambda x_3^3}\big(44^2H-33^2H\big),\\
&(\nabla_3W)_{2323}=-(\nabla_3W)_{2424}=\tfrac{3}{4\Lambda x_3^3}\big(2(33^2H-44^2H)-x_3(344^3H-333^3H)\big),\\
&(\nabla_3W)_{2324}=\tfrac{3}{2\Lambda x_3^3}\big(234^2H+x_3433^2H\big),\\
&(\nabla_4W)_{2323}=-(\nabla_4W)_{2424}=-\tfrac{3}{4\Lambda x_3^3}\big(-434^2H+x_3(444^3H-433^3H)\big),\\
&(\nabla_4W)_{2324}=-\tfrac{3}{2\Lambda x_3^3}\big(33^2H-44^2H-x_3344^3H\big).
\end{aligned}
$$

Setting $\nabla W=0$, we immediately get equations (7), which characterize conformally flat Siklos metrics. $\square$

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