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# $C$-$fp_n$-INJECTIVE AND $C$-$fp_n$-FLAT MODULES

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Abstract. Let $C = {}_SC_R$ be a (faithfully) semidualizing bimodule. This paper begins with the introduction of the concepts of $C$-$fp_n$-injective $R$-modules and $C$-$fp_n$-flat $S$-modules. Subsequently, we investigate various properties associated with classes of modules characterized by $C$-$fp_n$-injective and $C$-$fp_n$-flat dimensions. For instance, we explore Foxby equivalence and the existence of preenvelopes and covers in relation to these classes of modules. Finally, we analyze the exchange properties of these classes and the connections between preenvelopes (or precovers) and Foxby equivalence, particularly within the context of almost excellent extensions of rings.

*Keywords*: $C$-$fp_n$-flat module; $C$-$fp_n$-injective module; Foxby equivalence; semidualizing bimodule.
*2020 MSC*: 16E10; 16E30; 16E65; 16P70.

## 1. Introduction

The notion of $fp_n$-injective (resp. $fp_n$-flat) modules was introduced by Wei and Zhang in [18] as a generalization of $fp$-injective and $FP_n$-injective (resp. $fp$-flat and $FP_n$-flat) modules, where $fp$-injective and $fp$-flat modules were studied by Garkusha and Generalov in [8], and also $FP_n$-injective and $FP_n$-flat modules were introduced by Bravo and Pérez in [2]. The extension of the concept of a semidualizing module to a pair of general associative rings was made by Holm and White, in [13]. Also, they studied Auslander and Bass classes under a semidualizing bimodule, and then the notions of $C$-flat, $C$-projective, and $C$-injective modules were introduced. Here, $C = {}_SC_R$ stands for a semidualizing bimodule. Recently, some the homological results about relative injective and flat modules according to semidualizing bimodules have been proved, see [11, 13, 19]) for more details.

In [19], Wu and Gao studied the notion of $C$-$FP_n$-injective (resp. $C$-$FP_n$-flat) modules. They investigated Foxby equivalence in relation to these modules, proved that the classes $\mathcal{FI}^n_C(R)$ and $\mathcal{FF}^n_C(S)$ are preenveloping and covering, where $\mathcal{FI}^n_C(R)$ and $\mathcal{FF}^n_C(S)$ are the classes of $C$-$FP_n$-injective $R$-modules and $C$-$FP_n$-flat $S$-modules.

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In Section 2, we state some required concepts and results. For example, the definitions of $fp$-injective, $fp$-flat, $FP_n$-injective, $FP_n$-flat, $fp_n$-injective, $fp_n$-flat, $C$-$FP_n$-injective, and $C$-$FP_n$-flat modules provide motivation for the definition of $C$-$fp_n$-injective and $C$-$fp_n$-flat modules as common generalizations of these concepts. It is noteworthy that this approach enables us to deal with several important concepts on homological theory comprehensively. So, in Section 3, we introduce and review the concepts of $C$-$fp_n$-injective (resp. $C$-$fp_n$-flat) modules as a common generalization of modules of the classes $\mathcal{FI}_C^n(R)$ and $\mathcal{WI}_C(R)$ (resp. $\mathcal{FF}_C^n(S)$ and $\mathcal{WF}_C(S)$), where $\mathcal{WI}_C(R)$ and $\mathcal{WF}_C(S)$ are considered as a class of $C$-weak injective $R$-modules and $C$-weak flat $S$-modules. Then we obtain some results of homological relationships between the classes $fp_nI(S)_{\leq k}$, $fp_nF(R)_{\leq k}$, $Cfp_nI(R)_{\leq k}$, $Cfp_nF(S)_{\leq k}$, $\mathcal{A}_C(R)$, and $\mathcal{B}_C(S)$, where these classes are the class of $S$-modules with $fp_n$-injective dimension at most $k$, the class of $R$-modules with $fp_n$-flat dimension at most $k$, the class of $R$-modules with $C$-$fp_n$-injective dimension at most $k$, the class of $S$-modules with $C$-$fp_n$-flat dimension at most $k$, Auslander class, and Bass class under faithfully semidualizing bimodules $C$, respectively. Then, using these results, we investigate Foxby equivalence relative to the theses classes, see Theorem 3.9. Also, we prove that the classes $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ are preenveloping and covering, see Theorem 3.17.

Section 4 considering faithfully semidualizing module $C$ is devoted to the exchange properties of these classes, as well as preenvelopes, precovers and Foxby equivalence, under change of rings, see Theorems 4.7, 4.8 and 4.17. Section 5 is dedicated to the conclusion.

## 2. Preliminaries

Throughout, $n$ is a positive integer, $R$ and $S$ are two fixed associative rings with units. all $R$- or $S$-modules are understood to be unital left $R$- or $S$-modules (unless specified otherwise). ${}_SM_R$ is used to denote that $M$ is an $(S, R)$-bimodule which means that $M$ is both a left $S$-module and a right $R$-module, and these structures are compatible. Also, right $R$- or $S$-modules are considered as left modules over the rings $R^{op}$ and $S^{op}$. We use Mod $R$ or Mod $S$ as a class of left $R$- or $S$-modules.

**Definition 2.1.** ( [2,10]) *An $R$-module $M$ is finitely $n$-presented if there is an exact sequence*

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_i \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

*where each $P_i$ is a finitely generated free $R$-module for all $0 \leq i \leq n$. $\mathcal{FP}_n(R)$ is considered as a class of all finitely $n$-presented $R$-modules. In case $n = 0$, $n = 1$ and $n = \infty$, $\mathcal{FP}_0(R)$, $\mathcal{FP}_1(R)$ and $\mathcal{FP}_\infty(R)$ are classes of all finitely generated, finitely presented and super finitely presented $R$-modules, respectively.* A ring $R$ is *$n$-coherent* if $\mathcal{FP}_n(R) \subseteq \mathcal{FP}_{n+1}(R)$.

**Definition 2.2.** ( [2,9,24]) *An $R$-module $M$ is $FP_n$-injective or $(n,0)$-injective (resp. $FP_n$-flat or $(n,0)$-flat) whenever* $\mathrm{Ext}^1_R(D,M)=0$ *(resp.* $\mathrm{Tor}^R_1(D,M)=0$*) for any $D\in\mathcal{FP}_n(R)$ (resp. $\mathcal{FP}_n(R^{op})$), and in case $n=\infty$, $M$ is weak injective (resp. weak flat). The symbols (resp. $\mathcal{FP}_n$-$\mathrm{Flat}(R)$) and $\mathcal{WI}(R)$ (resp. $\mathcal{WF}(R)$) are classes of these modules, respectively.*

**Definition 2.3.** ( [18]) *An $R$-module $M$ is $fp_n$-injective (resp. $fp_n$-flat) when for every exact sequence $0\longrightarrow K_1\longrightarrow K_2$ with $K_1,K_2\in\mathcal{FP}_n(R)$ (resp. $\mathcal{FP}_n(R^{op})$), the induced sequence* $\mathrm{Hom}_R(K_1,M)\longrightarrow\mathrm{Hom}_R(K_2,M)\longrightarrow 0$ *(resp. $0\longrightarrow K_1\otimes_R M\longrightarrow K_2\otimes_R M$) is exact. $fp_nI(R)$ (resp. $fp_nF(R)$) is considered as a class of these modules.*

By [2, Proposition 1.7(1)], $\mathcal{FP}_n$-$\mathrm{Inj}(R)\subseteq fp_mI(R)$ (resp. $\mathcal{FP}_n$-$\mathrm{Flat}(R)\subseteq fp_mF(R)$) for any $m\geq n$. But the opposite is not true, see Example 3.3.

**Definition 2.4.** ( [11])

*(i) An $(S,R)$-bimodule $C={}_SC_R$ is semidualizing when the following conditions hold:*
- $(a_1)$ *$C\in\mathcal{FP}_\infty(S)$;*
- $(a_2)$ *$C\in\mathcal{FP}_\infty(R^{op})$;*
- $(b_1)$ *The homothety map ${}_S\gamma:{}_SS_S\longrightarrow\mathrm{Hom}_{R^{op}}(C,C)$ is an isomorphism;*
- $(b_2)$ *The homothety map $\gamma_R:{}_RR_R\longrightarrow\mathrm{Hom}_S(C,C)$ is an isomorphism;*
- $(c)$ $\mathrm{Ext}^i_S(C,C)=0=\mathrm{Ext}^i_{R^{op}}(C,C)$ *for all $i\geq 1$.*

*A semidualizing bimodule ${}_SC_R$ is faithfully semidualizing whenever* $\mathrm{Hom}_S(C,Y)=0$ *(resp.* $\mathrm{Hom}_{R^{op}}(C,X)=0$*), then $Y=0$ (resp. $X=0$) for every module ${}_SY$ (resp. $X_R$).*

*(ii) The Auslander class $\mathcal{A}_C(R)$ (resp. Bass class $\mathcal{B}_C(S)$ ) with respect to $C$ consists of all $R$-modules $A$ (resp. $S$-modules $B$) such that for all $i\geq 1$*
- $(A_1)$ $\mathrm{Tor}^R_i(C,A)=0$ *(resp.* $\mathrm{Ext}^i_S(C,B)=0$*);*
- $(A_2)$ $\mathrm{Ext}^i_S(C,C\otimes_R A)=0$ *(resp.* $\mathrm{Tor}^R_i(C,\mathrm{Hom}_S(C,B))=0$*);*
- $(A_3)$ *The natural evaluation homomorphism $\mu_A:A\longrightarrow\mathrm{Hom}_S(C,C\otimes_R A)$ (resp. $\nu_B:C\otimes_R\mathrm{Hom}_S(C,B)\longrightarrow B$) is an isomorphism of $R$-modules (resp. $S$-modules).*

*By ( [13, Proposition 4.1]), we have the following equivalence:*

$$\mathcal{A}_C(R)\ \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}}\ \mathcal{B}_C(S)\quad(\sim)$$

**Definition 2.5.** ( [19]) *An $R$-module $M$ is $C$-$FP_n$-injective if $M=\mathrm{Hom}_S(C,X)$ for some $X\in\mathcal{FP}_n$-$\mathrm{Inj}(S)$. An $S$-module $N$ is $C$-$FP_n$-flat if $N=C\otimes_R Y$ for some $Y\in\mathcal{FP}_n$-$\mathrm{Flat}(R)$. In case $n=\infty$, $M\in\mathcal{WI}_C(R)$ and $N\in\mathcal{WF}_C(S)$.*

**Definition 2.6.** Suppose that $\mathcal{Y} = \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} U \longrightarrow 0$ is a projective resolution of $U$ in Mod $R$. Then $\mathcal{Y}$ is said to be $\mathcal{Y}$-finitely presented when $U, \mathrm{Ker}(f_i) \in \mathcal{FP}_1(R)$ for any $i \geq 0$.

There are examples of $\mathcal{Y}$-finitely presented, see Example 3.5(ii).

The following proposition is similar to Proposition 7.2 and Remark 4 from [13], but for non-commutative rings.

**Proposition 2.7.** *The following assertions are true.*

*(i)* $A \in \mathcal{A}_C(R)$ *if and only if* $A^* \in \mathcal{B}_C(R^{op})$*;*
*(ii)* $B \in \mathcal{B}_C(R)$ *if and only if* $B^* \in \mathcal{A}_C(R^{op})$*.*

**Definition 2.8.** The $fp_n$*-injective dimension* of an $S$-module $M$ is defined such that $fp_n.\mathrm{id}_S(M) \leq k$ when there is an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0$$

in Mod $S$, where $I_i \in fp_nI(S)$ for all $0 \leq i \leq k$. Also, the $fp_n$*-flat dimension* of an $R$-module $N$ is defined such that $fp_n.\mathrm{fd}_R(N) \leq k$ when there is an exact sequence

$$0 \longrightarrow J_k \longrightarrow J_{k-1} \longrightarrow \cdots \longrightarrow J_1 \longrightarrow J_0 \longrightarrow N \longrightarrow 0$$

in Mod $R$, where $J_i \in fp_nF(R)$ for all $0 \leq i \leq k$. Set $fp_n.\mathrm{id}_S(M) = \infty$ (resp. $fp_n.\mathrm{fd}_R(N) = \infty$) if no such $k$ exists.

$fp_n.id_S(M) \leq 0$ if and only if $M \in fp_nI(S)$, and $fp_n.\mathrm{fd}_R(N) \leq 0$ if and only if $N \in fp_nF(R)$.

In the next lemma under a faithfully semidualizing bimodule $C = {}_SC_R$, we show that $fp_nI(S)_{\leq k} \subseteq \mathcal{B}_C(S)$ and $fp_nF(R)_{\leq k} \subseteq \mathcal{A}_C(R)$.

**Lemma 2.9.** *The following assertions are true.*

*(i)* $fp_nI(S)_{\leq k} \subseteq \mathcal{B}_C(S)$*;*
*(ii)* $fp_nF(R)_{\leq k} \subseteq \mathcal{A}_C(R)$*.*

*Proof.* (i). First, we prove that for $k = 0$, $fp_nI(S) \subseteq \mathcal{WI}(S)$. Consider, $\mathcal{Y}$-finitely presented $\mathcal{Y} = \cdots \longrightarrow P_j \longrightarrow P_{j-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow U \longrightarrow 0$ in Mod $S$. Then we have $0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow U \longrightarrow 0$, where $K_0, P_0, U \in \mathcal{FP}_n(R)$. So if $I \in fp_nI(S)$, then $\mathrm{Hom}_S(P_0, I) \longrightarrow \mathrm{Hom}_S(K_0, I) \longrightarrow 0$ is exact. Hence $\mathrm{Ext}^1_S(U, I) = 0$, and then $I \in \mathcal{WI}(S)$. Consequently, $fp_nI(S) \subseteq \mathcal{B}_C(S)$ from [11, Theorem 2.2]. So for $M \in fp_nI(S)_{\leq k}$ we have

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0,$$

where $I_i \in \mathcal{B}_C(S)$ for all $0 \leq i \leq k$. Therefore by [13, Corollary 6.3], $M \in \mathcal{B}_C(S)$.

(ii). Assume that $N \in fp_nF(R)_{\leq k}$. Then we have

$$0 \longrightarrow J_k \longrightarrow J_{k-1} \longrightarrow \cdots \longrightarrow J_1 \longrightarrow J_0 \longrightarrow N \longrightarrow 0,$$

where $J_i \in fp_nF(R)$ for all $0 \leq i \leq k$. Hence by [17, Lemma 3.53], we have

$$0 \longrightarrow N^* \longrightarrow J_0^* \longrightarrow J_1^* \longrightarrow \cdots \longrightarrow J_{k-1}^* \longrightarrow J_k^* \longrightarrow 0,$$

where $J_i^* \in fp_nI(R^{op})$ for all $0 \leq i \leq k$ from [18, Proposition 2.4(2)]. By (i), $J_i^* \in \mathcal{B}_C(R^{op})$, and then from [13, Corollary 6.3] and Proposition 2.7, we deduce that $N^* \in \mathcal{B}_C(R^{op})$ if and only if $N \in \mathcal{A}_C(R)$. □

## 3. Modules with $C$-$fp_n$-injective and $C$-$fp_n$-flat dimensions

First, we have the following definition under a (faithfully) semidualizing bimodule $C = {}_SC_R$.

**Definition 3.1.** *An $R$-module $M$ is $C$-$fp_n$-injective if $M = \mathrm{Hom}_S(C, X)$ for some $X \in fp_nI(S)$. An $S$-module $N$ is $C$-$fp_n$-flat if $N = C \otimes_R Y$ for some $Y \in fp_nF(R)$. We set*

$$Cfp_nI(R) = \{\mathrm{Hom}_S(C, X) : X \in fp_nI(S)\}$$

*and*

$$Cfp_nF(S) = \{C \otimes_R Y : Y \in fp_nF(R)\}.$$

*Remark* 3.2. *(i)* $\mathcal{FI}_C^n(R) \subseteq Cfp_mI(R)$ (resp. $\mathcal{FF}_C^n(S) \subseteq Cfp_mF(S)$) for any $m \geq n$ (see [2, Proposition 1.7(1)]). But not conversely, see (Example 3.3);

*(ii)* $Cfp_nI(R) \subseteq Cfp_mI(R)$ (resp. $Cfp_nF(S) \subseteq Cfp_mF(S)$) for any $m \geq n$, and so we have

$$Cfp_1I(R) \subseteq Cfp_2I(R) \subseteq \cdots \subseteq Cfp_nI(R) \subseteq Cfp_{n+1}I(R) \subseteq \cdots$$

and

$$Cfp_1F(S) \subseteq Cfp_2F(S) \subseteq \cdots \subseteq Cfp_nF(S) \subseteq Cfp_{n+1}F(S) \subseteq \cdots;$$

*(iii)* $M \in Cfp_\infty I(R)$ (resp. $Cfp_\infty F(S)$) if and only if $M \in \mathcal{WI}_C(R)$ (resp. $\mathcal{WF}_C(S)$).

Recall that a ring $R$ is said to be an *$(n, 0)$-ring* or *$n$-regular ring* when every $R$-module in $\mathcal{FP}_n(R)$ is projective (see [15, 24]).

**Example 3.3.** Suppose that $K$ is a field, $E$ is a $K$-vector space with infinite rank, and $A$ is a Noetherian ring of global dimension 0. Suppose also that $B = K \ltimes E$ is the trivial extension of $K$ by $E$ and $R = A \times B$ is the direct product of $A$ and $B$. By [15, Theorem 3.4(3)], $R$ is a $(2, 0)$-ring which is not a $(1, 0)$-ring. Thus, for every $M$ in Mod $R$ and every $L \in \mathcal{FP}_2(R)$, $\mathrm{Ext}_R^1(L, M) = 0$ (resp. $\mathrm{Tor}_1^R(L, M) = 0$). Hence $M \in \mathcal{FP}_2$-Inj$(R)$ (resp. $\mathcal{FP}_2$-Flat$(R)$), and so $M \in fp_2I(R)$ (resp. $fp_2F(R)$). On the other hand, there is an $R$-module which is not in $\mathcal{FP}_1$-Inj$(R)$ (resp. $\mathcal{FP}_1$-Flat$(R)$), since if every $R$-module is in $\mathcal{FP}_1$-Inj$(R)$ (resp. $\mathcal{FP}_1$-Flat$(R)$), [24, Theorem 3.9] implies that $R$ is a $(1, 0)$-ring, which is a contradiction. Therefore, if $C = R = S$, then every $R$-module is in $Cfp_2I(R)$ and $Cfp_2F(R)$, and there exists an $R$-module which is not in $\mathcal{FI}_C^1(R)$ (resp. $\mathcal{FF}_C^1(R)$).

**Definition 3.4.** *Let $C = {}_SC_R$ be a faithfully semidualizing bimodule. The $C$-$fp_n$-injective dimension of an $R$-module $M$ is defined such that $Cfp_n.\mathrm{id}_R(M) \leq k$ when there is an exact sequence*

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0$$

*in Mod $R$, where $I_i \in Cfp_nI(R)$ for all $0 \leq i \leq k$. Also, the $C$-$fp_n$-flat dimension of an $S$-module $N$ is defined such that $Cfp_n.\mathrm{fd}_S(N) \leq k$ when there is an exact sequence*

$$0 \longrightarrow J_k \longrightarrow J_{k-1} \longrightarrow \cdots \longrightarrow J_1 \longrightarrow J_0 \longrightarrow N \longrightarrow 0$$

*in Mod $S$, where where $J_i \in Cfp_nF(S)$ for all $0 \leq i \leq k$. Set $Cfp_n.\mathrm{id}_S(M) = \infty$ (resp. $Cfp_n.\mathrm{fd}_R(N) = \infty$) if no such $k$ exists.*

$Cfp_n.\mathrm{id}_R(M) \leq 0$ if and only if $M \in Cfp_nI(R)$, and $Cfp_n.\mathrm{fd}_S(N) \leq 0$ if and only if $N \in Cfp_nF(S)$.

The finitely presented dimension of an $R$-module $A$ is defined as $\mathrm{f.p.dim}_R(A) = \inf\{n \mid$ there is an exact sequence $\mathrm{P}_{n+1} \to \mathrm{P}_n \to \cdots \to \mathrm{P}_1 \to \mathrm{P}_0 \to \mathrm{A} \to 0$ in Mod $R$, where $P_i$ is projective for all $0 \leq i \leq n+1$, and $P_n, P_{n+1} \in \mathcal{FP}_0(R)\}$. So $\mathrm{f.p.dim}(R) = \sup\{\mathrm{f.p.dim}_R(A) \mid A \in \mathcal{FP}_0(R)\}$. $\mathrm{w.gl.dim}(R)$ and $\mathrm{gl.dim}(R)$ are the weak global dimension and global dimension of a ring $R$, respectively. Also, a ring $R$ is said to be an $(a,b,c)$-ring, whenever $\mathrm{w.gl.dim}(R) = a$, $\mathrm{gl.dim}(R) = b$ and $\mathrm{f.p.dim}(R) = c$ (see [16]).

**Example 3.5.** Let $R_1 = k[[x_1, x_2, x_3, x_4]]$ be the ring of power series in 4 indeterminates over a field $k$ and that $R_2$ is a valuation ring with global dimension 4.

*(i)* From [1, Example 2.2], $R = R_1 \oplus R_2$ is $(4,4,5)$-ring and coherent. So, $\mathrm{w.gl.dim}(R) = \mathrm{gl.dim}(R) = 4$, and then for every $M$ in Mod $R$, we have $0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow I_3 \longrightarrow D \longrightarrow 0$, where $I_i \in fp_nI(R)$ and injective for all $0 \leq i \leq 3$. One easily gets that $D \in fp_nI(R)$. Therefore if $C = R = S$, we deduce that $Cfp_n.\mathrm{id}_R(M) \leq 4$. Similarly, it follows that $Cfp_n.\mathrm{fd}_R(M) \leq 4$;

*(ii)* From (i), $\mathrm{f.p.dim}(R) = 5$, and hence there is $U \in \mathcal{FP}_0(R)$ such that $\mathrm{f.p.dim}_R(U) = 5$. Thus, we have a projective resolution of $U$

$$P_6 \longrightarrow P_5 \longrightarrow P_4 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow U \longrightarrow 0,$$

where $P_5, P_6 \in \mathcal{FP}_0(R)$. Also, since $R$ is coherent, it follows that $K_4 := \mathrm{Im}(P_5 \to P_4)$ is $\mathcal{Y}$-finitely presented.

- Up to the end of the section, $C$ is a faithfully semidualizing bimodule.

**Lemma 3.6.** *The following assertions are true.*

*(i)* $Cfp_nI(R)_{\leq k} \subseteq \mathcal{A}_C(R)$;
*(ii)* $Cfp_nF(S)_{\leq k} \subseteq \mathcal{B}_C(S)$.

*Proof.* (i). Let $N \in Cfp_nI(R)$. Then $N = \mathrm{Hom}_S(C, X)$ for some $X \in fp_nI(S)$. By Lemma 2.9(i), $X \in \mathcal{B}_C(S)$ and so $N \in \mathcal{A}_C(R)$ from [11, Lemma 2.9(1)]. Now, if $M \in Cfp_nI(R)_{\leq k}$, then we have

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0,$$

where each $I_i \in Cfp_nI(R)$ and each $I_i \in \mathcal{A}_C(R)$ for all $0 \leq i \leq k$. Hence by [13, Corollary 6.3], $M \in \mathcal{A}_C(R)$.

(ii). The proof is similar to the first part. □

In the following, we investigate Foxby equivalence relative to the classes $Cfp_nI(R)$ and $Cfp_nF(S)$ as a generalization of Foxby equivalence relative to the classes $\mathcal{FI}^n_C(R)$ and $\mathcal{FF}^n_C(S)$ in [19].

**Proposition 3.7.** *We have the following equivalences:*

*(i)* $$Cfp_nI(R)_{\leq k} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\underset{\sim}{\rightleftarrows}}} fp_nI(S)_{\leq k}\ ;$$

*(ii)* $$fp_nF(R)_{\leq k} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\underset{\sim}{\rightleftarrows}}} Cfp_nF(S)_{\leq k}\ .$$

*Proof.* (i). Assume that $M \in Cfp_nI(R)_{\leq k}$. Then, we have

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow 0,$$

where $I_i \in Cfp_nI(R)$ for all $0 \leq i \leq k$. Thus, $I_i = \mathrm{Hom}_S(C, X)$ for some $X \in fp_nI(S)$. By Lemma 2.9(i), $X \in \mathcal{B}_C(S)$, and then $C \otimes_R \mathrm{Hom}_S(C, X) \cong X$. So $C \otimes_R I_i \in fp_nI(S)$ and also from Lemma 3.6(i), $I_i \in \mathcal{A}_C(R)$, and so $\mathrm{Tor}^R_j(C, I_i) = 0$ for all $j \geq 1$. By Lemma 3.6(i), $M \in \mathcal{A}_C(R)$ and hence $\mathrm{Tor}^R_j(C, M) = 0$ for all $j \geq 1$. Therefore, we obtain

$$0 \longrightarrow C \otimes_R M \longrightarrow C \otimes_R I_0 \longrightarrow C \otimes_R I_1 \longrightarrow \cdots \longrightarrow C \otimes_R I_{k-1} \longrightarrow C \otimes_R I_k \longrightarrow 0,$$

which shows that $C \otimes_R M \in fp_nI(S)_{\leq k}$. Now, assume that $N \in fp_nI(S)_{\leq k}$. Then we have

$$0 \longrightarrow N \longrightarrow I'_0 \longrightarrow I'_1 \longrightarrow \cdots \longrightarrow I'_{k-1} \longrightarrow I'_k \longrightarrow 0,$$

where $I'_i \in fp_nI(S)$ for all $0 \leq i \leq k$. For all $0 \leq i \leq k$, from Lemma 2.9(i), $I'_i \in \mathcal{B}_C(S)$, and so $\mathrm{Ext}^j_S(C, I'_i) = 0$ for all $j \geq 1$. Also, by Lemma 2.9(i), $N \in \mathcal{B}_C(S)$ and hence $\mathrm{Ext}^j_S(C, N) = 0$ for all $j \geq 1$. Therefore, we obtain the exact sequence

$$0 \to \mathrm{Hom}_S(C, N) \to \mathrm{Hom}_S(C, I'_0) \to \cdots \to \mathrm{Hom}_S(C, I'_{k-1}) \to \mathrm{Hom}_S(C, I'_k) \to 0$$

in Mod $R$ which shows that $\mathrm{Hom}_S(C, N) \in Cfp_nI(R)_{\leq k}$. Note that, if $M \in Cfp_nI(R)_{\leq k}$, then from Lemma 3.6(i), $M \in \mathcal{A}_C(R)$, and if $N \in fp_nI(S)_{\leq k}$, then by Lemma 2.9(i), $N \in \mathcal{B}_C(S)$. Hence we have the natural isomorphisms $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$ and $C \otimes_R \mathrm{Hom}_S(C, N) \cong N$.

(ii). The proof is similar to the first part. □

**Corollary 3.8.** *We have the following equivalences:*

(i) $$Cfp_nI(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} fp_nI(S);$$ ($\sim$)

(ii) $$fp_nF(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} Cfp_nF(S).$$ ($\sim$)

*Proof.* Put $k = 0$ in Proposition 3.7. $\square$

By using Lemma 3.6, Proposition 3.7, and Corollary 3.8, we get the first main result of this section.

**Theorem 3.9.** (Foxby Equivalence) *We have the following equivalences:*

$$\begin{tikzcd}
fp_nF(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[d, hook] & Cfp_nF(S) \arrow[l, shift left, "\mathrm{Hom}_S(C,-)", "\sim"'] \arrow[d, hook] \\
fp_nF(R)_{\leq k} \arrow[r, shift left, "C\otimes_R -"] \arrow[d, hook] & Cfp_nF(S)_{\leq k} \arrow[l, shift left, "\mathrm{Hom}_S(C,-)", "\sim"'] \arrow[d, hook] \\
\mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R -"] & \mathcal{B}_C(S) \arrow[l, shift left, "\mathrm{Hom}_S(C,-)", "\sim"'] \\
Cfp_nI(R)_{\leq k} \arrow[r, shift left, "C\otimes_R -"] \arrow[u, hook] & fp_nI(S)_{\leq k} \arrow[l, shift left, "\mathrm{Hom}_S(C,-)", "\sim"'] \arrow[u, hook] \\
Cfp_nI(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[u, hook] & fp_nI(S). \arrow[l, shift left, "\mathrm{Hom}_S(C,-)", "\sim"'] \arrow[u, hook]
\end{tikzcd}$$

**Corollary 3.10.** *Let $M$ be in Mod $R$ and $N$ be in Mod $S$. Then the following assertions are true.*

*(i)* $M \in Cfp_nI(R)_{\leq k}$ *if and only if* $M \in \mathcal{A}_C(R)$ *and* $C\otimes_R M \in fp_nI(S)_{\leq k}$*;*

*(ii)* $N \in Cfp_nF(S)_{\leq k}$ *if and only if* $N \in \mathcal{B}_C(S)$ *and* $\mathrm{Hom}_S(C, N) \in fp_nF(R)_{\leq k}$.

*Proof.* (i). ($\Rightarrow$) This follows from Lemma 3.6(i) and Theorem 3.9.

($\Leftarrow$) If $M \in \mathcal{A}_C(R)$ and $C \otimes_R M \in fp_nI(S)_{\leq k}$, then $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$ and, by Theorem 3.9, $\mathrm{Hom}_S(C, C \otimes_R M) \in Cfp_nI(R)_{\leq k}$. Thus $M \in Cfp_nI(R)_{\leq k}$.

(ii). The proof is similar to the first part. $\square$

**Corollary 3.11.** *Let $X$ be in Mod $S$ and $Y$ be in Mod $R$. Then the following assertions are true.*

*(i)* $\mathrm{Hom}_S(C, X) \in Cfp_nI(R)_{\leq k}$ *if and only if* $X \in fp_nI(S)_{\leq k}$*;*

*(ii)* $C \otimes_R Y \in Cfp_nF(S)_{\leq k}$ *if and only if* $Y \in fp_nF(R)_{\leq k}$.

*Proof.* (i). Assume that $\mathrm{Hom}_S(C, X) \in Cfp_nI(R)_{\leq k}$. Then, by Corollary 3.10(i), $\mathrm{Hom}_S(C, X) \in \mathcal{A}_C(R)$. Thus, from [11, Lemma 2.9(1)], $X \in \mathcal{B}_C(S)$ and hence $C \otimes_R \mathrm{Hom}_S(C, X) \cong X$. Therefore $X \in fp_nI(S)_{\leq k}$ by Theorem 3.9.

(ii). The proof is similar to the first part. □

We denote the character module of $M$ by $M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ [17, Page 135].

**Proposition 3.12.** *Suppose that* $M$ *is in Mod* $R$ *and* $N$ *is Mod* $S$. *Then the following assertions are true.*

*(i)* $M \in Cfp_nI(R)_{\leq k}$ *if and only if* $M^* \in Cfp_nF(R^{op})_{\leq k}$*;*
*(ii)* $N \in Cfp_nF(S)_{\leq k}$ *if and only if* $N^* \in Cfp_nI(S^{op})_{\leq k}$.

*Proof.* (i). We argue by induction on $k$. Assume that $k = 0$. ($\Rightarrow$). Assume that $M \in Cfp_nI(R)$. Then $M = \mathrm{Hom}_S(C, X)$ for some $X \in fp_nI(S)$. From [18, Proposition 2.4(1)], $X^* \in fp_nF(S^{op})$. Thus $M^* \in Cfp_nF(R^{op})$ because $M^* = \mathrm{Hom}_S(C, X)^* \cong C \otimes_{S^{op}} X^*$ by [17, Lemma 3.55 and Proposition 2.56]. ($\Leftarrow$) Now, assume that $M^* \in Cfp_nF(R^{op})$. Then, from Corollary 3.10(ii), $M^* \in \mathcal{B}_C(R^{op})$ and $\mathrm{Hom}_{R^{op}}(C, M^*) \in fp_nF(S^{op})$. Also, by [17, Proposition 2.56 and Theorem 2.76], $(C \otimes_R M)^* \cong \mathrm{Hom}_{R^{op}}(C, M^*)$ and so $C \otimes_R M \in fp_nI(S)$ from [18, Proposition 2.4(1)]. Since $M^* \in \mathcal{B}_C(R^{op})$, $M^* \cong C \otimes_{S^{op}} \mathrm{Hom}_{R^{op}}(C, M^*) \cong C \otimes_{S^{op}} (C \otimes_R M)^* \cong \mathrm{Hom}_S(C, C \otimes_R M)^*$ from [17, Proposition 2.56, Theorem 2.76, and Lemma 3.55]. Hence $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$ by [17, Lemma 3.53]. Thus $M \in Cfp_nI(R)$.

Suppose that $k > 0$ and that $k-1$ is settled. Assume that $M \in Cfp_nI(R)_{\leq k}$. Then we have the exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow L \longrightarrow 0,$$

where $I \in Cfp_nI(R)$ and $L \in Cfp_nI(R)_{\leq k-1}$. Since $I^* \in Cfp_nF(R^{op})$, and by [17, Lemma 3.53],

$$0 \longrightarrow L^* \longrightarrow I^* \longrightarrow M^* \longrightarrow 0$$

is an exact sequence in Mod $R^{op}$, we deduce that $M \in Cfp_nI(R)_{\leq k}$ if and only if $L \in Cfp_nI(R)_{\leq k-1}$ if and only if $L^* \in Cfp_nF(R^{op})_{\leq k-1}$ if and only if $M^* \in Cfp_nF(R^{op})_{\leq k}$.

(ii). The proof is similar to the first part. □

**Corollary 3.13.** *Suppose that* $M$ *is Mod* $R$ *and* $N$ *is Mod* $S$. *Then the following assertions are true.*

*(i)* $M \in Cfp_nI(R)_{\leq k}$ *if and only if* $M^{**} \in Cfp_nI(R)_{\leq k}$*;*
*(ii)* $N \in Cfp_nF(S)_{\leq k}$ *if and only if* $N^{**} \in Cfp_nF(S)_{\leq k}$.

*Proof.* This is followed by Proposition 3.12. □

**Proposition 3.14.** *The following assertions are true.*

(i) *The class* $Cfp_nI(R)_{\leq k}$ *is closed under direct summands, direct products, and direct sums;*
(ii) *The class* $Cfp_nF(S)_{\leq k}$ *is closed under direct summands, direct products, and direct sums.*

*Proof.* (i). Assume that $M \in Cfp_nI(R)_{\leq k}$ and $M'$ is a direct summand of $M$. Then, by Corollary 3.10(i), $M \in \mathcal{A}_C(R)$ and $C \otimes_R M \in fp_nI(S)_{\leq k}$, and also there is $M''$ in Mod $R$ such that $M \cong M' \oplus M''$. From [13, Proposition 4.2(a)], it follows that $M' \in \mathcal{A}_C(R)$. Also, by [17, Theorem 2.65], we have $C \otimes_R M \cong (C \otimes_R M') \oplus (C \otimes_R M'')$ which shows from [18, Proposition 2.3(1)] that $C \otimes_R M' \in fp_nI(S)_{\leq k}$. Thus $M' \in Cfp_nI(R)_{\leq k}$ by Corollary 3.10(i).
Now, assume that $\{M_j\}_{j\in J}$ is a family in $Cfp_nI(R)_{\leq k}$. Then, by Corollary 3.10(i), $M_j \in \mathcal{A}_C(R)$ and $C \otimes_R M_j \in fp_nI(S)_{\leq k}$ for all $j \in J$. Hence, from [13, Proposition 4.2(a)], $\prod_{j\in J} M_j \in \mathcal{A}_C(R)$ (resp. $\bigoplus_{j\in J} M_j \in \mathcal{A}_C(R)$). Also, we have

$$0 \longrightarrow C \otimes_R M_j \longrightarrow I_{0j} \longrightarrow I_{1j} \longrightarrow \cdots \longrightarrow I_{k-1j} \longrightarrow I_{kj} \longrightarrow 0,$$

where $I_{ij} \in fp_nI(S)$ for all $0 \leq i \leq k$. So

$$0 \longrightarrow \prod_{j\in J}(C \otimes_R M_j) \longrightarrow \prod_{j\in J} I_{0j} \longrightarrow \cdots \longrightarrow \prod_{j\in J} I_{k-1j} \longrightarrow \prod_{j\in J} I_{kj} \longrightarrow 0$$

is an exact sequence, where by [18, Proposition 2.3(1)], $\prod_{j\in J} I_{ij} \in fp_nI(S)$ for all $0 \leq i \leq k$, and so $\prod_{j\in J}(C \otimes_R M_j) \in fp_nI(S)_{\leq k}$. Similarly, $\bigoplus_{j\in J}(C \otimes_R M_j) \in fp_nI(S)_{\leq k}$. Since $C \in \mathcal{FP}_1(R)$, from [4, Lemma 2.10(2)] we have $C \otimes_R (\prod_{j\in J} M_j) \cong \prod_{j\in J}(C \otimes_R M_j)$, and then $C \otimes_R (\prod_{j\in J} M_j) \in fp_nI(S)_{\leq k}$. Also, $C \otimes_R (\bigoplus_{j\in J} M_j) \in fp_nI(S)$ by [17, Theorem 2.65]. Thus $\prod_{j\in J} M_j \in Cfp_nI(R)_{\leq k}$ (resp. $\bigoplus_{j\in J} M_j \in Cfp_nI(R)_{\leq k}$) by Corollary 3.10(i).

(ii). By using [17, Theorem 2.30 and Corollary 2.32] and [4, Lemma 2.9], the proof is similar to the first part. □

Suppose that $\mathcal{F}$ is a class of $R$-modules and that $M$ is in Mod $R$. A morphism $f : F \longrightarrow M$ (resp. $f : M \longrightarrow F$) with $F \in \mathcal{F}$ is said to be an $\mathcal{F}$*-precover* (resp. $\mathcal{F}$*-preenvelope*) of $M$ when $\mathrm{Hom}_R(F', F) \longrightarrow \mathrm{Hom}_R(F', M) \longrightarrow 0$ (resp. $\mathrm{Hom}_R(F, F') \longrightarrow \mathrm{Hom}_R(M, F') \longrightarrow 0$) is exact for all $F' \in \mathcal{F}$. Assume that $f : F \longrightarrow M$ (resp. $f : M \longrightarrow F$) is an $\mathcal{F}$-precover (resp. $\mathcal{F}$-preenvelope) of $M$. Then $f$ is said to be an $\mathcal{F}$*-cover* (resp. $\mathcal{F}$*-envelope*) of $M$ whenever every morphism $g : F \longrightarrow F$ such that $fg = f$ (resp. $gf = f$) is an isomorphism. The class $\mathcal{F}$ is said to be (*pre*)*covering* (resp. (*pre*)*enveloping*) if each $R$-module has an $\mathcal{F}$-(pre)cover (resp. $\mathcal{F}$-(pre)envelope) (see [6, Definitions 5.1.1 and 6.1.1]).

A *duality pair* over $R$ is a pair $(\mathcal{M}, \mathcal{N})$, where $\mathcal{M}$ is a class of Mod $R$ and $\mathcal{N}$ is a class of Mod $R^{op}$, subject to the following conditions:

(i) For $M$ in Mod $R$, one has $M \in \mathcal{M}$ if and only if $M^* \in \mathcal{N}$;
(ii) $\mathcal{N}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{M}, \mathcal{N})$ is said to be *(co)product-closed* when $\mathcal{M}$ is closed under (co)products in the category of all $R$-modules (see [12, Definition 2.1]).

**Corollary 3.15.** *$(Cfp_nI(R)_{\leq k}, Cfp_nF(R^{op})_{\leq k})$ and $(Cfp_nF(S)_{\leq k}, Cfp_nI(S^{op})_{\leq k})$ are duality pairs.*

*Proof.* From Proposition 3.12, $M \in Cfp_nI(R)_{\leq k}$ (resp. $Cfp_nF(S)_{\leq k}$) if and only if $M^* \in Cfp_nF(R^{op})_{\leq k}$ (resp. $Cfp_nI(S^{op})_{\leq k}$). Also, by Proposition 3.14, $Cfp_nF(R^{op})_{\leq k}$ (resp. $Cfp_nI(S^{op})_{\leq k}$) is closed under direct summands and direct sums. Thus the assertions follow. □

We say that $Y \leq X$ is a *pure $R$-submodule* of $X$, $X/Y$ is a *pure quotient* of $X$, and $X$ is a *pure extension* of $Y$ and $X/Y$ if

$$0 \longrightarrow A \otimes_R Y \longrightarrow A \otimes_R X \longrightarrow A \otimes_R X/Y \longrightarrow 0$$

is an exact sequence for every $A$ in Mod $R^{op}$, equivalently, if

$$0 \longrightarrow \mathrm{Hom}_R(B, Y) \longrightarrow \mathrm{Hom}_R(B, X) \longrightarrow \mathrm{Hom}_R(B, X/Y) \longrightarrow 0$$

is an exact sequence for every $B \in \mathcal{FP}_1(R)$ [6, Definition 5.3.6].

The next corollary shows that $Cfp_nI(R)$ and $Cfp_nF(S)$ are also closed under pure submodules, pure quotients, and pure extensions.

**Corollary 3.16.** *Suppose that $Y \leq X$ is a pure $R$-submodule (resp. $S$-submodule) of $X$. Then the following assertions are true.*

*(i) $X \in Cfp_nI(R)$ if and only if $Y \in Cfp_nI(R)$ and $X/Y \in Cfp_nI(R)$;*
*(ii) $X \in Cfp_nF(S)$ if and only if $Y \in Cfp_nF(S)$ and $X/Y \in Cfp_nF(S)$.*

*Proof.* The assertion is followed by Corollary 3.15 and [12, Theorem 3.1]. □

In the second main result of this section, by the use of duality pairs, we show that $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ are preenveloping and covering.

**Theorem 3.17.** *The classes $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ are preenveloping and covering.*

*Proof.* By Corollary 3.15, it is clear that $(Cfp_nI(R)_{\leq k}, Cfp_nF(R^{op})_{\leq k})$ and $(Cfp_nF(S)_{\leq k}, Cfp_nI(S^{op})_{\leq k})$ are duality pairs. Also, from Proposition 3.14, the classes $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ are closed under direct products and direct sums. Therefore, from [12, Theorem 3.1], the classes $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$ are preenveloping and covering. □

## 4. $C$-$fp_n$-injective and $C$-$fp_n$-flat dimension of modules with respect to change of rings

In this section, we consider, $R = S$ and we assume that $S' \geq R$ is a unitary ring extension. The ring $S'$ is said to be right $R$-projective, [21, 23] in case, for any $M_{S'}$ in Mod $S'$ with an $N_{S'}$ in Mod $S'$, $N_R \mid M_R$ implies $N_{S'} \mid M_{S'}$, where $N \mid M$ means $N$ is a direct summand of $M$. $S'$ is said to be a finite normalizing extension of $R$ when there are elements $a_1, \cdots, a_n \in S'$ such that

$a_1 = 1$, $S^{'} = Ra_1 + \cdots + Ra_n$. A finite normalizing extension $S^{'} \geq R$ is said to be an almost excellent extension in case ${}_RS^{'}$ is flat, $S^{'}_R$ is projective, and the ring $S^{'}$ is right $R$-projective. An almost excellent extension $S^{'} \geq R$ is an excellent extension in case both ${}_RS^{'}$ and $S^{'}_R$ are free modules with a common basis $\{a_1, \cdots, a_n\}$.

In this section, we investigate modules of $Cfp_nI(R)_{\leq k}$ and also, modules of $Cfp_nF(R)_{\leq k}$ under an almost excellent extension of rings, where $C$ is a faithfully semidualizing $R$-module. Throughout this section, $S^{'} \geq R$ is an almost excellent extension.

**Lemma 4.1.** *The following assertions are true.*

*(i) If* $X \in fp_nI(R)_{\leq k}$, *then* $\mathrm{Hom}_R(S^{'}, X) \in fp_nI(S^{'})_{\leq k}$;

*(ii) If* $X \in fp_nF(R)_{\leq k}$, *then* $(S^{'} \otimes_R X) \in fp_nF(S^{'})_{\leq k}$.

*Proof.* (i). Consider, the exact sequence $0 \longrightarrow K_1 \longrightarrow K_2$, where $K_1, K_2 \in \mathcal{FP}_n(S^{'})$. By [20, Theorem 5], $K_1, K_2 \in \mathcal{FP}_n(R)$. If $k = 0$, then $X \in fp_nI(R)$. We prove that $\mathrm{Hom}_R(S^{'}, X) \in fp_nI(S^{'})$. We have the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{S'}(K_2, \mathrm{Hom}_R(S^{'}, X)) & \longrightarrow & \mathrm{Hom}_{S'}(K_1, \mathrm{Hom}_R(S^{'}, X)) & & \\
\downarrow \cong & & \downarrow \cong & & \\
\mathrm{Hom}_R(K_2, X) & \longrightarrow & \mathrm{Hom}_R(K_1, X) & \longrightarrow & 0,
\end{array}$$

and so, the sequence

$$\mathrm{Hom}_{S'}(K_2, \mathrm{Hom}_R(S^{'}, X)) \longrightarrow \mathrm{Hom}_{S'}(K_1, \mathrm{Hom}_R(S^{'}, X)) \longrightarrow 0$$

is exact and hence $\mathrm{Hom}_R(S^{'}, X) \in fp_nI(S^{'})$.

Now, let $X \in fp_nI(R)_{\leq k}$. Then we have

$$0 \longrightarrow X \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_k \longrightarrow 0,$$

where $X_i \in fp_nI(R)$ for all $0 \leq i \leq k$. Since $S^{'}_R$ is projective, there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(S^{'}, X) \longrightarrow \mathrm{Hom}_R(S^{'}, X_0) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(S^{'}, X_k) \longrightarrow 0$$

in Mod $S^{'}$, where $\mathrm{Hom}_R(S^{'}, X_i) \in fp_nI(S^{'})$ for all $0 \leq i \leq k$. So, it follows that $\mathrm{Hom}_R(S^{'}, X) \in fp_nI(S^{'})_{\leq k}$.

(ii). By Definition 2.8 and [18, Proposition 2.4(1)], it follows that for $Y$ in Mod $R$, $Y \in fp_nI(R)_{\leq k}$ if and only if $Y^* \in fp_nF(R^{op})_{\leq k}$ and $Y \in fp_nF(R)_{\leq k}$ if and only if $Y^* \in fp_nI(R^{op})_{\leq k}$. Thus if $X \in fp_nF(R)_{\leq k}$, then $X^* \in fp_nI(R^{op})_{\leq k}$. Hence by (i) and [17, Proposition 2.56 and Theorem 2.76], $(S^{'} \otimes_R X)^* \cong \mathrm{Hom}_R(S^{'}, X^*) \in fp_nI(S^{'op})_{\leq k}$, and therefore $(S^{'} \otimes_R X) \in fp_nF(S^{'})_{\leq k}$. □

**Lemma 4.2.** *Suppose that* $C$ *is a (faithfully) semidualizing in Mod* $R$. *Then* $C \otimes_R S^{'}$ *is a faithfully semidualizing* $S$*-module.*

*Proof.* By [5, Lemma 3.4], $C \otimes_R S^{'}$ is a semidualizing in Mod $S^{'}$. Assume that $\mathrm{Hom}_{S'}(C \otimes_R S^{'}, N) = 0$ for a $N$ in Mod $S^{'}$. Then $0 = \mathrm{Hom}_{S'}(C \otimes_R S^{'}, N) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_{S'}(S^{'}, N)) \cong \mathrm{Hom}_R(C, N)$, and so $N = 0$. □

**Proposition 4.3.** *The following assertions are true.*

*(i) If $M \in Cfp_nI(R)_{\leq k}$, then $\mathrm{Hom}_R(S^{'}, M) \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$;*
*(ii) If $M \in Cfp_nF(R)_{\leq k}$, then $(S^{'} \otimes_R M) \in (C \otimes_R S^{'})fp_nF(S^{'})_{\leq k}$.*

*Proof.* (i). Assume that $M \in Cfp_nI(R)_{\leq k}$. If $k = 0$, then $M = \mathrm{Hom}_R(C, X)$ for some $X \in fp_nI(R)$. We have

$$\begin{aligned}\mathrm{Hom}_R(S^{'}, M) &\cong \mathrm{Hom}_R(S^{'}, \mathrm{Hom}_R(C, X))\\ &\cong \mathrm{Hom}_R(C \otimes_R S^{'}, X)\\ &\cong \mathrm{Hom}_R(C \otimes_R S^{'} \otimes_{S'} S^{'}, X)\\ &\cong \mathrm{Hom}_{S'}(C \otimes_R S^{'}, \mathrm{Hom}(S^{'}, X)).\end{aligned}$$

Since by Lemma 4.1, $\mathrm{Hom}_R(S^{'}, X) \in fp_nI(S^{'})$ and by Lemma 4.2, $C \otimes_R S^{'}$ is semidualizing $S^{'}$-module, we deduce that $\mathrm{Hom}_{S'}(C \otimes_R S^{'}, \mathrm{Hom}(S^{'}, X)) \in (C \otimes_R S^{'})fp_nI(S^{'})$. Therefore $\mathrm{Hom}_R(S^{'}, M) \in (C \otimes_R S^{'})fp_nI(S^{'})$. Also, if $M \in Cfp_nI(R)_{\leq k}$, it simply follows that $\mathrm{Hom}_R(S^{'}, M) \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$.
(ii). The proof is similar to the first part. □

In the following, we give equivalent conditions for modules of $Cfp_nI(R)_{\leq k}$ and also, modules of $Cfp_nF(R)_{\leq k}$ under almost excellent extension of rings.

**Proposition 4.4.** *Suppose that $M$ is in Mod $S^{'}$. Then the following assertions are equivalent:*

*(i) $M \in Cfp_nI(R)_{\leq k}$;*
*(ii) $\mathrm{Hom}_R(S^{'}, M) \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$;*
*(iii) $M \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$.*

*Proof.* (i)⇒(ii). This follows from Proposition 4.3(i).
(ii)⇒(iii). By [21, Lemma 1.1], ${}_{S'}M$ is isomorphic to a direct summand of $S^{'}$-module $\mathrm{Hom}_R(S^{'}, M)$. Then by (2) and Proposition 3.14(i), $M \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$.
(iii)⇒ (i). Assume that $k = 0$. Then $M \in (C \otimes_R S^{'})fp_nI(S^{'})$, and so $M = \mathrm{Hom}_{S'}(C \otimes_R S^{'}, X)$ for some $X \in fp_nI(S^{'})$. We have $M = \mathrm{Hom}_{S'}(C \otimes_R S^{'}, X) \cong \mathrm{Hom}_R(C, \mathrm{Hom}_{S'}(S^{'}, X)) \cong \mathrm{Hom}_R(C, X)$. We show that $X \in fp_nI(R)$. Assume that $0 \longrightarrow K_1 \longrightarrow K_2$ is an exact sequence in Mod $R$, where $K_1, K_2 \in \mathcal{FP}_n(R)$. Since $S^{'}$ is a flat $R$-module, we have that $0 \longrightarrow K_1 \otimes_R S^{'} \longrightarrow K_2 \otimes_R S^{'}$ is an exact sequence in Mod $S^{'}$, where $K_1 \otimes_R S^{'}, K_1 \otimes_R S^{'} \in \mathcal{FP}_n(S^{'})$ by [20,

Lemma 4]. We have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{S'}(K_2 \otimes_R S', X) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{S'}(K_1 \otimes_R S', X) \arrow[r] \arrow[d, "\cong"] & 0 \\
\mathrm{Hom}_R(K_2, X) \arrow[r] & \mathrm{Hom}_R(K_1, X).
\end{tikzcd}$$

So, the sequence $\mathrm{Hom}_R(K_2, X) \longrightarrow \mathrm{Hom}_R(K_1, X) \longrightarrow 0$ is exact, and then $X \in fp_nI(R)$. Therefore, we get $M \in Cfp_nI(R)$. Also, if $M \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$, it simply follows that $M \in Cfp_nI(R)_{\leq k}$. □

**Proposition 4.5.** *Suppose that $M$ is Mod $S^{'}$. Then the following assertions are equivalent:*

*(i)* $M \in Cfp_nF(R)_{\leq k}$;
*(ii)* $(S^{'} \otimes_R M) \in (C \otimes_R S^{'})fp_nF(S^{'})_{\leq k}$;
*(iii)* $M \in (C \otimes_R S^{'})fp_nF(S^{'})_{\leq k}$.

*Proof.* By Propositions 4.4 and 3.12 and [17, Proposition 2.56 and Theorem 2.76], $M \in Cfp_nF(R)_{\leq k}$ if and only if $M^* \in Cfp_nI(R^{op})_{\leq k}$ if and only if $\mathrm{Hom}_R(S^{'}, M^*) \in (C \otimes_R S^{'})fp_nI({S^{'}}^{op})_{\leq k}$ if and only if $(S^{'} \otimes_R M)^* \in (C \otimes_R S^{'})fp_nI({S^{'}}^{op})_{\leq k}$ if and only if $(S^{'} \otimes_R M) \in (C \otimes_R S^{'})fp_nF(S^{'})_{\leq k}$. Also, $M \in Cfp_nF(R)_{\leq k}$ if and only if $M^* \in Cfp_nI(R^{op})_{\leq k}$ if and only if $M^* \in (C \otimes_R S^{'})fp_nI({S^{'}}^{op})_{\leq k}$ if and only if $M \in (C \otimes_R S^{'})fp_nF(S^{'})_{\leq k}$. □

**Corollary 4.6.** *Suppose that $R$ is an $n$-coherent ring. Then the following assertions are true.*

*(i) The class $(C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$ is closed under extensions and cokernels of monomorphisms;*
*(ii) The class $(C \otimes_R S^{'})fp_nF(S^{'})_{\leq k}$ is closed under extensions and kernels of epimorphisms.*

*Proof.* (i). Consider, the exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in Mod $S^{'}$, where $A, C \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$. Then by Proposition 4.4, $A, C \in Cfp_nI(R)_{\leq k}$. So by Remark 3.2(ii) and [19, Theorem 4.9], $B \in Cfp_nI(R)_{\leq k}$, and then $B \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$ from Proposition 4.4. Similarly, if $B, C \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$, then $A \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$.

(ii). The proof is similar to the first part by using Proposition 4.5 and [19, Theorem 4.8]. □

**Theorem 4.7.** *The class $(C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$ is preenveloping and precovering.*

*Proof.* Assume that $M$ is in Mod $S^{'}$. We show that $M$ has a $(C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$-preenvelope. Since $M$ is Mod $R$, then by Theorem 3.17, $M$ has a $Cfp_nI(R)_{\leq k}$-preenvelope. Assume that $R$-homomorphism $\alpha : M \longrightarrow N$ is a $Cfp_nI(R)_{\leq k}$-preenvelope of $M$. So by Proposition 4.3(1), $\mathrm{Hom}_R(S^{'}, N) \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$.

We prove that $\alpha_*\lambda_M : M \longrightarrow \mathrm{Hom}_R(S', N)$ is a $(C\otimes_R S')fp_nI(S')_{\leq k}$-preenvelope of $S'$-module $M$, where $\lambda_M : M \longrightarrow \mathrm{Hom}_R(S', M)$ and $\alpha_* : \mathrm{Hom}_R(S', M) \longrightarrow \mathrm{Hom}_R(S', N)$. If $L \in (C \otimes_R S')fp_nI(S')_{\leq k}$, and $\beta : M \longrightarrow L$ is an $S'$-homomorphism, then by Proposition 4.4, $L \in Cfp_nI(R)_{\leq k}$, and so there is an $R$-homomorphism $\gamma : N \longrightarrow L$ such that $\beta = \gamma\alpha$. Thus, we have the following commutative diagram:

$$\begin{tikzcd} {}_{S'}M \arrow[r, shift left, "\lambda_M"] \arrow[d, "\beta"] & \mathrm{Hom}_R(S', M) \arrow[l, shift left, "\pi_M"] \arrow[r, "\alpha_*"] \arrow[d, "\beta_*"] & \mathrm{Hom}_R(S', N) \arrow[d, "1"] \\ {}_{S'}L \arrow[r, shift left, "\lambda_L"] & \mathrm{Hom}_R(S', L) \arrow[l, shift left, "\pi_L"] & \mathrm{Hom}_R(S', N) \arrow[l, "\gamma_*"] \end{tikzcd}.$$

So, we have $(\pi_L\gamma_*)(\alpha_*\lambda_M) = \pi_L(\gamma_*\alpha_*)\lambda_M = \pi_L(\gamma\alpha)_*\lambda_M = \pi_L(\beta)_*\lambda_M = \pi_L\lambda_L\beta = \beta$. Therefore, we get that every $S'$-module $M$ has a $(C\otimes_R S')fp_nI(S')_{\leq k}$-preenvelope. Similarly, $(C \otimes_R S')fp_nI(S')_{\leq k}$ is precovering. $\square$

**Theorem 4.8.** *The class $(C \otimes_R S')fp_nF(S')_{\leq k}$ is preenveloping and precovering.*

*Proof.* Assume that $M$ is in Mod $S'$. We prove that $M$ has a $(C\otimes_R S')fp_nF(S')_{\leq k}$-preenvelope. Since $M$ is in Mod $R$, then by Theorem 3.17, $M$ has a $Cfp_nF(R)_{\leq k}$-preenvelope. Assume that $R$-homomorphism $\alpha : M \longrightarrow N$ is a $Cfp_nF(R)_{\leq k}$-preenvelope of $M$. Then by Proposition 4.3(2), $(S'\otimes_R N) \in (C\otimes_R S')fp_nF(S')_{\leq k}$. We show that $(S' \otimes_R \alpha)l_M : M \longrightarrow S' \otimes_R N$ is a $(C \otimes_R S')fp_nF(S')_{\leq k}$-preenvelope of $S'$-module $M$, where $l_M : M \longrightarrow (S' \otimes_R M)$ and $S' \otimes_R \alpha : S' \otimes_R M \longrightarrow S' \otimes_R N$. If $L \in (C \otimes_R S')fp_nF(S')_{\leq k}$ and $\beta : M \longrightarrow L$ is an $S'$-homomorphism, then by Proposition 4.5, $L \in Cfp_nF(R)_{\leq k}$, and so there is an $R$-homomorphism $\gamma : N \longrightarrow L$ such that $\beta = \gamma\alpha$. Thus, we have the following commutative diagram:

$$\begin{tikzcd} {}_{S'}M \arrow[r, shift left, "l_M"] \arrow[d, "\beta"] & S' \otimes_R M \arrow[l, shift left, "\tau_M"] \arrow[r, "S'\otimes_R\alpha"] \arrow[d, "S'\otimes_R\beta"] & S' \otimes_R N \arrow[d, "1"] \\ {}_{S'}L \arrow[r, shift left, "l_L"] & S' \otimes_R L \arrow[l, shift left, "\tau_L"] & S' \otimes_R N \arrow[l, "S'\otimes_R\gamma"] \end{tikzcd}.$$

Thus, we have $\tau_L(S' \otimes_R \gamma)(S' \otimes_R \alpha)l_M = \tau_L(S' \otimes_R \gamma\alpha)l_M = \tau_L l_L\beta = \beta$, and so every $S'$-module $M$ has a $(C \otimes_R S')fp_nF(S')_{\leq k}$-preenvelope. Similarly, $(C \otimes_R S')fp_nF(S')_{\leq k}$ is precovering. $\square$

**Corollary 4.9.** *The following assertions are equivalent:*

(i) *Every $S'$-module has a monic $(C \otimes_R S')fp_nI(S')_{\leq k}$-cover;*
(ii) *Every $S'^{op}$-module has an epic $(C \otimes_R S')fp_nF(S'^{op})_{\leq k}$-preenvelope;*

(iii) *Every quotient in* $(C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$ *is in* $(C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$;
(iv) *Every submodule of* $(C\otimes_R S^{'})fp_nF(S^{'^{op}})_{\leq k}$ *is in* $(C\otimes_R S^{'})fp_nF(S^{'^{op}})_{\leq k}$.

*Moreover, if $R$ is an $n$-coherent ring, then the above conditions are also equivalent to:*

(v) *The kernel of any* $Cfp_nI(R)_{\leq k}$*-precover of any* $R$*-module is in* $Cfp_nI(R)_{\leq k}$;
(vi) *The cokernel of any* $Cfp_nF(R^{op})_{\leq k}$*-preenvelope of any* $R^{op}$*-module is in* $Cfp_nF(R^{op})_{\leq k}$.

*Proof.* (i)⇔(iii). We first prove that $(C\otimes_R S^{'})fp_nI(S^{'})_{\leq k}$ is closed under direct sums. Assume that $\{M_j\}_{j\in J}$ is a family in Mod $S^{'}$ such that every $M_j \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$. Then by Proposition 4.4, $M_j \in Cfp_nI(R)_{\leq k}$, and then by Proposition 3.14(i), $\bigoplus_{j\in J} M_j \in Cfp_nI(R)_{\leq k}$, and hence by Proposition 4.4, $\bigoplus_{j\in J} M_j \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$. Therefore [7, Proposition 4] shows that (i) and (iii) are equivalent.

(ii)⇔(iv). The proof is similar to that of (i)⇔(iii) by using Propositions 3.14(ii), 4.5 and [3, Theorem 2].

(iii)⇒(iv). Assume that $N \in (C\otimes_R S^{'})fp_nF(S^{'^{op}})_{\leq k}$ and $N^{'}$ is a submodule of $N$. From the short exact sequence

$$0 \longrightarrow N^{'} \longrightarrow N \longrightarrow N/N^{'} \longrightarrow 0,$$

we get the short exact sequence

$$0 \longrightarrow (N/N^{'})^* \longrightarrow N^* \longrightarrow N^{'^*} \longrightarrow 0.$$

By Propositions 4.5 and 3.12(ii), $N \in Cfp_nF(R^{op})_{\leq k}$ if and only if $N^* \in Cfp_nI(R)_{\leq k}$ if and only if $N^* \in (C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$. Then by (iii) and Proposition 4.4, $N^{'^*} \in (C\otimes_R S^{'})fp_nI(S^{'})_{\leq k}$ if and only if $N^{'^*} \in Cfp_nI(R)_{\leq k}$, and consequently by Propositions 3.12(i) and 4.5 , $N' \in Cfp_nF(R^{op})_{\leq k}$ if and only if $N^{'} \in (C \otimes_R S^{'})fp_nF(S^{'^{op}})_{\leq k}$. Similarly, (iv)⇒(iii) is also proved.

(i)⇒(v). Assume that $M$ is in Mod $S^{'}$ and that, by Theorem 4.7, $f : F \longrightarrow M$ is a $(C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$-precover of $M$. Assume also that $g : E \longrightarrow M$ is a monic $(C \otimes_R S^{'})fp_nI(S^{'})_{\leq k}$-cover of $M$. Then [6, Lemma 8.6.3] implies that $\mathrm{Ker}(f) \oplus E \cong F$. By Proposition 4.4, $F \in Cfp_nI(R)_{\leq k}$, and so by Proposition 3.14(i), $\mathrm{Ker}(f) \in Cfp_nI(R)_{\leq k}$.

(ii)⇒(vi). The proof is similar to that of (i)⇒(v) by using the dual of [6, Lemma 8.6.3].

(vi)⇒(iv). Assume that $N \in (C \otimes_R S^{'})fp_nF(S^{'^{op}})_{\leq k}$ and that $N^{'}$ is a submodule of $N$. Assume also that, by Theorem 4.8, $f : N^{'} \longrightarrow F$ is a $(C \otimes_R S^{'})fp_nF(S^{'^{op}})_{\leq k}$-preenvelope of $N^{'}$. Then we have the following commutative diagram

$$\begin{tikzcd} & N' \arrow[r, "f"] \arrow[d, equal] & F \arrow[r] \arrow[d] & \mathrm{Coker}(f) \arrow[r] & 0 \\ 0 \arrow[r] & N' \arrow[r] & N & & \end{tikzcd}$$

with exact rows. In particular, the sequence

$$0 \longrightarrow N^{'} \longrightarrow F \longrightarrow \mathrm{Coker}(f) \longrightarrow 0$$

is exact, and then by Remark 3.2(ii) and Corollary 4.6(ii), $N' \in (C \otimes_R S^{'}) fp_n F(S^{'op})_{\leq k}$.

(v)⇒(iii). The proof is similar to that of (vi)⇒(iv) by using Corollary 4.6(i). □

In the next proposition, we investigate the homological behavior of Auslander and Bass classes under almost excellent extension of rings.

**Proposition 4.10.** *The following assertions are true.*

*(i) If $A \in \mathcal{A}_C(R)$, then $(S^{'} \otimes_R A) \in \mathcal{A}_{C \otimes_R S'}(S^{'})$;*
*(ii) If $B \in \mathcal{B}_C(R)$, then $\mathrm{Hom}_R(S^{'}, B) \in \mathcal{B}_{C \otimes_R S'}(S^{'})$.*

*Proof.* (i). There exists an exact sequence in Mod $R$

$$\cdots \longrightarrow P_{j+1} \longrightarrow P_j \longrightarrow P_{j-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \longrightarrow 0,$$

where for all $j \geq 0$, $P_j \in \mathcal{FP}_0(R)$ and free . Since $A \in \mathcal{A}_C(R)$, we have the exact sequence

$$\cdots \longrightarrow P_{j+1} \otimes_R A \longrightarrow P_j \otimes_R A \longrightarrow \cdots \longrightarrow P_0 \otimes_R A \longrightarrow C \otimes_R A \longrightarrow 0,$$

in Mod $R$ and since $S^{'}$ is flat $R$-module, we have the commutative diagram

$$\begin{tikzcd} S^{'} \otimes_R (P_{j+1} \otimes_R A) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & S^{'} \otimes_R (C \otimes_R A) \arrow[r] \arrow[d, "\cong"] & 0 \\ (P_{j+1} \otimes_R S^{'}) \otimes_{S'} (S^{'} \otimes_R A) \arrow[r] & \cdots \arrow[r] & (C \otimes_R S^{'}) \otimes_{S'} (S^{'} \otimes_R A) \arrow[r] & 0, \end{tikzcd}$$

and so $\mathrm{Tor}_j^{S^{'}}(C \otimes_R S^{'}, S^{'} \otimes_R A) = 0$ for any $j \geq 0$.

On the other hand, $C \otimes_R A \in \mathcal{B}_C(R)$ by [13, Proposition 4.1]. Thus we have

$$0 \longrightarrow \mathrm{Hom}_R(C, C \otimes_R A) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(P_{j+1}, C \otimes_R A) \longrightarrow \cdots,$$

and hence by [17, Lemma 4.86], we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & S' \otimes_R \mathrm{Hom}_R(C, C\otimes_R A) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & S' \otimes_R \mathrm{Hom}_R(P_{j+1}, C\otimes_R A) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_R(C, S' \otimes_R (C\otimes_R A)) \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathrm{Hom}_R(P_{j+1}, S' \otimes_R (C\otimes_R A)) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}_{S'}(C\otimes_R S', (C\otimes_R S')\otimes_{S'} (S'\otimes_R A)) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_{S'}(P_{j+1}\otimes_R S', (C\otimes_R S')\otimes_{S'} (S'\otimes_R A)).
\end{tikzcd}$$

Hence $\mathrm{Ext}^j_{S'}(C\otimes_R S', (C\otimes_R S')\otimes_{S'} (S'\otimes_R A)) = 0$, and also

$S'\otimes_R A \cong S'\otimes_R \mathrm{Hom}_R(C, C\otimes_R A) \cong \mathrm{Hom}_{S'}(C\otimes_R S', (C\otimes_R S')\otimes_{S'} (S'\otimes_R A))$. Therefore $(S'\otimes_R A) \in \mathcal{A}_{C\otimes_R S'}(S')$.

(ii). Assume that $B \in \mathcal{B}_C(R)$. Then by Proposition 2.7(ii), $B^* \in \mathcal{A}_C(R^{op})$. So $(S'\otimes_{R^{op}} B^*) \in \mathcal{A}_{C\otimes_{R^{op}} S'}(S'^{op})$ by (i). By [20, Theorem 5], $S'$ is in $\mathcal{FP}_1(R)$, and so [17, Lemma 3.55] implies that $\mathrm{Hom}_{R^{op}}(S', B)^* \in \mathcal{A}_{C\otimes_{R^{op}} S'}(S'^{op})$. Consider, $\mathcal{Y}$-finitely presented $\mathcal{Y} = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \longrightarrow 0$ in Mod $R$. Then by Lemma 4.2, $\mathcal{Y}\otimes_R S'$ is a $\mathcal{Y}\otimes_R S'$-finitely presented, and then similar to the proof of Proposition 2.7(ii), $\mathrm{Hom}_R(S', M) \in \mathcal{B}_{C\otimes_R S'}(S')$. □

**Corollary 4.11.** *The following assertions are true.*

*(i)* $fp_nF(S')_{\leq k} \subseteq \mathcal{A}_{C\otimes_R S'}(S')$;

*(ii)* $fp_nI(S')_{\leq k} \subseteq \mathcal{B}_{C\otimes_R S'}(S')$.

*Proof.* (i). Assume that $M \in fp_nF(S')_{\leq k}$. Then we have

$$0 \longrightarrow J_k \longrightarrow J_{k-1} \longrightarrow \cdots \longrightarrow J_1 \longrightarrow J_0 \longrightarrow M \longrightarrow 0,$$

where $J_i \in fp_nF(S')$ for all $0 \leq i \leq k$. By [18, Proposition 3.2], $J_i \in fp_nF(R)$. So we obtain that $M \in fp_nF(R)_{\leq k}$. Thus by Lemma 2.9(ii), $M \in \mathcal{A}_C(R)$, and hence by Proposition 4.10(i), $(S'\otimes_R M) \in \mathcal{A}_{C\otimes_R S'}(S')$. From [21, Lemma 1.1], we see that $S'$-module $M$ is isomorphic to a direct summand of $S'\otimes_R M$. Therefore [13, Proposition 4.2] implies that $M \in \mathcal{A}_{C\otimes_R S'}(S')$.

(ii). The proof is similar to the first part. □

**Lemma 4.12.** *The following assertions are true.*

*(i)* $(C\otimes_R S')fp_nI(S')_{\leq k} \subseteq \mathcal{A}_{C\otimes_R S'}(S')$;

*(ii)* $(C\otimes_R S')fp_nF(S')_{\leq k} \subseteq \mathcal{B}_{C\otimes_R S'}(S')$.

*Proof.* (i). Assume that $M \in (C\otimes_R S')fp_nI(S')_{\leq k}$. Then from Proposition 4.4, $M \in Cfp_nI(R)_{\leq k}$, and so $M \in \mathcal{A}_C(R)$ by Lemma 3.6(i). Thus from Proposition 4.10(i), $(S'\otimes_R M) \in \mathcal{A}_{C\otimes_R S'}(S')$. By [21, Lemma 1.1], $M$ is isomorphic to a direct summand of $S'\otimes_R M$, and consequently by [13, Proposition 4.2], $M \in \mathcal{A}_{C\otimes_R S'}(S')$. Similarly, (ii) will be proved. □

In the following, we investigate Foxby equivalence relative to the class $(C\otimes_R S^{'})fp_nI(S^{'})_{\leq k}$ with the class $fp_nI(S^{'})_{\leq k}$ and the class $(C\otimes_R S^{'})fp_nF(S^{'})_{\leq k}$ with the class $fp_nF(S^{'})_{\leq k}$, where $S^{'}\geq R$ is an almost excellent extension.

**Proposition 4.13.** *We have the following equivalences:*

*(i)* $$(C\otimes_R S^{'})fp_nI(S^{'})_{\leq k} \underset{\mathrm{Hom}_{S'}(C\otimes_R S^{'},-)}{\overset{(C\otimes_R S^{'})\otimes_{S'}-}{\rightleftarrows}} fp_nI(S^{'})_{\leq k};$$

*(ii)* $$fp_nF(S^{'})_{\leq k} \underset{\mathrm{Hom}_{S'}(C\otimes_R S^{'},-)}{\overset{(C\otimes_R S^{'})\otimes_{S'}-}{\rightleftarrows}} (C\otimes_R S^{'})fp_nF(S^{'})_{\leq k}.$$

*Proof.* (i). Assume that $M\in (C\otimes_R S^{'})fp_nI(S^{'})_{\leq k}$. Then we have

$$0\longrightarrow M\longrightarrow I_0\longrightarrow I_1\longrightarrow\cdots\longrightarrow I_{k-1}\longrightarrow I_k\longrightarrow 0,$$

where $I_i\in (C\otimes_R S^{'})fp_nI(S^{'})$ for all $0\leq i\leq k$. By Proposition 4.4, each $I_i\in Cfp_nI(R)$, and so by Proposition 3.7(i) and [18, Proposition 3.2], $C\otimes_R I_i\in fp_nI(R)$ if and only if $C\otimes_R I_i\in fp_nI(S^{'})$. On the other hand, by Proposition 4.4, $M\in Cfp_nI(R)_{\leq k}$, and then by Lemma 3.6(i), $M, I_i\in \mathcal{A}_C(R)$. Thus, we have

$$0\longrightarrow C\otimes_R M\longrightarrow C\otimes_R I_0\longrightarrow C\otimes_R I_1\longrightarrow\cdots\longrightarrow C\otimes_R I_{k-1}\longrightarrow C\otimes_R I_k\longrightarrow 0,$$

where $C\otimes_R I_i\in Cfp_nI(S^{'})$ for all $0\leq i\leq k$, and hence $(C\otimes_R S^{'})\otimes_{S'} M\cong C\otimes_R M\in fp_nI(S^{'})_{\leq k}$.

Also, $M\in \mathcal{A}_{C\otimes_R S'}(S^{'})$ by Lemma 4.12(i). So we have $M\cong \mathrm{Hom}_{S'}(C\otimes_R S^{'},(C\otimes_R S^{'})\otimes_{S'} M)$.

Now, assume that $N\in fp_nI(S^{'})_{\leq k}$. Then we have

$$0\longrightarrow N\longrightarrow X_0\longrightarrow X_1\longrightarrow\cdots\longrightarrow X_{k-1}\longrightarrow X_k\longrightarrow 0,$$

where $X_i\in fp_nI(S^{'})$ for all $0\leq i\leq k$. By [18, Proposition 3.2], $X_i\in fp_nI(R)$. Therefore $N\in fp_nI(R)_{\leq k}$. Thus by Proposition 3.7(i), $\mathrm{Hom}_R(C,N)\in Cfp_nI(R)_{\leq k}$. We have $\mathrm{Hom}_{S'}(C\otimes_R S^{'},N)\cong \mathrm{Hom}_R(C,\mathrm{Hom}_{S'}(S^{'},N))\cong \mathrm{Hom}_R(C,N)$. Hence $\mathrm{Hom}_{S'}(C\otimes_R S^{'},N)\in Cfp_nI(R)_{\leq k}$, and therefore by Proposition 4.4, $\mathrm{Hom}_{S'}(C\otimes_R S^{'},N)\in (C\otimes_R S^{'})fp_nI(S^{'})_{\leq k}$.

(ii). The proof is similar to the first part. □

In the following, we give equivalent conditions with modules of the classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ under almost excellent extension of rings.

**Proposition 4.14.** *Suppose that $A$ is in Mod $S^{'}$. Then the following assertions are equivalent:*

*(i)* $A\in \mathcal{A}_C(R)$;

*(ii)* $(S^{'}\otimes_R A)\in \mathcal{A}_{C\otimes_R S'}(S^{'})$;

*(iii)* $A \in \mathcal{A}_{C\otimes_R S'}(S^{'})$.

*Proof.* (i)⇒(ii). It is clear by Proposition 4.10(i).

(ii)⇒(iii). By [21, Lemma 1.1], ${}_{S'}A$ is isomorphic to a direct summand of $S^{'}$-module $S^{'} \otimes_R A$. Thus by [13, Proposition 4.2(1)], $A \in \mathcal{A}_{C\otimes_R S'}(S^{'})$.

(iii)⇒(i). Assume that $A \in \mathcal{A}_{C\otimes_R S'}(S^{'})$. Then $\mathrm{Tor}_j^{S^{'}}(C \otimes_R S^{'}, A) = 0$ for any $j \geq 1$. Also, we have $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \longrightarrow 0$, where for all $i \geq 0$, $P_i \in \mathcal{FP}_0(R)$ and free. Hence, we have the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & (P_1 \otimes_R S^{'}) \otimes_{S'} A \arrow[r] \arrow[d, "\cong"] & (P_0 \otimes_R S^{'}) \otimes_{S'} A \arrow[r] \arrow[d, "\cong"] & (C \otimes_R S^{'}) \otimes_{S'} A \arrow[r] \arrow[d, "\cong"] & 0 \\
\cdots \arrow[r] & P_1 \otimes_R A \arrow[r] & P_0 \otimes_R A \arrow[r] & C \otimes_R A \arrow[r] & 0,
\end{tikzcd}$$

where the first line is exact by (iii), and so the second line is also exact, and hence $\mathrm{Tor}_j^R(C, A) = 0$ for any $j \geq 1$.

On the other hand, $\mathrm{Ext}^j_{S'}(C \otimes_R S^{'}, (C \otimes_R S^{'}) \otimes_{S'} A) = 0$ for any $j \geq 1$. Since $S^{'}$ is a flat $R$-module, the sequence $\cdots \longrightarrow P_1 \otimes_R S^{'} \longrightarrow P_0 \otimes_R S^{'} \longrightarrow C \otimes_R S^{'} \longrightarrow 0$ is exact. Thus, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{S'}(C \otimes_R S^{'}, (C \otimes_R S^{'}) \otimes_{S'} A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{S'}(P_0 \otimes_R S^{'}, (C \otimes_R S^{'}) \otimes_{S'} A) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_{S'}(C \otimes_R S^{'}, C \otimes_R A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{S'}(P_0 \otimes_R S^{'}, C \otimes_R A) \arrow[r] \arrow[d, "\cong"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_R(C, C \otimes_R A) \arrow[r] & \mathrm{Hom}_R(P_0, C \otimes_R A) \arrow[r] & \cdots ,
\end{tikzcd}$$

where the first and second lines are exact by (iii), and so the third one is also exact, and hence $\mathrm{Ext}^j_R(C, C \otimes_R A) = 0$ for any $j \geq 1$.

Also by (iii) and [17, Theorem 2.75], we have

$$A \cong \mathrm{Hom}_{S'}(C\otimes_R S^{'}, (C\otimes_R S^{'})\otimes_{S'} A) \cong \mathrm{Hom}_{S'}(C\otimes_R S^{'}, C\otimes_R A) \cong \mathrm{Hom}_R(C, C\otimes_R A).$$

Consequently, $A \in \mathcal{A}_C(R)$. □

**Proposition 4.15.** *Suppose that $B$ is in Mod $S^{'}$. Then the following assertions are equivalent:*

*(i)* $B \in \mathcal{B}_C(R)$;

*(ii)* $\mathrm{Hom}_R(S^{'}, B) \in \mathcal{B}_{C\otimes_R S'}(S^{'})$;

*(iii)* $B \in \mathcal{B}_{C\otimes_R S'}(S^{'})$.

*Proof.* The proof is similar to the proof of Proposition 4.14. □

Under change of rings, Auslander and Bass classes are equivalent under the pair of functors.

**Proposition 4.16.** *There are the following equivalences:*

$$\begin{tikzcd} \mathcal{A}_{C\otimes_R S'}(S^{'}) \arrow[r, shift left, "(C\otimes_R S^{'})\otimes_{S'}-"] \arrow[r, phantom, "\sim"] & \mathcal{B}_{C\otimes_R S'}(S^{'}) \arrow[l, shift left, "\mathrm{Hom}_{S'}(C\otimes_R S^{'},-)"] \end{tikzcd}.$$

*Proof.* From Proposition 4.14, $A \in \mathcal{A}_{C\otimes_R S'}(S^{'})$ if and only if $A \in \mathcal{A}_C(R)$. Thus by [13, Proposition 4.1], $(C \otimes_R A) \in \mathcal{B}_C(R)$, and so $(C \otimes_R S^{'}) \otimes_{S'} A \cong (C\otimes_R A) \in \mathcal{B}_{C\otimes_R S'}(S^{'})$ by Proposition 4.15. Also, we have $A \cong \mathrm{Hom}_R(C, C\otimes_R A) \cong \mathrm{Hom}_{S'}(C \otimes_R S^{'}, (C \otimes_R S^{'}) \otimes_{S'} A)$.

On the other hand, by Proposition 4.15, $B \in \mathcal{B}_{C\otimes_R S'}(S^{'})$ if and only if $B \in \mathcal{B}_C(R)$. Thus from [13, Proposition 4.1], $\mathrm{Hom}_R(C, B) \in \mathcal{A}_C(R)$, and so $\mathrm{Hom}_{S'}(C\otimes_R S^{'}, B) \cong \mathrm{Hom}_R(C, B) \in \mathcal{A}_{C\otimes_R S'}(S^{'})$ by Proposition 4.14 and [17, Theorem 2.75]. Also, we have

$$B \cong C \otimes_R \mathrm{Hom}_R(C, B) \cong (C \otimes_R S^{'}) \otimes_{S'} \mathrm{Hom}_{S'}(C \otimes_R S^{'}, B).$$

Therefore the assertion holds. □

By using Corollary 4.11, Lemma 4.12 and Propositions 4.10, 4.13, 4.16, we get Foxby Equivalence under an almost excellent extension:

**Theorem 4.17.** (Foxby Equivalence under almost excellent extension of rings) *We have the following equivalences:*

$$\begin{tikzcd}
fp_nF(S^{'}) \arrow[r, shift left, "(C\otimes_R S^{'})\otimes_{S'}-"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & (C \otimes_R S^{'})fp_nF(S^{'}) \arrow[l, shift left, "\mathrm{Hom}_{S'}(C\otimes_R S^{'},-)"] \arrow[d, hook] \\
fp_nF(S^{'})_{\leq k} \arrow[r, shift left, "(C\otimes_R S^{'})\otimes_{S'}-"] \arrow[r, phantom, "\sim"] \arrow[d, hook] & (C \otimes_R S^{'})fp_nF(S^{'})_{\leq k} \arrow[l, shift left, "\mathrm{Hom}_{S'}(C\otimes_R S^{'},-)"] \arrow[d, hook] \\
\mathcal{A}_{C\otimes_R S'}(S^{'}) \arrow[r, shift left, "(C\otimes_R S^{'})\otimes_{S'}-"] \arrow[r, phantom, "\sim"] & \mathcal{B}_{C\otimes_R S'}(S^{'}) \arrow[l, shift left, "\mathrm{Hom}_{S'}(C\otimes_R S^{'},-)"] \\
(C \otimes_R S^{'})fp_nI(S^{'})_{\leq k} \arrow[r, shift left, "(C\otimes_R S^{'})\otimes_{S'}-"] \arrow[r, phantom, "\sim"] \arrow[u, hook] & fp_nI(S^{'})_{\leq k} \arrow[l, shift left, "\mathrm{Hom}_{S'}(C\otimes_R S^{'},-)"] \arrow[u, hook] \\
(C \otimes_R S^{'})fp_nI(S^{'}) \arrow[r, shift left, "(C\otimes_R S^{'})\otimes_{S'}-"] \arrow[r, phantom, "\sim"] \arrow[u, hook] & fp_nI(S^{'}) \arrow[l, shift left, "\mathrm{Hom}_{S'}(C\otimes_R S^{'},-)"] \arrow[u, hook]
\end{tikzcd}.$$

**Example 4.18.** Let $R$ be a $(1, 2, 3)$-ring, (for example, $R = k[[x_1]] \oplus R^{'}$, where $R^{'}$ is a valuation ring with global dimension 2), see [16, Proposition 3.7]. So, $\mathrm{w.gl.dim}(R) = 1$ and $\mathrm{gl.dim}(R) = 2$. If $S^{'}$ is the ring $M_n(R)$ of

$n$ by $n$ matrices over a ring $R$, then by [14, Example 1], $S^{'}$ is almost excellent extensions of $R$. In case $R = C$, for every $M$ in Mod $R$, one easily gets that $Cfp_n.\mathrm{id}_R(M) \leq 2$ and $Cfp_n.\mathrm{fd}_R(M) \leq 1$. So by Proposition 4.3, $(C \otimes_R S^{'})fp_n.\mathrm{id}_{S'}(\mathrm{Hom}_R(S^{'}, M)) \leq 2$ and $(C \otimes_R S^{'})fp_n.\mathrm{fd}_{S'}(S^{'} \otimes_R M) \leq 1$. Also, for every $M$ in Mod $S^{'}$, it follows that $(C \otimes_R S^{'})fp_n.\mathrm{id}_{S'}(M) \leq 2$ and $(C \otimes_R S^{'})fp_n.\mathrm{fd}_{S'}(M) \leq 1$. Hence by Theorem 4.11, $fp_n.\mathrm{id}_{S'}((C \otimes_R S^{'}) \otimes_{S'} M) \leq 2$ and $fp_n.\mathrm{fd}_{S'}(\mathrm{Hom}_{S'}(C \otimes_R S^{'}, M)) \leq 1$.

## 5. Conclusion

Employing relative homological techniques, we have expanded several fundamental homological concepts to incorporate findings related to semidualizing modules. This paper focuses on the over rings $R$ and $S$, and examines the $fp_n$-injective $R$-modules and $fp_n$-flat $S$-modules under a (faithfully) semidualizing bimodule $C = {}_SC_R$. So, we introduce and analyze the $C$-$fp_n$-injective $R$-modules and $C$-$fp_n$-flat $S$-modules. Our investigation includes a study of Foxby equivalence, duality pairs, preenvelopes and precovers concerning a pairs of the classes $Cfp_nI(R)_{\leq k}$ and $Cfp_nF(S)_{\leq k}$, and also the study of the same features under the change of rings. Given the importance of these topics in the field of homological algebra, in the continuation of the future research work, we suggest further investigation into the Gorenstein properties of these modules and the relative cotorsion modules.

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