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HERMITE-HADAMARD INEQUALITIES FOR EXPONENTIAL TYPE HARMONICALLY $(\alpha, s)_h$-CONVEX FUNCTIONS

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Abstract. In this paper, the authors study and introduce some new integral forms of Hermite-Hadamard inequalities in the form of harmonically convex functions known as exponential type harmonically $(\alpha, s)_h$-convex function. Additionally, several special characteristics of this class of functions are examined. More precisely, the authors provide some properties and characteristics related to the Hermite-Hdamard inequality for harmonically $(\alpha, s)_h$-convex function, applications of this work with certain examples are made to establish results obtained.

Keywords: Hermite-Hadamard inequality, convex functions of the exponential kind, harmonically convex functions, and convex functions.

2020 MSC: 28C15, 28C20, 58C35, 46G12.

1. Introduction

Convex functions because of their fundamental applications in several branches of mathematics, has been studied by different researchers and generalized in different forms to Hermite-Hadamard inequalities. Their contributions to knowledge cannot be understated. A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is considered convex if and only if

$$f(\lambda x + (1-\lambda)y)) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1}$$

For every point x, $y \in I$, and $\lambda \in [0, 1]$,
Concave function, is a version of (1) that is reversed. Convex function is an important aspect of mathematics, especially to functional analysis and optimization. Numerous scholars have examined convexity theory in different forms and has been seen to have some significant connections with the theory of inequalities (see, [6 - 12]), few of the related inequalities to this are Hermite-Hadamard inequality, Hardy's inequality and Jensen's inequality. Hermite Hadamard inequality remains crucial and useful in studying convexity, this is stated as:

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Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function defined on the interval I of all real numbers and $m, n \in I$ with $m < n$, then

$$f(\frac{m+n}{2}) \leq \frac{1}{n-m}\int_m^n f(x)dx \leq \frac{f(m)+f(n)}{2} \tag{2}$$

holds. Applications of Hermite Hadamard inequality is known in several new classes of convex functions, this has generated different achievements of Hermite Hadamard's inequalities in various forms. For instance, Ozcan and Butt introduced Hermite-Hadamard type inequalities for multiplicatively harmonic convex functions[10], Hermite-Hadamard inequalities for exponential type multiplicatively convex functions was established by Ozcan[11], Muhammad et al.[6] came up with some Hermite-Hadamard integral inequalities in multiplicative calculus, while Iscan[1] discovered the class of harmonically convex functions in which Hermite-Hadamard type inequalities for this class of functions are studied. Recently, convex functions was introduced to exponential functions by Mahir and Iscan[5] and this is called exponential type convexity and some related inequalities.

Based on the existing literature, we noted that exponential convex function is yet to be introduced to harmonically convex functions, hence, the questions: will all properties and theorems proved on all classess of exponential convex functions be satisfied by exponential harmonically convex function, can we get a new class of harmonically convex function that generalize all other classess of exponential functions? Therefore, this paper aimed at introducing a new class of exponential convex function called exponential type harmonically $(\alpha, s)_h-$ convex function and established certain properties on some integral inequalities for this class of function as an answer to the raised questions. In order to show the existence of exponential functions on harmonically convex function, we related our results with existing results in literature. With the results obtained, we showed exponential type harmonically $(\alpha, s)_h-$ convex functions to possess all properties established by other classes of convex functions including exponential convex function. Necessary areas where this can be applied are equally given with example. The new concept of exponential function introduced can be linked to some areas under metric fixed point, the results obtained improved and generalized existing ones in literature.

We considered few basic concepts and results of harmonically convex functions necessary to this work.

Definition 1.1([1]) : Suppose $I \subset (0, \infty)$ to be a real interval, then the function $f : I \to \mathbb{R}$ is called harmonically convex function if

$$f(\frac{xy}{\lambda x + (1-\lambda)y}) \leq \lambda f(y) + (1-\lambda)f(x) \tag{3}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2([2]): The function $f : I \subset (0,\infty) \to \mathbb{R}$ is called harmonically s-convex function in second sense, if

$$f(\frac{xy}{\lambda x + (1-\lambda)y}) \leq \lambda^s f(y) + (1-\lambda)^s f(x) \tag{4}$$

and $s \in (0,1]$, for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.3([9]): The function $f : I \subset (0,\infty) \to \mathbb{R}$ is called harmonically s-convex function in first sense, if

$$f(\frac{xy}{\lambda x + (1-\lambda)y}) \leq \lambda^s f(y) + (1-\lambda^s) f(x) \tag{5}$$

given $s \in [0,1]$ and for all $x, y \in I$ with $\lambda \in [0,1]$.

Definition 1.4([3]): The function $f : (0, z^*] \to \mathbb{R}$, is called harmonically (α, m)-convex function, if

$$f(\frac{mxy}{m\lambda x + (1-\lambda)y} \leq f((\frac{\lambda}{y} + \frac{1-\lambda}{mx})^{-1}) \leq \lambda^\alpha f(y) + m(1-\lambda^\alpha) f(x) \tag{6}$$

for all $x, y \in (0, z^*]$, with $z^* > 0$ *and* $\lambda \in [0,1]$. Where $m \in (0,1], \alpha \in [0,1]$ and $\lambda \in [0,1]$.

Note that if $m = 1$ *and* $\alpha = s$ in Definition 1.4, the inequality (6) results to the definition of harmonically s-convex function in first sense.

Definition 1.5([4]): The function $f : I \subset (0,\infty) \to \mathbb{R}$ is called harmonically (α, s)-convex function, if

$$f(\frac{xy}{\lambda x + (1-\lambda)y} = f((\frac{\lambda}{x} + \frac{1-\lambda}{y})^{-1}) \leq \lambda^{\alpha s} f(x) + (1-\lambda^\alpha)^s f(y) \tag{7}$$

such that for all $x, y \in (0, b^*]$, with $b^* < 0, \lambda \in [0,1]$ and $\alpha \in [0,1]$.

Definition 1.6([4]): The function $f : I \subset (0,\infty) \to \mathbb{R}$ is called harmonically P-function, if

$$f(\frac{xy}{\lambda x + (1-\lambda)y}) \leq f(x) + f(y) \tag{8}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

The following theorems and lemma are needed for our main results.

Theorem 1.7([1]) Suppose $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ to be harmonically convex and $m, n \in I$, where $a < n$. If $f \in L[m,n]$, then we have the following inequalities to be satisfied:

$$f(\frac{2mn}{m+n}) \leq \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx \leq \frac{f(m)+f(n)}{2}. \tag{9}$$

Theorem 1.8([1]) Suppose $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ to be a differentiable function on $I^o, m, n \in I$ such that $m < n$ and $f' \in L[m,n]$. If $\mid f' \mid^q$ is harmonically convex on $[m,n]$ for $q \geq 1$, then we have

$$\begin{aligned} &|\frac{f(m)+f(n)}{2} - \frac{mn}{n-m}\int_m^n \frac{f)(x)}{x^2}dx| \\ &\leq \frac{mn(n-m)}{2}\xi_1^{1-\frac{1}{q}}(\xi_2|f'(m)|^q + \xi_3|f'(n)|^q)^{\frac{1}{q}}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \xi_1 &= \frac{1}{mn} - \frac{2}{(n-m)^2}\ln(\frac{(m+n)^2}{4mn}) \\ \xi_2 &= -\frac{1}{n(n-m)} + \frac{3m+n}{(n-m)^3}\ln(\frac{(m+n)^2}{4mn}), \end{aligned} \tag{11}$$

and

$$\xi_3 = \frac{1}{m(n-m)} + \frac{3n+m}{(n-m)^3}\ln(\frac{(m+n)^2}{4mn}). \tag{12}$$

Theorem 1.9([1]). Suppose $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ to be a differentiable function defined on $I^0, a, b \in I$ such that $m < n$ and $f' \in L[m,n]$. If $|f'|^q$ is harmonically convex on $[m,n]$ for all $p, q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. Then the following hold:

$$\begin{aligned} &|\frac{f(m)+f(n)}{2} - \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx| \\ &\leq \frac{mn(n-m)}{2}(\frac{1}{p+1})^{\frac{1}{p}}(\eta_1|f'(m)|^q + \eta_2|f'(n)|^q)^{\frac{1}{q}}, \end{aligned} \tag{13}$$

where

$$\eta_1 = \frac{m^{2-2q} + n^{1-2q}[(n-m)(1-2q) - m]}{2(n-m)^2(1-q)(1-2q)} \tag{14}$$

and

$$\eta_2 = \frac{n^{2-2q} - m^{1-2q}[(n-m)(1-2q) + m]}{2(n-m)^2(1-q)(1-2q)}. \tag{15}$$

Definition 1.10([13]). Assume f and g are two functions, if

$$(f(x)-f(y))(g(x)-g(y)) \geq 0 \tag{16}$$

is satisfied for all $x, y \in \mathbb{R}$, then we called f and g similarly ordered functions.

Definition 1.11([14]). The function $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ is called (α, s)-convex if for all $x, y \in I$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda^{\alpha s} f(x) + (1-\lambda^{\alpha})^{s} f(y) \tag{17}$$

and $\lambda \in (0,1)$, where $s \in [-1,1]$ and $\alpha \in (0,1]$.

Lemma 1.12([1]). Suppose $f : I \subset (0,\infty) \longrightarrow \mathbb{R}$ to be a differentiable function on I^0 and $m, n \in I$ with $m < n$, if $f' \in L[m,n]$. Then

$$\begin{aligned}&\frac{f(m)+f(n)}{2} - \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx \\ &= \frac{ab(n-m)}{2}\int_m^n \frac{1-2\lambda}{(\lambda n+(1-\lambda)m)^2} f'(\frac{mn}{\lambda n+(1-\lambda)m})d\lambda.\end{aligned} \tag{18}$$

2. Main Results

In this section, we relate harmonically convex function to exponential type harmonically $(\alpha, s)_h$-convex function, we examined the questions raised above by proving some integral inequalities on the newly introduced class of exponential convex functions.

Definition 2.1. A nonnegative function $f : I \longrightarrow \mathbb{R}$ is said to be exponential type harmonically $(\alpha, s)_h$-convex function, if

$$\begin{aligned}&f(\frac{xy}{\lambda y+(1-\lambda)x}) = f(\frac{\lambda}{x}+\frac{1-\lambda}{y})^{-1} \\ &\leq (e^{h(1-\lambda^{\alpha})^{s}}-1)f(x) + (e^{h(\lambda^{\alpha})^{s}}-1)f(y).\end{aligned} \tag{19}$$

where $(\alpha, s) \in (0,1]$, for all $x, y \in I$ and $h(\cdot) > 0, \lambda \in (0,1)$. If definition (19) is reverse, then f is called exponential type harmonically $(\alpha, s)_h$-concave functions.

Some noted remarks are :

On the condition $\frac{xy}{\lambda y+(1-\lambda)x} \leq \lambda x+(1-\lambda)y$ and if $h = I$ giving that $\lambda > \lambda^2$, $\lambda^2 \leq \lambda$ which implies $h(\lambda) = \lambda$, $\alpha = s = 1$ in inequality (19), then it becomes exponential type-convex function[5].

If $(e^{h(1-\lambda)} - 1) = 1$, $(e^{h(\lambda)} - 1) = 1$ and $\alpha = s = 1$ in equation (19), it result to the definition of harmonically P-convex function[4].

If $e^{h(\cdot)} = h(\cdot) + 1$, which implies $e^{h(\lambda^\alpha)^s} - 1 = h(\lambda^\alpha)^s$ in inequality (19), then it gives the definition of harmonically (α, s)-convex function[12].

The following proposition is useful :

Proposition 2.2. Let $f : I \subset [0,\infty) \longrightarrow \mathbb{R}$ be a mapping and I be the real interval,
(i) If f is (α, s)-convex or harmonically (α, s)-convex function, then f is exponential type harmonically $(\alpha, s)_h$-convex function.

(ii) If f is exponential type harmonically $(\alpha, s)_h$-convex and non increasing function, then f is harmonically (α, s)-convex and (α, s)-convex function.

Proof . For all $\lambda \in [0, 1]$ and $x, y \in I$, we have

$$e^{h(1-\lambda)h(\lambda)(x-y)^2} - 1 \geqslant 0. \tag{20}$$

The following inequality holds:

$$\frac{xy}{\lambda y+(1-\lambda)x} \leq e^{h(\lambda)x+h(1-\lambda)y} - 1 \tag{21}$$

with the stated inequality in (21) the proof is complete.

Remark 2.3. Every non decreasing harmonically (α, s)-convex function and non decreasing convex function is also exponential type harmonically $(\alpha, s)_h$-convex function.

Example 2.4. The function $f : (0,\infty) \longrightarrow \mathbb{R}$, $f(x) = x$ is a non decreasing-convex function and also harmonically α, s-convex function for $\alpha = 1 = s$, giving $e^{h(\lambda^\alpha)s} - 1 = h(\lambda^\alpha)^s$.

Proposition 2.5. Suppose f and g are exponential type harmonically (α, s)-convex functions. If f and g are similarly ordered functions and $e^{h(\lambda^\alpha)^s + h(1-\lambda^\alpha)^s} \leq 1$ with $e^{h(\lambda^\alpha)^s} - 1 \geq 0$, $e^{h(m)} + e^{h(n)} \geq 2$, then the product fg is also exponential type harmonically $(\alpha, s)_h$-convex function.

Proof. Let f and g be exponential type harmonically $(\alpha, s)_h$-convex functions. Then

$$
\begin{aligned}
& f(\frac{mn}{\lambda n+(1-\lambda)m})g(\frac{mn}{\lambda n+(1-\lambda)m}) \\
& \leq [(e^{h(\lambda^{\alpha})^{s}}-1)f(m)+(e^{h(1-\lambda^{\alpha})^{s}}-1)f(n)] \\
& \times[(e^{h(\lambda^{\alpha})^{s}}-1)g(a)+(e^{h(1-\lambda^{\alpha})^{s}}-1)g(n)] \\
& = [(e^{h(\lambda^{\alpha})^{s}}-1)(e^{h(\lambda^{\alpha})^{s}}-1)(f(m)g(m))] \\
& +(e^{h(\lambda^{\alpha})^{s}}-1)(e^{h(1-\lambda^{\alpha})s}-1)[f(m)g(n)] \\
& +(e^{h(1-\lambda^{\alpha})^{s}}-1)(e^{h(\lambda^{\alpha})^{s}}-1)[f(n)g(m)] \\
& +(e^{h(1-\lambda^{\alpha})^{s}}-1)(e^{h(1-\lambda^{\alpha})^{s}}-1)[f(n)g(n)]
\end{aligned} \tag{22}
$$

and so

$$
\begin{aligned}
& f(\frac{mn}{\lambda n+(1-\lambda)m})g(\frac{mn}{\lambda n+(1-\lambda)m}) \\
& \leq (e^{h(\lambda^{2\alpha})^{s}}-1)[f(m)g(m)] \\
& +(e^{h(\lambda^{\alpha})^{s(h(1-\lambda^{\alpha})^{s}}}-1)[f(m)g(n)] \\
& +(e^{h(1-\lambda^{\alpha})^{s}(h(\lambda^{\alpha})^{s})}-1)[f(n)g(m)] \\
& +(e^{h(1-\lambda^{\alpha})^{2s}}-1)[f(n)g(n)].
\end{aligned} \tag{23}
$$

Taking common terms in the second and third term of inequalities (22) gives

$$
\begin{aligned}
f(\frac{mn}{\lambda n+(1-\lambda)m})g(\frac{mn}{\lambda n+(1-\lambda)m}) \quad \leq \quad & (e^{h(\lambda^{2\alpha})^{s}}-1)[f(m)g(m)] \\
& +(e^{h(\lambda^{\alpha})^{s(h(1-\lambda^{\alpha}))^{s}}}-1)[f(m)g(m)] \\
& +(e^{h(1-\lambda^{\alpha})^{s}h(\lambda^{\alpha})^{s}}-1)[f(n)g(n)] \\
& +(e^{h(1-\lambda^{\alpha})^{2s}}-1)[f(n)g(n)] \\
= \quad & (e^{h(\lambda^{\alpha})^{s}}-1)[f(m)g(m)] \\
& +(e^{h(1-\lambda^{\alpha})^{s}}-1)[f(n)g(n)][(e^{h(\lambda^{\alpha})^{s}}-1) \\
& +(e^{h(1-\lambda^{\alpha})^{s}}-1)]
\end{aligned}
$$

This gives ;

$$f(\frac{mn}{\lambda n+(1-\lambda)m})g(\frac{mn}{\lambda n+(1-\lambda)m}) = (e^{h(\lambda^{\alpha})^{s}}-1)[f(m)g(m) +(e^{h(1-\lambda^{\alpha})^{s}}-1)(f(m)g(n))].$$

Hence the proof.

Proposition 2.6. Suppose $f:[m,n]\subset(0,\infty)\longrightarrow\mathbb{R}$ and $g:[\frac{1}{n}\ ,\ \frac{1}{m}]\longrightarrow\mathbb{R}$ are defined as $g(x)=\frac{1}{x}$ then f is exponential type harmonically $(\alpha,s)_h$-convex on $[m,n]$, where $\alpha\in(0,1]$, $h(\cdot)>0, s\in[0,1]$ if and only if g is harmonically (α,s)-convex on $[\frac{1}{n}\ ,\ \frac{1}{m}]$

Proof : Taking $x=\frac{1}{q}$ and $y=\frac{1}{p}$
we have

$$\begin{aligned} f(\frac{1}{\frac{\lambda}{q}+\frac{(1-\lambda)}{p}}) &= f(\frac{1}{\lambda\frac{1}{q}+(1-\lambda)\frac{1}{p}}) \\ &= f((\frac{\lambda}{q}+\frac{(1-\lambda)}{p})^{-1}) \\ &= (\lambda f(q)+(1-\lambda)f(p). \end{aligned}$$

f giving as exponential type harmonically $(\alpha,s)_h$-convex function, implies

$$f(\frac{1}{\frac{\lambda}{q}+\frac{(1-\lambda)}{p}}) \leq (e^{h(\lambda^{\alpha})^{s}}-1)f(q)+(e^{h(1-\lambda^{\alpha})^{s}}-1)f(p). \tag{24}$$

Following suitable composition of f and g, results to

$$f(\frac{1}{\frac{\lambda}{q}+\frac{(1-\lambda)}{p}}) = (e^{h(\lambda^{\alpha})^{s}}-1)(fog)(x)+(e^{h(1-\lambda^{\alpha})^{s}}-1)(fog)(y) \tag{25}$$

for all $x,y\in[\frac{1}{n},\frac{1}{m}]$, $\lambda\in[0,1]$, where $q,p\in[m,n]$.

This justified the necessary condition , to validate the sufficient condition;

for all $x,y\in[m,n]$ and $\lambda\in[0,1]$, we have,

$$f(\frac{\lambda}{x}+\frac{(1-\lambda)}{y})^{-1}=(fog)(\lambda\frac{1}{x}+(1-\lambda)\frac{1}{y}) \tag{26}$$

Since f is exponential type harmonically $(\alpha, s)_h$-convex, inequality (26) results to;

$$f(\frac{\lambda}{x} + \frac{(1-\lambda)}{y})^{-1} = (e^{h(\lambda^\alpha)^s} - 1)f(x) + (e^{h(1-\lambda^\alpha)^s} - 1)f(y) \tag{27}$$

hence the proof is complete.

Now we consider the following:

Theorem 2.7 Suppose $f : I \longrightarrow \mathbb{R}$ is exponential harmonically $(\alpha, s)_h$-convex with $\alpha \in (0, 1]$. If $0 < m < \infty$ and $f \in L[m, n]$, where $h(\cdot) > 0$. Then

$$\frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx \leq \frac{f(m+f(n))}{2}[\int_m^n (e^{h(\lambda^\alpha)^s} - 1) + (e^{h(1-\lambda^\alpha)^s} - 1)]d\lambda \tag{28}$$

Proof.
Since f is exponential type harmonically $(\alpha, s)_h$-convex, we have for all $x, y, \in I$

$$f(\frac{mn}{\lambda n + (1-\lambda)m}) \leq (e^{h(\lambda^\alpha)^s} - 1)f(m) + (e^{h(1-\lambda^\alpha)^s} - 1)f(n) \tag{29}$$

and

$$f(\frac{mn}{\lambda m + (1-\lambda)n}) \leq (e^{h(\lambda^\alpha)^s} - 1)f(n) + (e^{h(1-\lambda^\alpha)^s} - 1)f(m) \tag{30}$$

for all $\lambda \in [0, 1]$, by addition of inequalities (29) and (30), we obtain;

$$\begin{aligned} & f(\frac{mn}{\lambda n + (1-\lambda)m}) + f(\frac{mn}{\lambda m + (1-\lambda)n}) \\ & \leq [f(m) + f(n)](e^{h(\lambda^\alpha)^s} - 1) + (e^{h(1-\lambda^\alpha)^s} - 1) \end{aligned} \tag{31}$$

integrating inequality (31) on $[0, 1]$, gives;

$$\int_0^1 (f(\frac{mn}{\lambda n + (1-\lambda)m})d\lambda + f(\frac{mn}{\lambda m + (1-\lambda)n})d\lambda$$
$$\leq [f(m) + f(n)] \int_m^n [(e^{h(\lambda^\alpha)^s} - 1) + (e^{h(1-\lambda^\alpha)^s} - 1)]$$

this further implies;

$$\frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx \leq \frac{f(m) + f(n)}{2}\int_0^1 [e^{h(\lambda^\alpha)^s} - 1) + (e^{h(1-\lambda^\alpha)^s} - 1)]d\lambda \tag{32}$$

This gives the require results.

Remark 2.8
(1) If $\alpha = s = 1$ and $(e^{h(\lambda^\alpha)^s} - 1) = h(\lambda^\alpha)^s$ in Theorem 2.7, inequality (28) then gives the right hand side of inequality (2);

(2) If $s = 0$ in Theorem 2.7, inequality(28) then yields the following results

(3) If $e^{h(m)} = h(m)+1$, which implies $e^{h(\lambda^{\alpha})^s} - 1 = h(\lambda^{\alpha})^s$ in Theorem 2.8, inequality (28) gives the result obtained in harmonically (α, s)-convex function[12].

Corollary 2.9
Let $f : L \longrightarrow \mathbb{R}$ be harmonically P-function where $m, n \in I$, $m < n$ and $f \in L[m, n]$. Then

$$\frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx \le f(m) + f(n). \tag{33}$$

Theorem 2.10 Suppose $f, g : I \longrightarrow \mathbb{R}$ are exponential type harmonically $(\alpha, s)_h$-convex functions, where $m, n \in I, m < n$, $\alpha \in (0, 1]$, with $h(\cdot) > 0$, $s \in [0, 1]$ and $e^{h(\lambda^{\alpha})^s + h(1-\lambda)^{\alpha}} \le 1$, $e^{h(m)} + e^{h(n)} \ge 2$. Let $fg \in L[m, n]$, then

$$\frac{mn}{n-m}\int_m^n (\frac{f(x)g(x}{x^2})dx \le \phi_1\rho_1 + \phi_2\rho_2 + \phi_3\rho_3 \tag{34}$$

where

$$\begin{aligned} \phi_1 &= f(m)g(m), \\ \phi_2 &= f(n)g(n), \\ \phi_3 &= f(m)g(n) + f(n)g(m), \end{aligned} \tag{35}$$

and

$$\begin{aligned} \rho_1 &= \textstyle\int_m^n (e^{h(\lambda^{2\alpha})^s} - 1)d\lambda \\ \rho_2 &= \textstyle\int_m^n (e^{h(1-\lambda^{\alpha})^{2s}} - 1)d\lambda \\ \rho_3 &= \textstyle\int_m^n (e^{h(\lambda^{\alpha})^s h(1-\lambda^{\alpha})^s} - 1). \end{aligned} \tag{36}$$

Proof. Let $f, g : I \longrightarrow \mathbb{R}$ be exponential type harmonically $(\alpha, s)_h$-convex functions, it follows that

$$\frac{mn}{n-m}\textstyle\int_m^n (\frac{f(x)g(x}{x^2})dx = \int_0^1 f(\frac{xy}{\lambda y + (1-\lambda)x})g(\frac{xy}{\lambda y + (1-\lambda)x})d\lambda$$

$$\le [(e^{h(\lambda^{\alpha})^s} - 1)f(m) + (e^{h(1-\lambda^{\alpha})^s} - 1)f(n)]$$

$$\times[(e^{h(\lambda^{\alpha})^s} - 1)g(m)$$

$$+(e^{h(1-\lambda^{\alpha})^s} - 1)g(n)]$$

Taking the Integral on interval $[0,1]$ gives;

$$\frac{mn}{n-m}\int_m^n(\frac{f(x)g(x}{x^2})dx \leq \int_0^1[(e^{h(\lambda^\alpha)^s}-1)f(m)+(e^{h(1-\lambda^\alpha)^s}-1)f(n)]$$

$$\times[(e^{h(\lambda^\alpha)^s}-1)g(m)+(e^{h(1-\lambda^\alpha)^s}-1)g(n)]d\lambda$$

$$=\int_0^1(e^{h(\lambda^{2\alpha})^s}-1)f(m)g(m)+(e^{h(1-\lambda^\alpha)^{2s}}-1)f(n)g(n)$$

$$+(e^{h(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)[f(m)g(n)+f(n)g(m)]d\lambda$$

$$=\phi_1\int_0^1(e^{h(\lambda^{2\alpha})^s}-1)d\lambda+\phi_2\int_0^1(e^{h(1-\lambda^\alpha)^{2s}}-1)d\lambda$$

$$+\phi_3\int_0^1(e^{h(\lambda^\alpha)^s h(1-\lambda^\alpha)^s}-1)d\lambda$$

so therefore, we have

$$\frac{mn}{n-m}\int_m^n(\frac{f(x)g(x}{x^2})dx \quad = \quad \phi_1\rho_1+\phi_2\rho_2+\phi_3\rho_3. \tag{37}$$

Hence the proof is complete.

Remark 2.11 If $e^{h(m)}=h(m)+1$, which implies $e^{h(\lambda^\alpha)^s}-1=h(\lambda^\alpha)^s$ in Theorem (2.10), inequality (34) gives the result for harmonically (α,s)-convex function[12].

Theorem 2.12 Suppose $f,g:I\longrightarrow\mathbb{R}$ is exponential type harmonically $(\alpha,s)_h$-convex functions such that f and g are similarly ordered functions and

$\int_0^1(e^{h(\lambda^\alpha)^s}-1)+(e^{h(1-\lambda^\alpha)^s}-1)\leq 1$, $(e^{h(\lambda^\alpha)^s}-1)\geq 0, h(\cdot)>0$ then,

$$\frac{mn}{n-m}\int_m^n(\frac{f(x)g(x}{x^2})dx \leq \phi_1\nu_1+\phi_2\nu_2 \tag{38}$$

where ϕ_1 and ϕ_2 are giving by (35) and

$$\begin{aligned}\nu_1 &= \int_0^1(e^{h(\lambda^\alpha)^s}-1)d\lambda\\ \nu_2 &= \int_0^1(e^{h(1-\lambda^\alpha)^s}-1)d\lambda.\end{aligned} \tag{39}$$

Proof. Since f and g are exponential type harmonically $(\alpha, s)_h$-convex and both are similarly ordered functions, we have for all $x, y \in I$

$$\frac{mn}{n-m}\int_m^n(\frac{f(x)g(x}{x^2})dx = \int_0^1 f(\frac{xy}{\lambda y+(1-\lambda)x}g(\frac{xy}{\lambda y+(1-\lambda)x}))d\lambda$$

$$\leq \int_0^1[(e^{h(\lambda^\alpha)^s}-1)f(m)+(e^{h(1-\lambda^\alpha)^s}-1)f(n)][(e^{h(\lambda^\alpha)^s}-1)g(m) \tag{40}$$

$$+(e^{h(1-\lambda^\alpha)^s}-1)g(n)]d\lambda$$

$$\begin{aligned} &\frac{mn}{n-m}\int_m^n(\frac{f(x)g(x}{x^2})dx = \int_0^1[(e^{h(\lambda^\alpha)^s}-1)f(m)g(m) \\ &+(e^{h(1-\lambda^\alpha)^s)}-1)f(n)g(n)[(e^{h(\lambda^\alpha)^s}-1)+(e^{h(1-\lambda^\alpha)^s}-1)] \\ &\leq \int_0^1(e^{h(\lambda^\alpha)^s}-1)f(m)g(m)+(e^{h(1-\lambda^\alpha)^s)}-1)f(n)g(n) \\ &\leq \int_0^1[(e^{h(\lambda^\alpha)^s}-1)\phi_1+(e^{h(1-\lambda^\alpha)^s)}-1)\phi_2 \end{aligned} \tag{41}$$

thus, we have

$$\frac{mn}{n-m}\int_0^1(\frac{f(x)g(x}{x^2})dx \quad \leq \quad \phi_1\nu_1+\phi_2\nu_2 \tag{42}$$

which is the required results.

Theorem 2.13. Suppose $f : I \longrightarrow \mathbb{R}$ to be a differential function on $I^0, m, n \in I$ with $m < n$ and $f' \in L[m, n]$. If $|f'|^q$ is exponential type harmonically $(\alpha, s)_h$-convex functions on $[m, n]$ for $q \geq 1$, with $\alpha \in (0, 1], h(\cdot) > 0$ and $s \in [0, 1]$. Then

$$\begin{aligned} &|\frac{f(m)+f(n)}{2}-\frac{mn}{n-m}\int_m^n\frac{f(x)}{x^2}dx| \\ &\leq \frac{mn(n-m)}{2}\xi_1^{1-\dfrac{1}{q}}[\chi_1|F'(m)|^q+\chi_2|F'(n)|^q]^{\dfrac{1}{q}}. \end{aligned} \tag{43}$$

where

$$\xi_1 \quad = \quad \frac{1}{mn}-\frac{2}{(n-m)^2}\ln\frac{(m+n)^2}{4mn} \tag{44}$$

and

$$\begin{aligned} \chi_1 &= \int_0^1\frac{|1-2\lambda|(e^{h(\lambda^\alpha)^s}-1)}{(\lambda n+(1-\lambda)m)^2}d\lambda \\ \chi_2 &= \int_0^1\frac{|1-2\lambda|(e^{h(1-\lambda^\alpha)^s}-1)}{(\lambda n+(1-\lambda)m)^2}d\lambda. \end{aligned} \tag{45}$$

Proof. Since $|F'|^q$ is exponential type harmonically $(\alpha, s)_h$-convex function, by appropriate use of Lemma 1.12 and power mean inequality , we obtain

$$|\frac{f(m+f(n))}{2} - \frac{mn}{n-a}\int_n^b \frac{f(x)}{x^2}|$$

$$\leq \frac{mn(n-m)}{2}\int_0^1 |\frac{1-2\lambda}{(\lambda n+(1-\lambda m)^2)}||f'(\frac{mn}{\lambda n+(1-\lambda)m})|d\lambda$$

$$\leq \frac{mn(n-m)}{2}(\int_0^1 |\frac{1-2\lambda}{(\lambda n+(1-\lambda)m)^2}|d\lambda)^{1-\frac{1}{q}}$$

$$\times(\int_0^1 |\frac{1-2\lambda}{(\lambda n+(1-\lambda)m)^2}||f'(\frac{mn}{\lambda n+(1-\lambda)m})|^q d\lambda)^{\frac{1}{q}}$$

since $|f'|^q$ is exponential type harmonically $(\alpha, s)_h$-convex, we have

$$|\frac{f(m+f(n))}{2} - \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}|$$

$$\leq \frac{mn(n-m)}{2}(\int_0^1 |\frac{1-2\lambda}{(\lambda n+(1-\lambda)m)^2}|d\lambda)^{1-\frac{1}{q}}$$

(46)

$$(\frac{|1-2\lambda}{(\lambda n+(1-\lambda)m)^2}[(e^{h(\lambda^\alpha)^s}-1)|f'(m)|^q+(e^{h(1-\lambda^\alpha)^s}-1)|f'(n)|^q]d\lambda)^{\frac{1}{q}}$$

$$= \frac{mn(n-m)}{2}\xi_1^{1-\frac{1}{q}}[\chi_1|f'(m)|^q+\chi_2|f'(n)|^q]^{\frac{1}{q}}.$$

hence the proof.

Corollary 2.14 Following the stated conditions in Theorem (2.13), if we take $q=1$ then the results becomes

(47)
$$|\frac{f(m+f(n))}{2} - \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx| \leq \frac{mn(n-m)}{2}(\Theta_1|f'(m)|+\Theta_2|f'(n)|)$$

where

(48)
$$\Theta_1 = \int_0^1 \frac{|1-2\lambda|(e^{h(\lambda^\alpha)^s}-1)}{(\lambda n+(1-\lambda)m)^2}d\lambda$$

$$\Theta_2 = \int_0^1 \frac{|1-2\lambda|(e^{h(1-\lambda^\alpha)^s}-1)}{(\lambda n+(1-\lambda)m)^2}d\lambda.$$

Remark 2.15

(1)Taking $\alpha = s = 1$ and $e^{h(m)} = h(m)+1$, which implies $e^{h(\lambda^\alpha)^s} - 1 = h(\lambda^\alpha)^s$

in Theorem 2.13, inequality (43) results to inequality (10) of Theorem 1.8 but in this case we won't only be having ξ_1 but both ξ_1 and ξ_2.

(2)If $\alpha = 1$ and $e^{h(\lambda^\alpha)^s} - 1 = h(\lambda^\alpha)^s$ in Theorem 2.13, then the results for harmonically s-convex functions in second sense will be obtain.

(3) If $e^{h(m)} = h(m) + 1$, which implies $e^{h(\lambda^\alpha)^s} - 1 = h(\lambda^\alpha)^s$ in Theorem 2.13, inequality (43) gives the result for harmonically (α, s)-convex function[12].

Corollary 2.16
Suppose $f : I \longrightarrow \mathbb{R}$ to be a differentiable function on $I^0, m, n \in I$ with $m < n$ and $f' \in L[m,n]$. If $|f'|^q$ is harmonically s-convex function in second sense on $[m,n]$ for $q > 1$, then

$$
\begin{aligned}
&|\frac{f(m)+f(n)}{2} - \frac{mn}{n-m}\int_m^n \frac{f(x)}{x^2}dx| \\
&\leq \frac{mn(n-m)}{2}\xi_1^{1-\frac{1}{q}}\zeta_1|f'(m)|^q + \zeta_2|f'(n)|^q
\end{aligned} \tag{49}
$$

where

$$
\begin{aligned}
\zeta_1 &= \int_0^1 \frac{|1-2\lambda|(e^{h(\lambda)^s}-1)}{(\lambda n + (1-\lambda)m)^2}d\lambda \\
\zeta_2 &= \int_0^1 \frac{|1-2\lambda|(e^{h(1-\lambda)^s}-1)}{(\lambda n + (1-\lambda)m)^2}d\lambda
\end{aligned} \tag{50}
$$

and ξ_1 is given in (44).

3. Applications

First, we give an illustration of the Hermite-Hadamard inequality for exponential functions.
If $f = exp$, the Hermite-Hadamard inequality defined in (2) results to;

$$e^{\frac{h((m+n)^\alpha)^s}{2}} < \frac{e^{h((n)^\alpha)^s} - e^{h((m)^\alpha)^s}}{n-m} < \frac{e^{h((m)^\alpha)^s} + e^{h((n)^\alpha)^s}}{2}, \quad m \neq n \in \mathbb{R} \text{ and } h(\cdot) > 0$$

that is,

$$\sqrt{mn} < \frac{m-n}{logm - logn} < \frac{m+n}{2}, \quad m \neq n \in (0,\infty).$$

Some applications of this work to special means are;

(1) The generalized logarithmic mean:

$$L_m = L_m(x,y) = [\frac{y^{m+1} - x^{m+1}}{(m+1)(y-x)}]^{\frac{1}{m}}, \quad m \in \mathbb{R}\{-1,0\}. \tag{51}$$

(2) The logarithmic mean:

$$M = M(x,y) = \frac{y-x}{\ln y - \ln u} \tag{52}$$

(3) The harmonic mean

$$M = M(a,b) = \frac{2ab}{a+b}, \quad a,b > 0 \tag{53}$$

Proposition 2.17. If $0 < x < y$, $q > 1$, such that $s \in (0,1)$ and $h(\cdot) > 0$. Then,

$$\begin{aligned} |M(m^{h(s/q+1)}, n^{h(s/q+1)}) \quad &- \quad K^2 L^{h(s/q-1)}_{h(s/q-1)}| \\ &\leq \quad \frac{mn(n-m)(s+q)}{2q} \xi_1^{1-\frac{1}{q}} (\varsigma_1 m^{h(s)} + \varsigma_2 n^{h(s)})^{\frac{1}{q}} \end{aligned}$$

Proof. It easily follows from inequality (49) of corollary 2.16,

for: $(0,\infty) \to \mathbb{R}$, $f(a) = a^{\overline{q+1}}{}^{s} / (\frac{s}{q+1})$. where ξ_1, ς_1 and ς_2 are as seen in (44) and (50).

Proposition 2.18. Let $a, b \in (0,1]$, such that $a < b$. Then, it follows that

$$2(e^{h(\lambda^\alpha)^s} - 2) \ln M(a,b) \leq \ln I(a,b) \leq \frac{\ln G(a,b)}{2(\sqrt{e^{h(\lambda^\alpha)^s}} - 1)}$$

hold.

Proof. This follows from inequalities (28) for the function

$$f(x) = -\ln x, \quad a \in (0,1] \text{ and } h(\cdot) > 0$$

4. Conclusion

This paper introduced new class of exponential function, called exponential type harmonically $(\alpha, s)_h$-convex function. Although, the Hermite-Hadamard inequalities had been generalized in many areas of convex function, the results obtained here generalized and improved some known results in literature in terms of exponential convex function, particularly some results obtained in harmonically (α, s)-convex function, we studied some new results of Hermite-Hadamard type on this class of function. Necessary applications were made to certain special means. As a result of some future studies, the newly introduced concept could be linked to mathematical modelling, in the context of functions, exponential decay, and exponential growth. We look forward to seeing interested researchers make application of this to some areas of analysis, like, fixed point theory, under the study of convex metric, in analyzing the relationship between convex metric space and exponential convex metric space. It is important to note that, exponential type harmonically $(\alpha, s)_h$-convex function possesses distinct properties that harmonically (α, s)-convex function and other classes of convex functions have.

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The authors declare no conflicts of interest.

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