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# FUZZY NEAR BEST APPROXIMATION AS A GENERALIZATION OF FUZZY BEST APPROXIMATION

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ABSTRACT. Given a fuzzy normed space, we will introduce the notion of fuzzy near best approximation as a generalization of the notion of fuzzy best approximation. Some basic properties are characterized and also many examples for illustration are presented. Also, the hereditary properties of the fuzzy near best approximation on direct sum and tensor product of linear spaces are discussed.
*Keywords*: Fuzzy near best approximation, Direct sum, Tensor product, Fuzzy norm.
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## 1. Introduction

The logic and theory of fuzzy sets were introduced by L. Zadeh [13] in 1956. Many structures and concepts have been researched and investigated by following fuzzy logic in all sciences. Along with classical logic, many mathematicians have done research in the field of fuzzy logic and fuzzy mathematics [4, 5, 9]. The theory of the best approximation as a part of the research field in functional analysis has been of interest to many researchers for a long time [6]. Recently, the fuzzy best approximation has attracted the attention of some researchers [2, 8, 10, 11]. The main motivation of this paper is a generalization of the notion of the fuzzy best approximation, which is called the fuzzy near best approximation. The organization of the paper is as follows. In section 2, some definitions and concepts in the classical and fuzzy best approximation theory are presented. In section 3, the definition of the fuzzy near best approximation and some of its basic properties are stated. In section 4, the hereditary properties of the fuzzy near best approximation on direct sum and tensor product of linear spaces are discussed. In section 5, the conclusion about the paper is stated.

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## 2. Preliminaries

Let $X$ be a normed linear space and $Y$ be a subset of $X$. If $x \in X$, then the distance of $x$ from $Y$ is denoted by $d(x,Y)$ that is

$$d\left(x,Y\right) = \inf\left\{\|x-y\| \ \ s.t. \ y \in Y\right\}.$$

An element $y \in Y$ is said to be the best approximation to $x \in X$ from $Y$ if $\|x-y\| = d(x,Y)$. The set of all best approximations to $x \in X$ from $Y$ is denoted by $P_Y(x)$. If for any $x \in X$, $P_Y(x) \neq \emptyset$, then we say that $Y$ is proximinal in $X$. Also, if for any $x \in X$, $P_Y(x)$ is singleton, then $Y$ is a Chebyshev subset of $X$. A sequence $\left\{y_n\right\}_n \subseteq Y$ is called a minimizing sequence for $x \in X$ if $\lim_{n\to\infty} \|x-y_n\| = d\left(x,Y\right)$ [6].

An element $y^n \in Y$ is said to be a near best approximation to $x$ within a relative distance $\rho \geq 0$ if,

$$\|x-y^n\| \leq (1+\rho)\left\|x-x^b\right\| = (1+\rho)d\left(x,Y\right)$$

where $x^b$ is the best approximation to $x$ from $Y$ [7]. The set of all near best approximations to $x \in X$ from $Y$ is denoted by $P^n_Y(x)$.
Let $\left\{X_i\right\}_{i\in I}$ be a family of linear spaces. Then the algebraic direct sum of the spaces $X_i$, i.e.,

$$\sum_{i\in I} X_i = \left\{x = \left(x_i\right)_{i\in I} \middle| x_i = 0 \quad \text{for all but finitely many} \quad i \in I\right\}$$

with the pointwise vector-space operations as follows, is a linear space,

$$x + y = \left(x_i + y_i\right)_{i\in I}$$

and

$$\alpha x = \left(\alpha x_i\right)_{i\in I}$$

for all $x, y \in \sum_{i\in I} X_i$ and $\alpha \in \mathbb{C}$ or $\mathbb{R}$ [1].
Also, let $X$ and $Y$ be linear spaces over $\mathbb{C}$ or $\mathbb{R}$. Then the algebraic tensor product of $X$ and $Y$ is denoted by $X \otimes Y$. If $X'$ and $Y'$ are the dual spaces of $X$ and $Y$ respectively, then for all $x \in X$ and $y \in Y$, the map $x \otimes y : X' \times Y' \longrightarrow \mathbb{C}\,(\text{or } \mathbb{R})$ defined by

$$\left(x \otimes y\right)\left(f,g\right) = f\left(x\right)g\left(y\right), f \in X', g \in Y'$$

is a bilinear map. For the basic properties concerning the tensor product of linear spaces, we refer the reader to [3].

**Definition 2.1.** [9]Let $X$ be a linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is said to be fuzzy norm on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$ :

1- $N\left(x,t\right) = 0$ for $t \leq 0$.
2- $N\left(x,t\right) = 1$ for every $t \in \mathbb{R}^+$ if and only if $x = 0$ .
3- $N\left(cx,t\right) = N\left(x, \frac{t}{|c|}\right)$ for every $c \neq 0$ and $t \in \mathbb{R}$.
4- $N\left(x+y, s+t\right) \geq \min\left\{N\left(x,s\right), N\left(y,t\right)\right\}$ for every $s, t \in \mathbb{R}$.
5- $N\left(x,.\right)$ is non-decreasing on $\mathbb{R}$ and $\lim_{t\to\infty} N\left(x,t\right) = 1$.

Note that by part 3 of Definition 2.1, $N(-x,t) = N(x,t)$ for all $x \in X$ and $t \in \mathbb{R}$.
The notion of $t-$best approximation in fuzzy metric spaces was first introduced by [11] and then followed by [2, 10, 12].

**Definition 2.2.** [8]Let $Y$ be a nonempty subset of a fuzzy normed space $(X, N)$. For $x \in X$ and $t \in \mathbb{R}$, let

$$d(Y, x, t) = \sup\{N(x-y, t), \quad y \in Y\}.$$

An element $y_0 \in Y$ is said to be a fuzzy best approximation to $x$ from $Y$ if

$$N(x-y_0, t) = d(Y, x, t),$$

for all $t \in \mathbb{R}$. The set of all fuzzy best approximations to $x$ from $Y$ is denoted by $P_Y^f(x)$.

## 3. Fuzzy near best approximation and its basic properties

In this section, we introduce the notion of fuzzy near best approximation and investigate some basic properties concerning this notion.

**Definition 3.1.** Let $X$ be a linear space and $Y$ be a subset of $X$. Also let $N : X \times \mathbb{R} \longrightarrow [0,1]$ be a fuzzy norm on $X$. An element $y_0 \in Y$ is said to be a fuzzy near best approximation to $x$ from $Y$ within a relative distance $\rho \geq 0$ if,

$$N(x-y_0, t) \geq N\left(x - Y, \frac{t}{1+\rho}\right)$$

for all $t \in \mathbb{R}$, where

$$N\left(x-Y, \frac{t}{1+\rho}\right) := \sup\left\{N\left(x-y, \frac{t}{1+\rho}\right), y \in Y\right\} = d(Y, x, \frac{t}{1+\rho}).$$

The set of all fuzzy near best approximations to $x$ from $Y$ within the relative distance $\rho$ is denoted by $P_Y^{fn(\rho)}(x)$.

*Remark* 3.2. Trivially the notion of fuzzy best approximation is nothing else than fuzzy near best approximation within the relative distance $\rho = 0$.

**Proposition 3.3.** *Let $(X, N)$ be a fuzzy normed space and $Y$ be a nonempty subset of $X$. Also let $x, y \in X$, $z \in Y, \rho \geq 0$ and $\alpha \in \mathbb{R}$. Then*

1- *If $P_Y^{fn(\rho)}(x) \neq \emptyset$, then $P_{\alpha Y}^{fn(\rho)}(\alpha x) = \alpha P_Y^{fn(\rho)}(x)$*
2- *If $\alpha \neq 0$, then $P_Y^{fn(\rho)}(\alpha x) = \alpha P_{\frac{Y}{\alpha}}^{fn(\rho)}(x)$*
3- $P_Y^{fn(\rho)}(x+z) = P_{Y-z}^{fn(\rho)}(x) + z$
4- $P_{Y+y}^{fn(\rho)}(x+y) = P_Y^{fn(\rho)}(x) + y$.

*Proof.* We'll prove the first part. The rest of the parts are easily verified. In the case where $\alpha = 0$, the equality trivially holds. For $\alpha \neq 0$, let $z_0 \in P_{\alpha Y}^{fn(\rho)}(\alpha x)$. Then

$$N(\alpha x - z_0, t) \geq N\left(\alpha x - \alpha Y, \frac{t}{1+\rho}\right), \quad (\forall t \in \mathbb{R}).$$

Therefore

$$N\left(x-\frac{z_0}{\alpha},\frac{t}{|\alpha|}\right)\geq N\left(x-Y,\frac{t}{|\alpha|\,(1+\rho)}\right),\quad (\forall t\in\mathbb{R})\,.$$

Replacing $t$ by $|\alpha|\,t$, we have

$$N\left(x-\frac{z_0}{\alpha},t\right)\geq N\left(x-Y,\frac{t}{1+\rho}\right),\quad (\forall t\in\mathbb{R})\,.$$

So $\frac{z_0}{\alpha}\in P_Y^{fn(\rho)}(x)$. Hence $z_0\in\alpha P_Y^{fn(\rho)}(x)$.
Conversely, let $z_0\in\alpha P_Y^{fn(\rho)}(x)$. Then $\frac{z_0}{\alpha}\in P_Y^{fn(\rho)}(x)$. Therefore

$$N\left(x-\frac{z_0}{\alpha},t\right)\geq N\left(x-Y,\frac{t}{1+\rho}\right),\quad (\forall t\in\mathbb{R})\,.$$

So

$$N\left(\alpha x-z_0,|\alpha|\,t\right)\geq N\left(x-Y,\frac{t}{1+\rho}\right),\quad (\forall t\in\mathbb{R})\,.$$

Replacing $t$ by $\frac{t}{|\alpha|}$, we have

$$\begin{aligned}N\left(\alpha x-z_0,t\right)&\geq N\left(x-Y,\frac{t}{|\alpha|\,(1+\rho)}\right)\\&=N\left(\alpha x-\alpha Y,\frac{t}{1+\rho}\right),\quad (\forall t\in\mathbb{R})\,.\end{aligned}$$

Then $z_0\in P_{\alpha Y}^{fn(\rho)}(\alpha x)$. So we can conclude that $P_{\alpha Y}^{fn(\rho)}(\alpha x)=\alpha P_Y^{fn(\rho)}(x)$. □

**Proposition 3.4.** *Let $(X,N)$ be a fuzzy normed linear space and $Y$ be a subset of $X$. Then every fuzzy best approximation to $x\in X$ from $Y$ is a fuzzy near best approximation to $x$ from $Y$ within every relative distance $\rho\geq 0$.*

*Proof.* Let $y_0\in P_Y^f(x)$. So $N(x-y_0,t)=N(x-Y,t)$ for all $t\in\mathbb{R}$. We shall show that $N(x-y_0,t)\geq N\left(x-Y,\frac{t}{1+\rho}\right)$ for all $t\in\mathbb{R}$ and every $\rho\geq 0$. Let $y\in Y$ and $t\in\mathbb{R}$. So by Definition 2.1 part 5,

$$N\left(x-y,\frac{t}{1+\rho}\right)\leq N(x-y,t)\leq N(x-Y,t)=N(x-y_0,t)\,.$$

It follows that $N\left(x-y,\frac{t}{1+\rho}\right)\leq N(x-y_0,t)$ for all $t\in\mathbb{R}$ and $y\in Y$. Hence

$$\sup\left\{N\left(x-y,\frac{t}{1+\rho}\right),\quad y\in Y\right\}\leq N(x-y_0,t)$$

for all $t\in\mathbb{R}$. Therefore

$$N\left(x-Y,\frac{t}{1+\rho}\right)\leq N(x-y_0,t)$$

for all $t \in \mathbb{R}$, providing $y_0 \in P_Y^{fn(\rho)}(x)$. □

**Proposition 3.5.** *Let $(X, N)$ be a fuzzy normed linear space, $Y$ be a subset of $X$, $x \in X$, and $\rho \geq 0$. If $Y$ is convex, then $P_Y^f(x)$ and $P_Y^{fn(\rho)}(x)$ are convex.*

*Proof.* Since the proof method is similar, we only deal with the proof of the convexity of $P_Y^{fn(\rho)}(x)$. Let $y_1, y_2 \in P_Y^{fn(\rho)}(x)$ and $0 < \alpha < 1$. So by Definition 2.1, for all $t \in \mathbb{R}$ we have

$$\begin{aligned}
&N\left(x - \left(\alpha y_1 + (1-\alpha) y_2\right), t\right) \\
&= N\left(\alpha x + (1-\alpha) x - \left(\alpha y_1 + (1-\alpha) y_2\right), \alpha t + (1-\alpha) t\right) \\
&= N\left(\alpha (x - y_1) + (1-\alpha)(x - y_2), \alpha t + (1-\alpha) t\right) \\
&\geq \min\left\{N\left(\alpha (x - y_1), \alpha t\right), N\left((1-\alpha)(x - y_2), (1-\alpha) t\right)\right\} \\
&= \min\left\{N\left(x - y_1, t\right), N\left(x - y_2, t\right)\right\} \\
&\geq N\left(x - Y, \frac{t}{1+\rho}\right).
\end{aligned}$$

Hence $\alpha y_1 + (1-\alpha) y_2 \in P_Y^{fn(\rho)}(x)$. □

The following example shows that the converse of Proposition 3.5 is not true in general.

**Example 3.6.** *Suppose that $X = \mathbb{R}$, $Y = \{1, 2\}$, $x = 0$, and $\rho = 0.5$. Also let*

$$N(x, t) = \begin{cases} 0, & t \leq |x| \\ 1, & t > |x| \end{cases}$$

*be a fuzzy norm on $X$. Clearly, $P_Y^f(0) = \{1\}$, $P_Y^{fn(\rho)}(0) = \{1\}$ and, the set $\{1\}$ is convex but, $Y$ is not convex.*

The next example shows that the notion of fuzzy near best approximation is different from the notion of fuzzy best approximation.

**Example 3.7.** *Suppose that $X = \mathbb{R}$, $Y = [1, 1.5]$, $x = 0$ and $\rho = 0.5$. Also let*

$$N(x, t) = \begin{cases} 0, & t \leq |x| \\ 1, & t > |x| \end{cases}$$

*be a fuzzy norm on $X$. Then, $P_Y^f(0) = \{1\}$ and $P_Y^{fn(0.5)}(0) = Y$. Generally for $0 \leq \rho \leq 0.5$, $P_Y^{fn(\rho)}(0) = [1, 1+\rho]$. Indeed, if $0 < t \leq 1$, then $N(1, t) = 0$ and for all $y \in Y$, $t \leq y$. So $N(y, t) = 0$ for all $y \in Y$. It follows that $N(Y, t) = 0$. Hence $N(1, t) = N(Y, t)$ for all $0 < t \leq 1$. If $t > 1$, then $N(1, t) = 1$ and so $N(Y, t) = 1$. Therefore $N(1, t) = N(Y, t)$ for all $t \in \mathbb{R}$. This shows that $1 \in P_Y^f(0)$. If $1 < y_0 \leq 1.5$, then for $t = 1 + \frac{y_0 - 1}{2}$, $N(y_0, t) = 0$ and $N(Y, t) = N(1, t) = 1$. It follows that $0 = N(y_0, t) \neq N(Y, t) = 1$ for $t = 1 + \frac{y_0 - 1}{2}$. So $y_0 \notin P_Y^f(0)$. Hence $P_Y^f(0) = \{1\}$.*

*Now we will show that* $P_Y^{fn(0.5)}(0) = [1, 1.5] = Y$*. Let* $y_0 \in [1, 1.5]$*. If* $t \leq y_0$*, then* $N(y_0, t) = 0$ *and* $\frac{2}{3}t \leq \frac{2}{3}y_0 \leq \frac{2}{3}\left(\frac{3}{2}\right) = 1$*. So* $N\left(y, \frac{2}{3}t\right) = 0$ *for all* $y \in [1, 1.5]$*. It follows that* $N\left(Y, \frac{2}{3}t\right) = 0$*. Hence* $N(y_0, t) \geq N\left(Y, \frac{2}{3}t\right)$ *for all* $t \leq y_0$*. If* $t > y_0$*, then* $N(y_0, t) = 1 \geq N\left(Y, \frac{2}{3}t\right)$*. Therefore* $N(y_0, t) \geq N\left(Y, \frac{2}{3}t\right)$ *for all* $t \in \mathbb{R}$*. Hence* $y_0 \in P_Y^{fn(0.5)}(0)$*. This shows that* $P_Y^{fn(0.5)}(0) = [1, 1.5] = Y$*.*

**Example 3.8.** *Suppose that* $X = \mathbb{R}$*,* $Y = (0, 1)$*,* $x = 0$ *and* $\rho \geq 0$*. Also let*

$$N(x, t) = \begin{cases} 0, & t \leq |x| \\ 1, & t > |x| \end{cases}$$

*be a fuzzy norm on* $X$*. Then* $P_Y^f(0) = P_Y^{fn(\rho)}(0) = \emptyset$*. Indeed, since* $P_Y^f(0) \subseteq P_Y^{fn(\rho)}(0)$*, it's enough to prove that* $P_Y^{fn(\rho)}(0) = \emptyset$*. Let* $y_0 \in Y$ *be an arbitrary element. Choose* $n \in \mathbb{N}$ *such that* $\frac{1}{n} < \frac{y_0}{1+\rho}$*. So if* $t = y_0$*, then*

$$0 = N(y_0, t) < N\left(\frac{1}{n}, \frac{t}{1+\rho}\right) = 1.$$

*It follows that*

$$0 = N(y_0, t) < N\left(Y, \frac{t}{1+\rho}\right) = 1.$$

*Hence* $y_0 \notin P_Y^{fn(\rho)}(0)$ *and so* $P_Y^{fn(\rho)}(0) = \emptyset$*.*

**Proposition 3.9.** *Let* $X$ *be a linear space,* $Y$ *be a subset of* $X$ *and* $N : X \times \mathbb{R} \longrightarrow [0, 1]$ *be a fuzzy norm on* $X$*. For any* $\rho \geq 0$*,* $x \in Y$*,* $P_Y^{fn(\rho)}(x) = \{x\}$*.*

*Proof.* Clearly, $N(x - x, t) = N(0, t) = 1 \geq N\left(x - Y, \frac{t}{1+\rho}\right)$ for all $t > 0$. Hence, $N(x - x, t) \geq N\left(x - Y, \frac{t}{1+\rho}\right)$ for all $t \in \mathbb{R}$. This shows that $x \in P_Y^{fn(\rho)}(x)$. Now let $y_0 \in P_Y^{fn(\rho)}(x)$. Then for all $t > 0$

$$\begin{aligned} N(x - y_0, t) &\geq N\left(x - Y, \frac{t}{1+\rho}\right) \\ &\geq N\left(x - x, \frac{t}{1+\rho}\right) \\ &= 1. \end{aligned}$$

Therefore $N(x - y_0, t) = 1$ for all $t > 0$. Hence, $x - y_0 = 0$. Then $y_0 = x$. □

**Theorem 3.10.** *Let $X$ be a linear space, $Y$ be a subset of $X$ and $N : X \times \mathbb{R} \longrightarrow [0, 1]$ be a fuzzy norm on $X$. For $x \in X$, if $\rho_1 \leq \rho_2$, then $P_Y^{fn(\rho_1)}(x) \subseteq P_Y^{fn(\rho_2)}(x)$.*

*Proof.* If $y_0 \in P_Y^{fn(\rho_1)}(x)$, then

$$N(x - y_0, t) \geq N\left(x - Y, \frac{t}{1+\rho_1}\right).$$

Since for all $t \in \mathbb{R}$ and $y \in Y$

$$N\left(x - y, \frac{t}{1+\rho_1}\right) \geq N\left(x - y, \frac{t}{1+\rho_2}\right),$$

$$N\left(x - Y, \frac{t}{1+\rho_1}\right) \geq N\left(x - Y, \frac{t}{1+\rho_2}\right).$$

Therefore

$$N(x - y_0, t) \geq N\left(x - Y, \frac{t}{1+\rho_2}\right).$$

It follows that $y_0 \in P_Y^{fn(\rho_2)}(x)$. □

The following example shows that the converse of Theorem 3.10 is not true.

**Example 3.11.** *Suppose that $X = \mathbb{R}$, $Y = \{1, 2\}$, $x = 0$, $\rho_1 = 0.5$, and $\rho_2 = 0.25$. Also let*

$$N(x, t) = \begin{cases} 0, & t \leq |x| \\ 1, & t > |x| \end{cases}$$

*be a fuzzy norm on $X$. Clearly, $\{1\} = P_Y^{fn(\rho_1)}(0) \subseteq P_Y^{fn(\rho_2)}(0) = \{1\}$ but, $\rho_1 > \rho_2$.*

**Theorem 3.12.** *Let $X$ be a linear space, $Y$ be a subset of $X$ and $N : X \times \mathbb{R} \longrightarrow [0, 1]$ be a fuzzy norm on $X$. If $x \in X$ and $\rho_1 \geq \rho_2 \geq \rho_3 \geq \ldots$ such that $\rho_m \longrightarrow \rho$ as $m \longrightarrow \infty$, then $P_Y^{fn(\rho)}(x) \subseteq \bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x)$. In particular, if $N(z, .)$ is lower semicontinuous at every $z \in x - Y$, then $P_Y^{fn(\rho)}(x) = \bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x)$.*

*Proof.* By Theorem 3.10, since $\rho_m \geq \rho$ for all $m \in \mathbb{N}$,

$$\begin{aligned} &P_Y^{fn(\rho)}(x) \subseteq P_Y^{fn(\rho_1)}(x) \\ &P_Y^{fn(\rho)}(x) \subseteq P_Y^{fn(\rho_2)}(x) \\ &\vdots \\ &P_Y^{fn(\rho)}(x) \subseteq P_Y^{fn(\rho_m)}(x), m \in \mathbb{N}. \end{aligned}$$

Then $P_Y^{fn(\rho)}(x) \subseteq \bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x)$.
If $y_0 \in \bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x)$ and $N(z,.)$ is lower semicontinuous for every $z \in x - Y$, then for all $m \in \mathbb{N}$ and for all $t \in \mathbb{R}$ we have

$$\begin{aligned} N(x-y_0,t) \geq & N\left(x-Y, \frac{t}{1+\rho_m}\right) \\ \geq & N\left(x-y, \frac{t}{1+\rho_m}\right), y \in Y. \end{aligned}$$

Since $N(x-y,.)$ is lower semicontinuous for all $x-y$ and $\dfrac{t}{1+\rho_m} \leq \dfrac{t}{1+\rho}$ for every $m \in \mathbb{N}$,

$$\begin{aligned} N(x-y_0,t) \geq & \lim_{m \longrightarrow \infty} N\left(x-y, \frac{t}{1+\rho_m}\right) \\ = & N\left(x-y, \frac{t}{1+\rho}\right), y \in Y. \end{aligned}$$

So

$$\begin{aligned} N(x-y_0,t) \geq & \sup\left\{N\left(x-y, \frac{t}{1+\rho}\right), y \in Y\right\} \\ = & N\left(x-Y, \frac{t}{1+\rho}\right). \end{aligned}$$

Hence $\bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x) \subseteq P_Y^{fn(\rho)}(x)$. It follows that $\bigcap_{m=1}^{\infty} P_Y^{fn(\rho_m)}(x) = P_Y^{fn(\rho)}(x)$. □

## 4. Fuzzy Near Best Approximation On Direct Sum And Tensor Product Of Linear Spaces

We use the following lemma in the proof of Proposition 4.2 and Theorem 4.6.

**Lemma 4.1.** *Let $0 \leq a_i \leq 1$ and $0 \leq b_i \leq 1$ for all $1 \leq i \leq m$. Then*

$$\begin{aligned} & \min\left\{\min\{a_i, b_i\} \middle| 1 \leq i \leq m\right\} \\ = & \min\left\{\min\left\{a_i \middle| 1 \leq i \leq m\right\}, \min\left\{b_i \middle| 1 \leq i \leq m\right\}\right\}. \end{aligned}$$

*Proof.* It's obvious. □

**Proposition 4.2.** *Let $\{(X_i, N_i)\}_{i \in I}$ be a family of fuzzy normed spaces. Then $\left(\sum_{i \in I} X_i, N\right)$ is a fuzzy normed space, where $N : \left(\sum_{i \in I} X_i\right) \times \mathbb{R} \longrightarrow [0,1]$ is defined by*

$$N\left((x_i)_{i \in I}, t\right) = \inf\left\{N_i(x_i,t) \middle| i \in I\right\}.$$

*Proof.* To prove the above proposition, we only prove conditions 4 and 5 of Definition 2.1 and we leave the rest to the reader. To prove the fourth part, suppose that $(x_i)_{i\in I}, (y_i)_{i\in I} \in \sum_{i\in I} X_i$ and $s, t \in \mathbb{R}$. So $x_i = 0$ and $y_i = 0$ for all but finitely many $i_k \in I, 1 \leq k \leq m$. Clearly if $s+t \leq 0$, then the inequality

$$N\left((x_i)_{i\in I} + (y_i)_{i\in I}, s+t\right) \geq \min\left\{N\left((x_i)_{i\in I}, s\right), N\left((y_i)_{i\in I}, t\right)\right\} \tag{1}$$

holds. Also if $s + t > 0$, $s \leq 0$ or $t \leq 0$, then obviously inequality 1 holds.

Let $s + t > 0, s > 0$ and $t > 0$. Then

$$\begin{aligned}
&N\left((x_i)_{i\in I} + (y_i)_{i\in I}, s+t\right)\\
&= N\left((x_i + y_i)_{i\in I}, s+t\right)\\
&\inf\left\{N_i\left(x_i + y_i, s+t\right) \middle| i \in I\right\}\\
&\geq \inf\left\{\min\left\{N_i\left(x_i, s\right), N_i\left(y_i, t\right)\right\} \middle| i \in I\right\}\\
&= \min\left\{\min\left\{N_{i_k}\left(x_{i_k}, s\right), N_{i_k}\left(y_{i_k}, t\right)\right\}, 1 \middle| 1 \leq k \leq m\right\}\\
&= \min\left\{\min\left\{N_{i_k}\left(x_{i_k}, s\right), N_{i_k}\left(y_{i_k}, t\right)\right\} \middle| 1 \leq k \leq m\right\}\\
&= \min\left\{\min\left\{N_{i_k}\left(x_{i_k}, s\right) \middle| 1 \leq k \leq m\right\}, \min\left\{N_{i_k}\left(y_{i_k}, t\right) \middle| 1 \leq k \leq m\right\}\right\}\\
&= \min\left\{\inf\left\{N_i\left(x_i, s\right) \middle| i \in I\right\}, \inf\left\{N_i\left(y_i, t\right) \middle| i \in I\right\}\right\}\\
&= \min\left\{N\left((x_i)_{i\in I}, s\right), N\left((y_i)_{i\in I}, t\right)\right\}.
\end{aligned}$$

To prove the fifth part, suppose that $(x_i)_{i\in I} \in \sum_{i\in I} X_i$. So $x_i = 0$ for all but finitely many $i_k \in I$, $1 \leq k \leq m$. Hence

$$\begin{aligned}
&\lim_{t\longrightarrow\infty} N\left((x_i)_{i\in I}, t\right)\\
&= \lim_{t\longrightarrow\infty}\left\{\inf\left\{N_i\left(x_i, t\right) \middle| i \in I\right\}\right\}\\
&= \lim_{t\longrightarrow\infty}\left\{\min\left\{N_{i_k}\left(x_{i_k}, t\right), 1 \middle| 1 \leq k \leq m\right\}\right\}\\
&= \lim_{t\longrightarrow\infty}\left\{\min\left\{N_{i_k}\left(x_{i_k}, t\right) \middle| 1 \leq k \leq m\right\}\right\}\\
&= \min\left\{\lim_{t\longrightarrow\infty} N_{i_k}\left(x_{i_k}, t\right) \middle| 1 \leq k \leq m\right\}\\
&= \min\{1\}\\
&= 1.
\end{aligned}$$

Since $N_i(x_i, .)$ is increasing for all $x_i \in X_i$, $N\left((x_i)_{i\in I}, .\right)$ is increasing for all $(x_i)_{i\in I}$. □

**Corollary 4.3.** *Let $N_i : X_i \times \mathbb{R} \longrightarrow [0,1]$ be a fuzzy norm on $X_i$ for $i = 1,2,\dots,n$. Then $N : X_1 \times X_2 \times \cdots \times X_n \times \mathbb{R} \longrightarrow [0,1]$ defined by*

$$N\left(\left(x_1,x_2,\dots,x_n\right),t\right) = \min\left\{N_1\left(x_1,t\right),N_2\left(x_2,t\right),\dots,N_n\left(x_n,t\right)\right\}$$

*is a fuzzy norm on $X_1 \times X_2 \times \cdots \times X_n$.*

**Proposition 4.4.** *Let $\{(X_i,N_i)\}_{i=1}^n$ be a finite family of fuzzy normed spaces and $N : X_1 \times X_2 \times \cdots \times X_n \times \mathbb{R} \longrightarrow [0,1]$ is defined by*

$$N\left(\left(x_1,x_2,\dots,x_n\right),t\right) = \min\left\{N_1\left(x_1,t\right),N_2\left(x_2,t\right),\dots,N_n\left(x_n,t\right)\right\}.$$

*Also let $Y_i \subseteq Xi$, $x_i \in X_i$ and $\rho_i \geq 0$ for all $1 \leq i \leq n$. Then*
(2)

$$P_{Y_1}^{fn(\rho_1)}\left(x_1\right) \times P_{Y_2}^{fn(\rho_2)}\left(x_2\right) \times \cdots \times P_{Y_n}^{fn(\rho_n)}\left(x_n\right) \subseteq P_{Y_1 \times Y_2 \times \cdots \times Y_n}^{fn(\max\{\rho_i\})}\left(x_1,x_2,\dots,x_n\right).$$

*Proof.* Let $(y_1,y_2,\dots,y_n) \in P_{Y_1}^{fn(\rho_1)}\left(x_1\right) \times P_{Y_2}^{fn(\rho_2)}\left(x_2\right) \times \cdots \times P_{Y_n}^{fn(\rho_n)}\left(x_n\right)$. Then $y_1 \in P_{Y_1}^{fn(\rho_1)}\left(x_1\right)$, $y_2 \in P_{Y_2}^{fn(\rho_2)}\left(x_2\right)$, $\dots$, $y_n \in P_{Y_n}^{fn(\rho_n)}\left(x_n\right)$. Therefore

$$N_1\left(x_1-y_1,t\right) \geq N_1\left(x_1-z_1,\frac{t}{1+\rho_1}\right) \geq N_1\left(x_1-z_1,\frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right)$$
$$N_2\left(x_2-y_2,t\right) \geq N_2\left(x_2-z_2,\frac{t}{1+\rho_2}\right) \geq N_2\left(x_2-z_2,\frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right)$$
$$\vdots$$
$$N_n\left(x_n-y_n,t\right) \geq N_n\left(x_n-z_n,\frac{t}{1+\rho_n}\right) \geq N_n\left(x_n-z_n,\frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right)$$

for every $z_i \in Y_i$ and $t \in \mathbb{R}$.
From previous lines we get that, for all $z_i \in Yi$,

$$\begin{aligned}\min_{1\leq i\leq n}\{N_i(x_i-y_i,t)\} &\geq \min_{1\leq i\leq n}\left\{N_i\left(x_i-z_i,\frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right)\right\}\\ &= N\left((x_1,\cdots,x_n)-(z_1,\cdots,z_n),\frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right)\end{aligned}$$

which gives

$$\begin{aligned}&N\left((x_1,\cdots,x_n)-(y_1,\cdots,y_n),t\right)\\ &=\min_{1\leq i\leq n}\{N_i(x_i-y_i,t)\}\\ &\geq \sup_{(z_1,z_2,\dots,z_n)\in Y_1\times Y_2\times\cdots\times Y_n}\left\{N\left(\left(x_1,x_2,\dots,x_n\right)-\left(z_1,z_2,\dots,z_n\right),\frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right)\right\}\\ &= N\left(\left(x_1,x_2,\dots,x_n\right)-Y_1\times Y_2\times\cdots\times Y_n,\frac{t}{1+\max_{1\leq i\leq n}\{\rho_i\}}\right).\end{aligned}$$

Hence

$$N\left(\left(x_1,x_2,\ldots,x_n\right)-\left(y_1,y_2,\ldots,y_n\right),t\right)$$
$$\geq N\left(\left(x_1,x_2,\ldots,x_n\right)-Y_1\times Y_2\times\cdots\times Y_n,\frac{t}{1+\max_{1\leq i\leq n}\left\{\rho_i\right\}}\right).$$

□

In the next example, we will show that the converse of inclusion 2 is not true in general.

**Example 4.5.** *Let $X_1 = X_2 = \mathbb{R}$, $Y_1 = Y_2 = [1, 3]$, and $N_i : X_i \times \mathbb{R} \longrightarrow [0, 1]$ is defined by*

$$N_i\left(\alpha,t\right)=\begin{cases}0, & t\leq|\alpha|\\ 1, & t>|\alpha|\end{cases}$$

*for $i = 1, 2$. Assume that $N\left(\left(x, y\right), t\right) = \min\left(N_1\left(x, t\right), N_2\left(y, t\right)\right)$, $\rho = 1$, $x_1 = 0$ and $x_2 = \frac{1}{2}$. Then we have*

$$P_{Y_1}^{fn(1)}\left(0\right)=[1,2]$$
$$P_{Y_2}^{fn(1)}\left(\frac{1}{2}\right)=[1,\frac{3}{2}]$$
$$P_{Y_1}^{fn(1)}\left(0\right)\times P_{Y_2}^{fn(1)}\left(\frac{1}{2}\right)=[1,2]\times[1,\frac{3}{2}].$$

*It is easy to see that*

$$N\left(\left(x,y\right),t\right)=\begin{cases}0, & t\leq\max\left(|x|,|y|\right)\\ 1, & t>\max\left(|x|,|y|\right)\end{cases}$$

*for all $x, y, t \in \mathbb{R}$. Also a sufficient effort can be applied to show that*

$$P_{Y_1\times Y_2}^{fn(1)}\left((0,\frac{1}{2})\right)=[1,2]\times[1,\frac{5}{2}].$$

**Theorem 4.6.** *Let $(X, N_1)$ and $(Y, N_2)$ be fuzzy normed spaces. Also let $B_X$ and $B_Y$ be the bases of $X$ and $Y$ respectively. Define $N : (X \otimes Y) \times \mathbb{R} \longrightarrow [0, 1]$ by*

$$N\left(\sum_{j=1}^{n}\alpha_jx_j\otimes y_j,t\right)=\min\left\{N_1\left(\alpha_jx_j,t\right),N_2\left(\alpha_jy_j,t\right)\middle|1\leq j\leq n\right\},$$

*where $x_j \in B_X, y_j \in B_Y, n \in \mathbb{N}, \alpha_j \in \mathbb{C}$ and $t \in \mathbb{R}$. Then $(X \otimes Y, N)$ is a fuzzy normed space.*

*Proof.* (1) : The proof that N is well-defined is based under the assumption that every $z \in X \otimes Y$ admits a unique representation $z = \sum_{j=1}^{n} \alpha_j x_j \otimes y_j$. Indeed, Let $(z, s) = (w, t) \in (X \otimes Y) \times \mathbb{R}$. So $z = w$ and $s = t$. Hence there exists an $n \in \mathbb{N}, \{x_j\}_{j=1}^n \subseteq B_X, \{y_j\}_{j=1}^n \subseteq B_Y$ and $\alpha_j, \beta_j \in \mathbb{C}$ such that $z = \sum_{j=1}^n \alpha_j x_j \otimes y_j$ and $w = \sum_{j=1}^n \beta_j x_j \otimes y_j$. It follows that $\alpha_j = \beta_j$ for all $1 \leq j \leq n$. Therefore

$$\min \left\{ N_1 \left(\alpha_j x_j, s\right), N_2 \left(\alpha_j y_j, s\right) \middle| 1 \leq j \leq n \right\}$$
$$= \min \left\{ N_1 \left(\beta_j x_j, t\right), N_2 \left(\beta_j y_j, t\right) \middle| 1 \leq j \leq n \right\},$$

providing $N(z, s) = N(w, t)$.

In the sequel, we will prove parts 2, 3, 4 and 5 of Definition 2.1.

(2) : Let $z = \sum_{j=1}^n \alpha_j x_j \otimes y_j$ and $N(z, t) = 1$ for all $t > 0$. So

$$\min \left\{ N_1 \left(\alpha_j x_j, t\right), N_2 \left(\alpha_j y_j, t\right) \middle| 1 \leq j \leq n \right\} = 1$$

for all $t > 0$. It follows that $N_1(\alpha_j x_j, t) = N_2(\alpha_j y_j, t) = 1$ for all $1 \leq j \leq n$ and for all $t > 0$. Hence $\alpha_j x_j = 0$ and $\alpha_j y_j = 0$ for all $1 \leq j \leq n$. Therefore $\alpha_j = 0$ for all $1 \leq j \leq n$. This shows that $z = 0 \otimes 0$. Also for all $t > 0$, since $0 \otimes 0 = 0x \otimes y$ for all $x \in B_X$ and $y \in B_Y$,

$$\begin{aligned} &N(0 \otimes 0, t) \\ &= N(0x \otimes y, t) \\ &= \min\{N_1(0, t), N_2(0, t)\} \\ &= \min\{1\} \\ &= 1. \end{aligned}$$

(3) : Let $c \neq 0$ and $\sum_{j=1}^n \alpha_j x_j \otimes y_j \in X \otimes Y$. So

$$\begin{aligned} &N\left(c \sum_{j=1}^n \alpha_j x_j \otimes y_j, t\right) \\ &= N\left(\sum_{j=1}^n c\alpha_j x_j \otimes y_j, t\right) \\ &= \min\left\{N_1\left(c\alpha_j x_j, t\right), N_2\left(c\alpha_j y_j, t\right) \middle| 1 \leq j \leq n\right\} \\ &= \min\left\{N_1\left(\alpha_j x_j, \frac{t}{|c|}\right), N_2\left(\alpha_j y_j, \frac{t}{|c|}\right) \middle| 1 \leq j \leq n\right\} \\ &= N\left(\sum_{j=1}^n \alpha_j x_j \otimes y_j, \frac{t}{|c|}\right), \end{aligned}$$

for all $t \in \mathbb{R}$.

(4) : Let $z, w \in X \otimes Y$ and $s, t \in \mathbb{R}$. Without loss of generality we can assume

that $z = \sum_{j=1}^{n} \alpha_j x_j \otimes y_j$ and $w = \sum_{j=1}^{n} \beta_j x_j \otimes y_j$, where $n \in \mathbb{N}, \alpha_j, \beta_j \in \mathbb{C}, \{x_j\}_{j=1}^{n} \subseteq B_X$ and $\{y_j\}_{j=1}^{n} \subseteq B_Y$. Indeed, if $z = \sum_{p=1}^{m_1} \alpha'_p x'_p \otimes y'_p$ and $w = \sum_{q=1}^{m_2} \alpha''_q x''_q \otimes y''_q$, then $X_1 = \operatorname{span}\left\{x'_p, x''_q \mid 1 \le p \le m_1, 1 \le q \le m_2\right\}$ and $Y_1 = \operatorname{span}\left\{y'_p, y''_q \mid 1 \le p \le m_1, 1 \le q \le m_2\right\}$ are finite dimensional vector spaces. So $X_1 \otimes Y_1$ is a finite dimensional subspace of $X \otimes Y$, with the basis $B_{X_1 \otimes Y_1} \subseteq B_X \otimes B_Y$. Since $z, w \in X_1 \otimes Y_1$, $z = \sum_{j=1}^{n} \alpha_j x_j \otimes y_j$ and $w = \sum_{j=1}^{n} \beta_j x_j \otimes y_j$, where $n = \dim X_1 \otimes Y_1, \alpha_j, \beta_j \in \mathbb{C}, \{x_j \otimes y_j\}_{j=1}^{n} \subseteq B_{X_1 \otimes Y_1} \subseteq B_X \otimes B_Y$.
Hence

$$
\begin{aligned}
&N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j + \sum_{j=1}^{n} \beta_j x_j \otimes y_j, s+t\right) \\
&= N\left(\sum_{j=1}^{n} (\alpha_j + \beta_j) x_j \otimes y_j, s+t\right) \\
&= \min\left\{N_1\left((\alpha_j + \beta_j) x_j, s+t\right), N_2\left((\alpha_j + \beta_j) y_j, s+t\right) \middle| 1 \le j \le n\right\} \\
&= \min\left\{N_1\left(\alpha_j x_j + \beta_j x_j, s+t\right), N_2\left(\alpha_j y_j + \beta_j y_j, s+t\right) \middle| 1 \le j \le n\right\} \\
&\ge \min\left\{\min\left(N_1\left(\alpha_j x_j, s\right), N_1\left(\beta_j x_j, t\right)\right), \min\left(N_2\left(\alpha_j y_j, s\right), N_2\left(\beta_j y_j, t\right)\right) \middle| 1 \le j \le n\right\} \\
&= \min\left\{\min\left(A_j, B_j\right), \min\left(C_j, D_j\right) \middle| 1 \le j \le n\right\} \\
&= \min\left\{\min\left\{A_j, C_j \middle| 1 \le j \le n\right\}, \min\left\{B_j, D_j \middle| 1 \le j \le n\right\}\right\} \\
&= \min\left\{N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j, s\right), N\left(\sum_{j=1}^{n} \beta_j x_j \otimes y_j, t\right)\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
A_j &= N_1\left(\alpha_j x_j, s\right), \\
B_j &= N_1\left(\beta_j x_j, t\right), \\
C_j &= N_2\left(\alpha_j y_j, s\right), \\
D_j &= N_2\left(\beta_j y_j, t\right).
\end{aligned}
$$

$(5)$ : Let $s < t$ and $z = \sum_{j=1}^{n} \alpha_j x_j \otimes y_j$. Clearly $N_1(\alpha_j x_j, s) \leq N_1(\alpha_j x_j, t)$ and $N_2(\alpha_j y_j, s) \leq N_2(\alpha_j y_j, t)$ for all $1 \leq j \leq n$. So

$$\begin{aligned} &\min\left\{ N_1(\alpha_j x_j, s), N_2(\alpha_j y_j, s) \middle| 1 \leq j \leq n \right\} \\ \leq &\min\left\{ N_1(\alpha_j x_j, t), N_2(\alpha_j y_j, t) \middle| 1 \leq j \leq n \right\}. \end{aligned}$$

It follows that

$$N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j, s\right) \leq N\left(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j, t\right).$$

Hence $N(z, .) : \mathbb{R} \longrightarrow [0,1]$ is increasing for all $z \in X \otimes Y$. Also

$$\begin{aligned} &\lim_{t \longrightarrow \infty} N(z,t) \\ &= \lim_{t \longrightarrow \infty} \min\left\{ N_1(\alpha_j x_j, t), N_2(\alpha_j y_j, t) \middle| 1 \leq j \leq n \right\} \\ &= \min\left\{ \lim_{t \longrightarrow \infty} N_1(\alpha_j x_j, t), \lim_{t \longrightarrow \infty} N_2(\alpha_j y_j, t) \middle| 1 \leq j \leq n \right\} \\ &= \min\{1\} \\ &= 1. \end{aligned}$$

□

**Theorem 4.7.** *Let $X$ and $Y$ be linear spaces and $N : (X \otimes Y) \times \mathbb{R} \longrightarrow [0,1]$ be a fuzzy norm. Then for all $x \in X \setminus \{0\}$ and $y \in Y \setminus \{0\}$, the maps $N_x : Y \times \mathbb{R} \longrightarrow [0,1]$ and $N_y : X \times \mathbb{R} \longrightarrow [0,1]$, where $N_x(z,t) = N(x \otimes z, t)$ and $N_y(w,t) = N(w \otimes y, t)$, are fuzzy norms on $Y$ and $X$ respectively.*

*Proof.* We only prove that $N_x : Y \times \mathbb{R} \longrightarrow [0,1]$ is a fuzzy norm for all $x \in X \setminus \{0\}$.
$(1)$ : Let $z \in Y$ and $t \leq 0$. So $N_x(z,t) = N(x \otimes z, t) = 0$.
$(2)$ : If $z = 0$, then $N_x(0,t) = N(x \otimes 0, t) = N(0 \otimes 0, t) = 1$ for all $t > 0$. Also if $N_x(z,t) = 1$ for all $t > 0$, then $N(x \otimes z, t) = 1$ for all $t > 0$. It follows that $x \otimes z = 0 \otimes 0$. Since $x \neq 0$, there exists $f \in X'$ such that $f(x) \neq 0$. Let $g \in Y'$ be an arbitrary element. As $x \otimes z$ is a bilinear map on $X' \times Y'$, $(x \otimes z)(f,g) = (0 \otimes 0)(f,g)$. So $f(x)\,g(z) = 0$ for all $g \in Y'$. Since $f(x) \neq 0$, $g(z) = 0$ for all $g \in Y'$. It follows that $z = 0$.

(3) : Let $c \neq 0$ and $z \in Y$. So

$$\begin{aligned} & N_x\left(cz,t\right) \\ =& N\left(x\otimes cz,t\right) \\ =& N\left(cx\otimes z,t\right) \\ =& N\left(x\otimes z,\frac{t}{|c|}\right) \\ =& N_x\left(z,\frac{t}{|c|}\right) \end{aligned}$$

for all $z \in Y$ and $t \in \mathbb{R}$.
(4) : Let $z_1, z_2 \in Y$ and $s, t \in \mathbb{R}$. So

$$\begin{aligned} & N_x\left(z_1+z_2,s+t\right) \\ =& N\left(x\otimes\left(z_1+z_2\right),s+t\right) \\ =& N\left(x\otimes z_1+x\otimes z_2,s+t\right) \\ \geq& \min\left(N\left(x\otimes z_1,s\right),N\left(x\otimes z_2,t\right)\right) \\ =& \min\left(N_x\left(z_1,s\right),N_x\left(z_2,t\right)\right). \end{aligned}$$

(5) : Let $s < t$ and $z \in Y$. So $N\left(x\otimes z,s\right) \leq N\left(x\otimes z,t\right)$. It follows that $N_x\left(z,s\right) \leq N_x\left(z,t\right)$. Hence $N_x\left(z,.\right) : \mathbb{R} \longrightarrow [0,1]$ is increasing and

$$\begin{aligned} & \lim_{t\longrightarrow\infty} N_x\left(z,t\right) \\ =& \lim_{t\longrightarrow\infty} N\left(x\otimes z,t\right) \\ =& 1 \end{aligned}$$

for all $z \in Y$. Therefore $N_x : Y \times \mathbb{R} \longrightarrow [0,1]$ is a fuzzy norm. □

**Example 4.8.** *Let $X = Y = \mathbb{R}^2$ be linear spaces over $\mathbb{R}$ with the bases $B_X = B_Y = \{e_1 = (1,0), e_2 = (0,1)\}$. Also let $N_1 = N_2 : \mathbb{R}^2 \times \mathbb{R} \longrightarrow [0,1]$ be defined by*

$$N_i\left(\sum_{j=1}^{2}\alpha_j e_j,t\right) = \begin{cases} 0, & t \leq \max\left(\left|\alpha_1\right|,\left|\alpha_2\right|\right) \\ 1, & t > \max\left(\left|\alpha_1\right|,\left|\alpha_2\right|\right) \end{cases}$$

*for $i = 1,2$. Clearly $N_1$ and $N_2$ are fuzzy norms on $\mathbb{R}^2$. According to Theorem 4.6, if $N : (X \otimes Y) \times \mathbb{R} \longrightarrow [0,1]$ is defined by*

$$\begin{aligned} & N\left(\alpha_1 e_1\otimes e_1+\alpha_2 e_1\otimes e_2+\alpha_3 e_2\otimes e_1+\alpha_4 e_2\otimes e_2,t\right) \\ &= \min\begin{pmatrix} N_1\left(\alpha_1 e_1,t\right), & N_1\left(\alpha_2 e_1,t\right), & N_1\left(\alpha_3 e_2,t\right), & N_1\left(\alpha_4 e_2,t\right) \\ N_2\left(\alpha_1 e_1,t\right), & N_2\left(\alpha_2 e_2,t\right), & N_2\left(\alpha_3 e_1,t\right), & N_2\left(\alpha_4 e_2,t\right) \end{pmatrix}, \end{aligned}$$

*then the equalities*

$$\begin{aligned} N_2(\alpha_1 e_1, t) &= N_1(\alpha_1 e_1, t), \\ N_2(\alpha_2 e_2, t) &= N_1(\alpha_2 e_1, t), \\ N_2(\alpha_3 e_1, t) &= N_1(\alpha_3 e_2, t), \\ N_2(\alpha_4 e_2, t) &= N_1(\alpha_4 e_2, t) \end{aligned}$$

*implies,*

$$\begin{aligned} & N(\alpha_1 e_1 \otimes e_1 + \alpha_2 e_1 \otimes e_2 + \alpha_3 e_2 \otimes e_1 + \alpha_4 e_2 \otimes e_2, t) \\ &= \min\{N_1(\alpha_1 e_1, t), N_1(\alpha_2 e_1, t), N_1(\alpha_3 e_2, t), N_1(\alpha_4 e_2, t)\} \\ &= \begin{cases} 0, & t \le \max(|\alpha_i|, 1 \le i \le 4) \\ 1, & t > \max(|\alpha_i|, 1 \le i \le 4). \end{cases} \end{aligned}$$

*Let* $x_0 = -\frac{1}{5}e_1, y_0 = -2e_2, K_1 = B_X, K_2 = B_Y, K = B_{X\otimes Y} = B_X \otimes B_Y$ *and* $\rho = \frac{1}{10}$. *Also let* $P_{K_1}^{fn\left(\frac{1}{10}\right)}(x_0)$ *and* $P_{K_2}^{fn\left(\frac{1}{10}\right)}(y_0)$ *be the set of all fuzzy near best approximations to* $x_0$ *and* $y_0$ *within the relative distance* $\rho = \frac{1}{10}$ *with respect to the fuzzy norms* $N_1$ *and* $N_2$ *respectively. If* $P_{K}^{fn\left(\frac{1}{10}\right)}(x_0 \otimes y_0)$ *is the set of all fuzzy near best approximations to* $x_0 \otimes y_0$ *within the relative distance* $\rho = \frac{1}{10}$ *with respect to the fuzzy norm* $N$*, then a straightforward calculation reveals that*

$$\begin{aligned} P_{K_1}^{fn\left(\frac{1}{10}\right)}(x_0) &= \{e_2\} \\ P_{K_2}^{fn\left(\frac{1}{10}\right)}(y_0) &= \{e_1\} \\ P_{K}^{fn\left(\frac{1}{10}\right)}(x_0 \otimes y_0) &= \{e_1 \otimes e_2\} \neq \{e_2 \otimes e_1\}. \end{aligned}$$

*This example shows that there is no relation between* $P_{K}^{fn\left(\frac{1}{10}\right)}(x_0 \otimes y_0)$ *and* $P_{K_1}^{fn\left(\frac{1}{10}\right)}(x_0) \otimes P_{K_2}^{fn\left(\frac{1}{10}\right)}(y_0)$.

## 5. Conclusion

In this paper some basic properties of a generalized notion of fuzzy best approximation are presented. Many hereditary properties concerning this notion are given. Also many examples for illustration are presented. In the future, we will look for deeper theoretical results and more detailed applications of this concept.

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