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GENERALIZED JORDAN TRIPLE (ζ,ξ)-DERIVATIONS ON SEMIPRIME RINGS

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Abstract. The purpose of this research is to demonstrate the following assertions: an additive mapping $\mathcal{H}$ is a generalized (ζ,ξ)-derivation associated with a (ζ,ξ)-derivation $\mathbf{h}$, where ζ,ξ are endomorphisms on a $(m+n+p-1)!$-torsion free semiprime ring $\mathcal{A}$. Here we prove another result in the setting of the generalized left (ζ,ξ)-derivation on $\mathcal{A}$.

Keywords: Semiprime ring, generalized (ζ,ξ)-derivation, (ζ,ξ)-derivations.
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1. Introduction and preliminaries

Throughout this research paper, $\mathcal{A}$ is used to represent an associative ring with unity e. A ring $\mathcal{A}$ is t-torsion free, where $t>1$ is an integer; in the case $tb=0$, it implies $b=0$ for all $b\in\mathcal{A}$. Notice that a ring $\mathcal{A}$ is termed prime if $a\mathcal{A}b=\{0\}$ implies either $a=0$ or $b=0$, and is recognized as semiprime if $b\mathcal{A}b=\{0\}$ implies $b=0$. Usually, the commutator $ab-ba$ will be denoted by $[a,b]$. An additive mapping $\mathbf{h}:\mathcal{A}\to\mathcal{A}$ is known as a derivation if $\mathbf{h}(ab)=\mathbf{h}(a)b+a\mathbf{h}(b)$ for all $a,b\in\mathcal{A}$ and is termed as a Jordan derivation if $\mathbf{h}(b^2)=\mathbf{h}(b)b+b\mathbf{h}(b)$ is true for all $b\in\mathcal{A}$.

Every derivation is a Jordan derivation, as is evident from the definition, but the converse is often not legitimate. An essential outcome due to [6], affirms that a Jordan derivation is a derivation on a prime ring with characteristic instead of two. The extension of this result is presented for a 2-torsion-free semiprime ring by Cusack [5]. A mapping $\mathcal{H}:\mathcal{A}\to\mathcal{A}$ is termed a generalized derivation if it is additive and there exists an associated derivation $\mathbf{h}$ on $\mathcal{A}$ such that $\mathcal{H}(ab)=\mathcal{H}(a)b+a\mathbf{h}(b)$ for all $a,b\in\mathcal{A}$ and particularly if $a=b$, then $\mathcal{H}$ is a generalized Jordan derivation associated with a Jordan derivation $\mathbf{h}$ on $\mathcal{A}$.

It is straightforward to authenticate that every generalized derivation is a generalized Jordan derivation, but generally, the converse is not true. Bresar [4]

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has begun studying Jordan triple derivations on rings, motivated by the Jordan derivation. A Jordan triple derivation is defined as an additive mapping $\mathbf{h} : \mathcal{A} \to \mathcal{A}$ satisfying $\mathbf{h}(aba) = \mathbf{h}(a)ba + a\mathbf{h}(b)a + ab\mathbf{h}(a)$ for all $a, b \in \mathcal{A}$. It is obvious that a Jordan triple derivation is a Jordan derivation. Inspired by Jordan triple derivation and the idea of generalized derivation, Jing and Lu [7] have presented a notion on a ring $\mathcal{A}$, which is known as a generalized Jordan triple derivation on rings as follows: an additive mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ is called a generalized Jordan triple derivation if there exists a Jordan triple derivation $\mathbf{h}$ on $\mathcal{A}$ such that $\mathcal{H}(aba) = \mathcal{H}(a)ba + a\mathbf{h}(b)a + ab\mathbf{h}(a)$ for all $a, b \in \mathcal{A}$. In [7], authors proved that every generalized Jordan triple derivation on a 2-torsion free prime ring $\mathcal{A}$ is a generalized derivation.

Example 1.1. *A mapping $d : R \to R$ is called a Generalized Jordan Triple (ζ, ξ)-Derivation if there exists another mapping $g : R \to R$ defined in a way that satisfies the following identity*

$$d(x^2y + yx^2) = \zeta\, d(x)d(y) + \xi\, g(x^2)y + \xi\, yg(x^2),$$

for all $x, y \in R$

2. On generalized (ζ, ξ)-derivation

If ζ and ξ are two endomorphisms on $\mathcal{A}$, then an additive mapping $\mathbf{h} : \mathcal{A} \to \mathcal{A}$ is said to be a (ζ, ξ)-derivation if $\mathbf{h}(ab) = \mathbf{h}(a)\zeta(b) + \xi(a)\mathbf{h}(b)$ for all $a, b \in \mathcal{A}$. An additive mapping $\mathbf{h} : \mathcal{A} \to \mathcal{A}$ is said to be a Jordan (ζ, ξ)-derivation if $\mathbf{h}(a^2) = \mathbf{h}(a)\zeta(a) + \xi(a)\mathbf{h}(a)$ for all $a \in \mathcal{A}$. An additive mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ is said to be a generalized (ζ, ξ)-derivation if there exists a (ζ, ξ)-derivation $\mathbf{h}$ on $\mathcal{A}$ such that $\mathcal{H}(ab) = \mathcal{H}(a)\zeta(b) + \xi(a)\mathbf{h}(b)$ for all $a, b \in \mathcal{A}$. An additive mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ is said to be a generalized Jordan (ζ, ξ)-derivation if there exists a Jordan (ζ, ξ)-derivation $\mathbf{h}$ on $\mathcal{A}$ such that $\mathcal{H}(a^2) = \mathcal{H}(a)\zeta(a) + \xi(a)\mathbf{h}(a)$ for all $a \in \mathcal{A}$. An additive mapping $\mathcal{H} : \mathcal{A} \to \mathcal{A}$ is known as a generalized Jordan triple (ζ, ξ)-derivation if there exists a Jordan triple (ζ, ξ)-derivation $\mathbf{h}$ on $\mathcal{A}$ such that $\mathcal{H}(aba) = \mathcal{H}(a)\zeta(ba) + \xi(a)\mathbf{h}(b)\zeta(a) + \xi(ab)\mathbf{h}(a)$ for all $a, b \in \mathcal{A}$, where ζ and ξ are two endomorphisms on $\mathcal{A}$.

Every Jordan triple (ζ, ξ)-derivations on a 2-torsion free semiprime ring is a (ζ, ξ)-derivations, and generalized Jordan triple (ζ, ξ)-derivation is a generalized (ζ, ξ)-derivation, as demonstrated by Liu and Shiue [9] in 2007. Then $\mathcal{H}$ is a generalized (ζ, ξ)-derivation associated with (ζ, ξ)-derivation $\mathbf{h}$. Inspired by the above result, if $\mathcal{H}$ is a generalized (ζ, ξ)-derivation (respectively generalized Jordan (ζ, ξ)-derivation) associated with a (ζ, ξ)-derivation (respectively Jordan (ζ, ξ)-derivation) $\mathbf{h}$ on $\mathcal{A}$, then the algebraic identity $\mathcal{H}(a^{m+n+p}) = \mathcal{H}(a^m)\zeta(a^{n+p}) + \xi(a^m)\mathbf{h}(a^n)\zeta(a^p) + \xi(a^{m+n})\mathbf{h}(a^p)$ holds for all $a \in \mathcal{A}$. However, this is not always true." Naturally, one could wonder, if the mappings $\mathcal{H}$

and $\mathbf{h}$ from a ring $\mathcal{A}$ to itself that meet the aforementioned generalized identity will lead to the same result as that: if m, n are positive fixed integers and $\mathcal{A}$ is a $(m+n-1)!$-torsion free semiprime ring. Let $\mathcal{H}, \mathbf{h} : \mathcal{A} \to \mathcal{A}$ be two additive mappings that satisfy $\mathcal{H}(a^{m+n}) = \mathcal{H}(a^m)\zeta(a^n) + \xi(a^m)\mathbf{h}(a^n)$ for all $a \in \mathcal{A}$, where ζ, ξ are automorphisms on $\mathcal{A}$. The response is positive. We have investigated under what circumstances this mathematical assertion is valid in this research study. More precisely, we established that if the additive mapping $\mathcal{H}$ fulfills $\mathcal{H}(a^{m+n+p}) = \mathcal{H}(a^m)\zeta(a^{n+p}) + \xi(a^m)\mathbf{h}(a^n)\zeta(a^p) + \xi(a^{m+n})\mathbf{h}(a^p)$ for all $a \in \mathcal{A}$, then it is a generalized derivation associated with a derivation $\mathbf{h}$ on $\mathcal{H}$, where $\mathcal{A}$ is a $(m+n+p-1)!$-torsion free semiprime ring and $p \geq 0$ and $m, n \geq 1$ are fixed integers.

Example 2.1. *Consider the ring $R = M_2(\mathbb{Z})$, the set of 2×2 matrices over integers. Let $\zeta = 1$, $\xi = 0$, and define $d : R \to R$ by*

$$d(A) = A + A^T,$$

where A^T is the transpose of A. Define $g : R \to R$ as the zero map (i.e., $g(A) = 0$ for all $A \in R$). In that case, d is a generalized (ζ, ξ)-derivation with $\zeta = 1$ and $\xi = 0$.

We start our research by considering the following:

Theorem 2.2. *Suppose that $p \geq 0$ and $m, n \geq 1$ are fixed integers and $\mathcal{A}$ is a $(m+n+p-1)!$-torsion free semiprime ring. If $\mathcal{H}, h : \mathcal{A} \to \mathcal{A}$ are additive mappings that satisfy $\mathcal{H}(a^{m+n+p}) = \mathcal{H}(a^m)\zeta(a^{n+p}) + \xi(a^m)\mathbf{h}(a^n)\zeta(a^p) + \xi(a^{m+n})\mathbf{h}(a^p)$ for all $a \in \mathcal{A}$, then $\mathcal{H}$ is generalized (ζ, ξ)-derivation associated with (ζ, ξ)-derivation $\mathbf{h}$ on $\mathcal{A}$, where ζ, ξ are endomorphisms on $\mathcal{A}$.*

Proof. Given that

(1)
$$\mathcal{H}(a^{m+n+p}) = \mathcal{H}(a^m)\zeta(a^{n+p}) + \xi(a^m)\mathbf{h}(a^n)\zeta(a^p) + \xi(a^{m+n})\mathbf{h}(a^p) \quad \text{for all } a \in \mathcal{A}.$$

Replacing a by e, we get $\mathbf{h}(e) = 0$. Replace a by $a + qe$ in (1) to find

$$\mathcal{H}(a^{m+n+p} + \tbinom{m+n+p}{1}(a^{m+n+p-1})qe + \cdots + \tbinom{m+n+p}{m+n+p-2}a^2q^{m+n+p-2}e + \tbinom{m+n+p}{m+n+p-1}aq^{m+n+p-1}e + q^{m+n+p}e) = \mathcal{H}(a^m + \tbinom{m}{1}a^{m-1}qe + \cdots + \tbinom{m}{m-2}a^2q^{m-2}e + \tbinom{m}{m-1}aq^{m-1}e + q^m e)\zeta(a^{n+p} + \tbinom{n+p}{1}a^{n+p-1}qe + \cdots + \tbinom{n+p}{n+p-2}a^2q^{n+p-2}e + \tbinom{n+p}{n+p-1}aq^{n+p-1}e + q^{n+p}e) + \xi(a^m + \tbinom{m}{1}a^{m-1}qe + \cdots + \tbinom{m}{m-2}a^2q^{m-2}e + \tbinom{m}{m-1}aq^{m-1}e + q^m e)\mathbf{h}(a^n + \tbinom{n}{1}a^{n-1}qe + \cdots + \tbinom{n}{n-2}a^2q^{n-2}e + \tbinom{n}{n-1}aq^{n-1}e + q^n e)\zeta(a^p + \tbinom{p}{1}a^{p-1}qe + \cdots + \tbinom{p}{p-2}a^2q^{p-2}e + \tbinom{p}{p-1}aq^{p-1}e + q^p e) + \xi(a^{m+n} + \tbinom{m+n}{1}(a^{m+n-1})qe + \cdots + \tbinom{m+n}{m+n-2}a^2q^{m+n-2}e + \tbinom{m+n}{m+n-1}aq^{m+n-1}e + q^{m+n}e)\mathbf{h}(a^p + \tbinom{p}{1}a^{p-1}qe + \cdots + \tbinom{p}{p-2}a^2q^{p-2}e + \tbinom{p}{p-1}aq^{p-1}e + q^p e)$$
for all $a \in \mathcal{A}$ and $q \geq 1$.

Rewrite the above expression by using (1) as

$$q\mathcal{P}_1(a,e)+q^2\mathcal{P}_2(a,e)+\cdots+q^{m+n+p-1}\mathcal{P}_{m+n+p-1}(a,e)=0,$$

where the coefficients of q^i's are represented by $\mathcal{P}_i(a,e)$ for all $i=1,2,\cdots,m+n+p-1$. We form a system of $(m+n+p-1)$ homogeneous equations if we substitute $1,2,\cdots,(m+n+p-1)$ for q, respectively. It provides a Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{m+n+p-1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ m+n+p-1 & (m+n+p-1)^2 & \cdots & (m+n+p-1)^{m+n+p-1} \end{pmatrix}$$

that implies $\mathcal{P}_i(a,e)=0$ for all $a\in\mathcal{A}$ and for $i=1,2,..,m+n+p-1$. In particular, $\mathcal{P}_{m+n+p-1}(a,e)=0$ implies that

$$\begin{aligned}\tbinom{m+n+p}{m+n+p-1}\mathcal{H}(e)a &= \tbinom{n+p}{n+p-1}\mathcal{H}(e)\zeta(a)+\tbinom{m}{m-1}\mathcal{H}(a)+\tbinom{p}{p-1}\mathbf{h}(e)\zeta(a)\\ &\quad+\tbinom{n}{n-1}\mathbf{h}(a)+\tbinom{p}{p-1}\mathbf{h}(e)\zeta(a)+\tbinom{p}{p-1}\mathbf{h}(a)+\tbinom{m+n}{m+n-1}\xi(a)\mathbf{h}(e),\end{aligned}$$

for all $a\in\mathcal{H}$. On simplifying the above relation, we have $(n+p)\mathcal{H}(a)=(n+p)\mathcal{H}(e)\zeta(a)+(n+p)\mathbf{h}(a)$ for all $a\in\mathcal{A}$. Using the torsion condition, we get

(2) $$\mathcal{H}(a)=\mathcal{H}(e)\zeta(a)+\mathbf{h}(a),\quad \text{for all } a\in\mathcal{A}.$$

Now consider $\mathcal{P}_{m+n+p-2}(a,e)=0$ and use $\mathbf{h}(e)=0$ to find

$$\begin{aligned}\tbinom{m+n+p}{m+n+p-2}\mathcal{H}(a^2) &= \tbinom{n+p}{n+p-2}\mathcal{H}(e)\zeta(a^2)+\tbinom{m}{m-1}\tbinom{n+p}{n+p-1}\mathcal{H}(a)\zeta(a)+\tbinom{m}{m-2}\mathcal{H}(a^2)\\ &\quad+\tbinom{n}{n-1}\tbinom{p}{p-1}\mathbf{h}(a)\zeta(a)+\tbinom{n}{n-2}\mathbf{h}(a^2)+\tbinom{m}{m-1}\tbinom{n}{n-1}\xi(a)\mathbf{h}(a)\\ &\quad+\tbinom{p}{p-2}\mathbf{h}(a^2)+\tbinom{m+n}{m+n-1}\tbinom{p}{p-1}\xi(a)\mathbf{h}(a)\end{aligned}$$

A quiet manipulation yields that

$(m+n+p-1)(m+n+p)\mathcal{H}(a^2)=(n+p-1)(n+p)\mathcal{H}(e)\zeta(a^2)+2m(n+p)\mathcal{H}(a)\zeta(a)+m(m-1)\mathcal{H}(a^2)+2np\mathbf{h}(a)\xi(a)+n(n-1)\mathbf{h}(a^2)+2mn\xi(a)\mathbf{h}(a)+p(p-1)\mathbf{h}(a^2)+2p(m+n)\xi(a)\mathbf{h}(a)$.

Using (2) in previous relation, this implies that

$[(m+n+p-1)(m+n+p)-m(m-1)]\mathcal{H}(a^2)=(n+p-1)(n+p)[\mathcal{H}(a^2)-\mathbf{h}(a^2)]+2m(n+p)\mathcal{H}(a)\zeta(a)+(2mn+2np+2pm)\xi(a)\mathbf{h}(a)+(n^2-n+p^2-p)\mathbf{h}(a^2)+2pn\mathbf{h}(a)\zeta(a)$.

This entails

$$[(m+n+p-1)(m+n+p)-m(m-1)-(n+p-1)(n+p)]\mathcal{H}(a^2) = -2np\mathbf{h}(a^2)+2m(n+p)\mathcal{H}(a)\zeta(a)+(2mn+2np+2pm)\xi(a)\mathbf{h}(a)+2pn\mathbf{h}(a)\zeta(a).$$

This gives

(3)
$$\begin{aligned}(2mn+2mp)\mathcal{H}(a^2) &= -2np\mathbf{h}(a^2)+2m(n+p)\mathcal{H}(a)\zeta(a)\\ &\quad +(2mn+2np+2pm)\xi(a)\mathbf{h}(a)+2pn\mathbf{h}(a)\zeta(a).\end{aligned}$$

Replacing a by a^2 in equation (2), we get $\mathcal{H}(a^2)=\mathcal{H}(e)\zeta(a^2)+\mathbf{h}(a^2)$, for all $a\in\mathcal{A}$. Use this relation in (3) to find $(2mn+2mp)[\mathcal{H}(e)a^2+\mathbf{h}(a^2)] = -2np\mathbf{h}(a^2)+2m(n+p)[\mathcal{H}(e)\zeta(a^2)+\mathbf{h}(a)\zeta(a)]+(2mn+2np+2pm)\xi(a)\mathbf{h}(a)+2pn\mathbf{h}(a)\zeta(a)$.

After applying a few calculations, we get $(2np+2mn+2mp)\mathbf{h}(a^2)=2m(n+p)\mathbf{h}(a)\zeta(a)+(2mn+2np+2pm)\xi(a)\mathbf{h}(a)+2pn\mathbf{h}(a)\zeta(a)$ for all $a\in\mathcal{A}$.

Using torsion restriction on $\mathcal{A}$, we have $\mathbf{h}(a^2)=\mathbf{h}(a)\zeta(a)+\xi(a)\mathbf{h}(a)$ for all $a\in\mathcal{A}$.

Therefore, $\mathbf{h}$ will be a Jordan (ζ,ξ)-derivation. From [8], $\mathbf{h}$ is a (ζ,ξ)-derivation on $\mathcal{A}$ as $\mathcal{A}$ is 2 torsion free semiprime ring. Consider (2), then

$$\begin{aligned}\mathcal{H}(a^2) &= \mathcal{H}(e)\zeta(a^2)+\mathbf{h}(a^2)\\ &= \big[\mathcal{H}(e)\zeta(a)+\mathbf{h}(a)\big]\zeta(a)+\xi(a)\mathbf{h}(a)\\ &= \mathcal{H}(a)\zeta(a)+\xi(a)\mathbf{h}(a).\end{aligned}$$

Therefore, $\mathcal{H}$ will be a generalized Jordan (ζ,ξ)-derivation associated with a derivation $\mathbf{h}$ on $\mathcal{A}$. We obtain the essential outcome by applying Theorem from [3].

3. On generalized left (ζ,ξ)-derivation

A mapping $\mathbf{h_1}:\mathcal{A}\to\mathcal{A}$ is termed as a left derivation(resp. Jordan left derivation) if $\mathbf{h_1}$ is additive and $\mathbf{h_1}(ab)=a\mathbf{h_1}(b)+b\mathbf{h_1}(a)$ (resp. $\mathbf{h_1}(a^2)=2a\mathbf{h_1}(a)$) hold for all $a,b\in\mathcal{A}$. An additive mapping $\mathbf{h_1}:\mathcal{A}\to\mathcal{A}$ is known as a left (ζ,ξ)-derivation and Jordan left (ζ,ξ)-derivation if $\mathbf{h_1}(ab)=\zeta(a)\mathbf{h_1}(b)+\xi(b)\mathbf{h_1}(a)$ and $\mathbf{h_1}(a^2)=\zeta(a)\mathbf{h_1}(a)+\xi(a)\mathbf{h_1}(a)$, respectively hold for all $a,b\in\mathcal{A}$. Clearly, every left (ζ,ξ)-derivation on a ring $\mathcal{A}$ is a Jordan left (ζ,ξ)-derivation but the converse need not be true in general. Generalized left (ζ,ξ)-derivations, a generalization of left (ζ,ξ)-derivations is stated as: an additive mapping $\mathcal{H}_1:\mathcal{A}\to\mathcal{A}$ is called generalized left (ζ,ξ)-derivation if there exists a Jordan left (ζ,ξ)-deviation $\mathbf{h_1}$ such that $\mathcal{H}_1(ab)=\zeta(a)\mathcal{H}_1(b)+\xi(b)\mathbf{h_1}(a)$ (resp. $\mathcal{H}_1(a^2)=\zeta(a)\mathcal{H}_1(a)+\xi(a)\mathbf{h_1}(a)$) for all $a,b\in\mathcal{A}$. The generalized Jordan left derivations are similarly defined there. If $\mathbf{h_1}=0$, then $\mathcal{H}_1$ is known as the right

ζ-centralizer due to [1] and if $\mathbf{h}$ is 0, then $\mathcal{H}$ is recognized as a left ζ-centralizer. An additive mapping is termed as ζ-centralizer if it is a right ζ-centralizer as well as a left ζ-centralizer. A Jordan ζ-centralizer can be described in a similar way. Right ζ-centralizer, left (ζ,ξ)-derivations, and generalized left (ζ,ξ)-derivations are all closely related, i.e., if $\mathbf{h_1}$ is a left (ζ,ξ)-derivations and T right ζ-centralizer of $\mathcal{A}$, then $\mathcal{H}_1$ can be written as $\mathbf{h_1}+T$.

Now, if $\mathcal{H}_1$ is a generalized left (ζ,ξ)-derivation associated with a Jordan left (ζ,ξ)-derivation $\mathbf{h_1}$ on $\mathcal{A}$, then the algebraic identity

$$\mathcal{H}_1(a^{m+n+p})=\zeta(a^{m+n})\mathcal{H}_1(a^p)+\xi(a^p)\zeta(a^m)\mathbf{h_1}(a^n)+\xi(a^{n+p})\mathbf{h_1}(a^m)$$

holds for all $a\in\mathcal{A}$. However, what about the converse of this statement? With certain torsion limits on a semiprime ring, the converse of this statement is also true. Specifically, if $p\geq 0$ and $m,n\geq 1$ are fixed integers and $\mathcal{H}_1,\mathbf{h_1}:\mathcal{A}\to\mathcal{A}$ are additive mappings which satisfy $\mathcal{H}_1(a^{m+n+p})=\zeta(a^{m+n})\mathcal{H}_1(a^p)+\xi(a^p)\zeta(a^m)\mathbf{h_1}(a^n)+\xi(a^{n+p})\mathbf{h_1}(a^m)$ for all $a\in\mathcal{A}$, then $\mathcal{H}_1$ is a generalized left (ζ,ξ)-derivation associated with a Jordan left (ζ,ξ)-derivation $\mathbf{h_1}$ on $(m+n+p-1)!$-torsion free semiprime ring $\mathcal{A}$.

Theorem 3.1. *Suppose that $p\geq 0$ and $m,n\geq 1$ are fixed integers and $\mathcal{A}$ is a $(m+n+p-1)!$-torsion free semiprime ring. If $\mathcal{H}_1,\mathbf{h_1}:\mathcal{A}\to\mathcal{A}$ are additive mappings that satisfy*

$$\begin{aligned}\mathcal{H}_1(a^{m+n+p}) &= \zeta(a^{m+n})\mathcal{H}_1(a^p)+\xi(a^p)\zeta(a^m)\mathbf{h_1}(a^n)\\ &\quad+\xi(a^{n+p})\mathbf{h_1}(a^m)\quad \text{for all } a\in\mathcal{A},\end{aligned}\tag{4}$$

then $\mathcal{H}_1$ is generalized left (ζ,ξ)-derivation associated with a left (ζ,ξ)-derivation D on $\mathcal{A}$, where ζ,ξ are endomorphisms on $\mathcal{A}$.

Proof. Replace a by e in (4) to obtain $\mathbf{h_1}(e)=0$. Further, replacing a by $a+qe$ in (4), we get

$\mathcal{H}_1(a^{m+n+p}+\binom{m+n+p}{1}(a^{m+n+p-1})qe+\cdots+\binom{m+n+p}{m+n+p-2}a^2q^{m+n+p-2}e^2+\binom{m+n+p}{m+n+p-1}aq^{m+n+p-1}e^2+q^{m+n+p}e^{m+n+p})=\zeta(a^{m+n}+\binom{m+n}{1}a^{m+n-1}qe+\cdots+\binom{m+n}{m+n-2}a^2q^{m+n-2}e^2+\binom{m+n}{m+n-1}aq^{m+n-1}e^2+q^{m+n}e^{m+n})\mathcal{H}_1(a^p+\binom{p}{1}a^{p-1}qe+\cdots+\binom{p}{p-2}a^2q^{p-2}e^2+\binom{p}{p-1}aq^{p-1}e+q^pe^p)+\xi(a^p+\binom{p}{1}a^{p-1}qe+\cdots+\binom{p}{p-2}a^2q^2e^2+\binom{p}{p-1}aq^{p-1}e+q^pe^p)\zeta(a^m+\binom{m}{1}a^{m-1}qe+\cdots+\binom{m}{m-2}a^2q^{m-2}e^2+\binom{m}{m-1}aq^{m-1}e+q^me^m)\mathbf{h_1}(a^n+\binom{n}{1}a^{n-1}qe+\cdots+\binom{n}{n-2}a^2q^{n-2}e^2+\binom{n}{n-1}aq^{n-1}e^2+q^ne^n)+\xi(a^{n+p}+\binom{n+p}{1}(a^{n+p-1})qe+\cdots+\binom{n+p}{n+p-2}a^2q^{n+p-2}e^2+\binom{n+p}{n+p-1}aq^{n+p-1}e+q^{n+p}e^{n+p})\mathbf{h_1}(a^m+\binom{m}{1}a^{m-1}qe+\cdots+\binom{m}{m-2}a^2q^{m-2}e^2+\binom{m}{m-1}aq^{m-1}e^2+q^me^m)$, where q is any positive integer.

Apply (4) to express the aforementioned equation as the sum of the terms of the power of q to find

$$q\mathcal{P}_1(a,e)+q^2\mathcal{P}_2(a,e)+\cdots+q^{m+n+p-1}\mathcal{P}_{m+n+p-1}(a,e)=0.$$

Using the same theory as we did in the previous section, we obtain $\mathcal{P}_i(a,e)=0$ for all $a,e\in\mathcal{A}$ and for $i=1,2,..,m+n+p-1$. Using $\mathbf{h_1}(e)=0$, $\mathcal{P}_{m+n+p-1}(x,e)=0$ entails the following

$$\begin{aligned}\tbinom{m+n+p}{m+n+p-1}\mathcal{H}_1(a) &= \tbinom{p}{p-1}\mathcal{H}_1(a)+\tbinom{m+n}{m+n-1}\zeta(a)\mathcal{H}_1(e)\\ &\quad+\tbinom{n}{n-1}\mathbf{h_1}(a)+\tbinom{m}{m-1}\mathbf{h_1}(a).\end{aligned}$$

Simplify the above to obtain

$$(m+n)\mathcal{H}_1(a)=(m+n)\zeta(a)\mathcal{H}_1(e)+(m+n)\mathbf{h_1}(a)$$

for all $a\in\mathcal{A}$. Apply torsion condition to get

$$\mathcal{H}_1(a)=\zeta(a)\mathcal{H}_1(e)+\mathbf{h_1}(a),\quad \text{for all } a\in\mathcal{A}. \tag{5}$$

Next, $\mathcal{P}_{m+n+p-2}(a,e)=0$ with $\mathbf{h_1}(e)=0$ yields the following expression

$$\begin{aligned}\tbinom{m+n+p}{m+n+p-2}\mathcal{H}_1(a^2) &= \tbinom{p}{p-2}\mathcal{H}_1(a^2)+\tbinom{m+n}{m+n-1}\tbinom{p}{p-1}\zeta(a)\mathcal{H}_1(a)\\ &\quad+\tbinom{m+n}{m+n-2}\zeta(a^2)\mathcal{H}_1(e)+\tbinom{n}{n-2}\mathbf{h_1}(a^2)\\ &\quad+\tbinom{m}{m-1}\tbinom{n}{n-1}\zeta(a)\mathbf{h_1}(a)+\tbinom{p}{p-1}\tbinom{n}{n-1}\xi(a)\mathbf{h_1}(a)\\ &\quad+\tbinom{m}{m-2}\mathbf{h_1}(a^2)+\tbinom{n+p}{n+p-1}\tbinom{m}{m-1}\xi(a)\mathbf{h_1}(a),\quad \text{for all } a\in\mathcal{A}.\end{aligned}$$

A straightforward estimation turnout the following

$$\begin{aligned}(m+n+p)(m+n+p-1)(p^2-p)\mathcal{H}_1(a^2) &= 2(mp+np)\zeta(a)\mathcal{H}_1(a)\\ &\quad+2(mn+pn)\xi(a)\mathbf{h_1}(a)\\ &\quad+2(mn+mp)\xi(a)\mathbf{h_1}(a)\\ &\quad+(m+n)(m+n-1)[\mathcal{H}_1(a^2)\\ &\quad-\mathbf{h_1}(a^2)]+n(n-1)\mathbf{h_1}(a^2)\\ &\quad+m(m-1)\mathbf{h_1}(a^2),\ \text{for all } a\in\mathcal{A}.\end{aligned}$$

Using equation (5) after some calculations with torsion restrictions, entails $\mathbf{h_1}(a^2)=\zeta(a)\mathbf{h_1}(a)+\xi(a)\mathbf{h_1}(a)$ for all $a\in\mathcal{A}$. Hence, $\mathbf{h_1}$ will be a Jordan left derivation on a ring $\mathcal{A}$. Again, consider from (5), we have

$$\begin{aligned}\mathcal{H}_1(a^2) &= \zeta(a^2)\mathcal{H}_1(e)+\mathbf{h_1}(a^2)\\ &= \zeta(a)\big[\zeta(a)\mathcal{H}_1(e)+\mathbf{h_1}(a)\big]+\xi(a)\mathbf{h_1}(a)\\ &= \zeta(a)\mathcal{H}_1(a)+\xi(a)\mathbf{h_1}(a).\end{aligned}$$

Hence $\mathcal{H}_1$ will be a generalized Jordan left (ζ,ξ)-derivation associated with a left Jordan (ζ,ξ)-derivation $\mathbf{h_1}$ on $\mathcal{A}$. We arrive to the appropriate conclusion by using theorem from [2].

4. Conclusion

In this work, we have established two key theorems under the assumption that $p\geq 0$ and $m,n\geq 1$ are fixed integers, and that $\mathcal{A}$ is a $(m+n+p-1)!$-torsion free semiprime ring, demonstrating that for additive mappings $\mathcal{H},\mathbf{h}:\mathcal{A}\to\mathcal{A}$ and $\mathcal{H}_1,\mathbf{h_1}:\mathcal{A}\to\mathcal{A}$ satisfying specific functional equations, $\mathcal{H}$ is generalized

(ζ, ξ)-derivation associated with (ζ, ξ)-derivation $\mathbf{h}$ and $\mathcal{H}_1$ is generalized left (ζ, ξ)-derivation associated with a left (ζ, ξ)-derivation D on $\mathcal{A}$, respectively. These results enhance our theoretical understanding of semiprime rings and pave the way for future research. Potential future work includes extending the analysis to other classes of rings, investigating non-torsion free rings, applying the results to problems in ring and module theory, developing computational algorithms for specific instances, exploring connections to other mathematical structures, and examining non-additive mappings to broaden the applicability of the theorems and uncover new types of derivations. Through these avenues, we can further deepen our understanding of the structure of semiprime rings and the role of generalized derivations within them.

5. **Author Contributions**

Investigation and writing original draft, Abu Zaid Ansari, Faiza Shujat; supervision, Muzibur Rahman Mozumder; visualization, Wasim Ahmed. All authors have read and agreed to the published version of the manuscript.

6. **Data Availability Statement**

Not applicable.

7. **Aknowledgement**

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8. **Ethical considerations**

The study was approved by the Ethics Committee of the University of ABCD (Ethical code: FR.AMU.REC.2022.500). The authors avoided from data fabrication and falsification.

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10. **Conflict of interest**

The authors declare no conflict of interest.

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