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ASYMMETRIC SMOOTH TRANSITION AUTOREGRESSIVE MODEL IN FORECASTING FINANCE RATE ON CONSUMER INSTALLMENT LOANS AT COMMERCIAL BANKS

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Abstract. Economic and finance time series are typically asymmetric and are expected to be modeled using asymmetric nonlinear time series models. The logistic smooth transition autoregressive, LSTAR, model which is an asymmetric type of the smooth transition autoregressive, is becoming popular in modeling economic and financial time series.
In this paper, we have considered the logistic smooth transition autoregressive model and have estimated unknown parameters based on the method of moment and modified maximum likelihood method. The performance of the proposed estimation methods is studied by simulation and is compared with the performance of maximum likelihood estimators. It shows that for large sample sizes, the modified maximum likelihood estimators usually have the lowest mean square error and bias.
We proposed a LSTAR model to finance rate on consumer installment loans at commercial banks and conclude that the estimated LSTAR model based on the modified maximum likelihood method has the lowest value of MSE.

Keywords: Asymmetric model, LSTAR model, Modified maximum likelihood, Method of moment, Parameter estimation.
2020 MSC: 62F10, 62M10.

1. Introduction

Nonlinear time series models have recently found widespread application in the analysis of economics and finance time series. One of the nonlinear time series models is the smooth transition autoregressive, STAR, model that is able to capture the movement of some economic variables, which adjusts every moment due to the behavior of economic agents. Chan and Tong [3] introduced the smooth transition autoregressive model into time series and used the cumulative function of the standard normal variable as the transition function in the model. Terasvirta [11] considered the specification, estimation and evaluation of the STAR model.

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The smooth transition autoregressive model implies the existence of nonlinear behavior in the time series. The logistic autoregressive model and exponential autoregressive model are most popular specifications of the STAR model which have asymmetric and symmetric properties, respectively. The logistic smooth transition autoregressive model, LSTAR, is characterized by the asymmetric properties which make it suitable for modeling specific economics and financial time series, see Yaya and Shitu [14].

The LSTAR model has been successfully applied by Terasvirta and Anderson [12] to characterize the different dynamics of industrial production indexes in a number of OECD countries during expansions and recessions. Feissolle [6] proposed the Bayesian estimation for nonlinear model by means of Monte Carlo integration with importance sampling and applied it to the LSTAR model with an artificial sample. Chan and McAleer [1] investigated the finite sample properties of maximum likelihood estimation of STAR and STAR-GARCH models through numerical simulation. Lopes and Salazar [7] proposed a Bayesian approach to the logistic smooth transition autoregressive model based on the novel reversible jump Markov chain Monte Carlo (RJMCMC) algorithm.

In nonlinear models such as STAR models, parameter estimation is not entirely straightforward. Calculation of maximum likelihood estimators of STAR model parameters can be problematic due to computational difficulties. So likelihood functions are estimated using the numerical methods. Dijk et al. [5] show that the convergence of the maximum likelihood estimator for STAR models is sensitive to initial values. The nature of the numerical difficulties using Monte Carlo simulation is studied by Chan and Theoharakis [2]. They show that the conventional optimization algorithms do not perform well in locating the global optimum of the associated likelihood function. Schleer [9] studied the starting-values for the estimation of vector STAR models based on a Monte Carlo. Midilic [8] presented a potential solution to these problems by using iteratively weighted least squares, IWLS, and compared its performance with other established algorithms. Saputro et al. [10] estimated logistic smooth transition autoregressive model parameter based on the Gauss-Newton method. The estimation of logistic smooth transition autoregressive parameter using Gauss-Newton method is an algorithm to minimize the sum of squared residue.

Tiku [13] developed the modified maximum likelihood method based on the linearization of intractable terms of the log-likelihood function using first-order Taylor series expansion and applied it to some non-normal time series models. Chung [4] obtained the modified maximum likelihood estimates by locating an optimal solution in the reduced parameter space and stated conditions under which the modified maximum likelihood method gives consistent and asymptotically normal estimators. Zamani and Sayyareh [15] have considered the first-order autoregressive model, where residual terms follow Exponential or Weibull family. They have derived modified maximum likelihood estimators of unknown parameters and have computed asymptotic distribution of modified

maximum likelihood estimators in both stationary and non-stationary models. In this paper, the logistic smooth transition autoregressive model is considered. Since the likelihood equations of this model have a nonlinear form and solving these equations by numerical and optimization methods are sensitive to initial values, therefore the method of moment and modified maximum likelihood method are proposed.
The rest of the paper is structured as follows: In Section 2, we estimate the parameters of the LSTAR model based on the method of moment and modified maximum likelihood method. In Section 3, the performance of the proposed estimation methods is studied by simulation. We illustrated our theoretical results with the analysis of a real dataset in Section 4.

2. LSTAR Model and Estimation Methods

The LSTAR model is defined as follows

$$\begin{aligned} y_t &= \alpha_0 + \alpha_1 y_{t-1} + (\beta_0 + \beta_1 y_{t-1}) \frac{1}{1+\exp\{-\gamma(y_{t-d}-c)\}} + \epsilon_t \\ &= \alpha_0 + \alpha_1 y_{t-1} + \beta_0 z_{t-d} + \beta_1 y_{t-1} z_{t-d} + \epsilon_t, \end{aligned} \tag{1}$$

where

$$z_{t-d} = \frac{1}{1+\exp\{-\gamma(y_{t-d}-c)\}},$$

y_t is the variable of interest, $\alpha = (\alpha_0, \alpha_1)$ and $\beta = (\beta_0, \beta_1)$ are autoregressive parameters and z_{t-d} denotes a continuous function, usually bounded between 0 and 1 and thus allowing for a smooth transition between regimes. The transition function z_{t-d} causes the nonlinear dynamics in the model. ϵ_t's are independent of y_{t-k} and z_{t-d}. The transition parameter $\{y_{t-d}, \gamma, c\}$ is a slope of parameter that determines the speed of transition between the two extreme regimes with low absolute values resulting in slower transition, where y_{t-d} is the transition variable, $\gamma > 0$ is a slope parameter and c is a location parameter. The value of $d (d > 0)$ is varied in order to improve nonlinearity in the system when it is not known prior to model estimation, see Terasvirta [11]. The ϵ_t's are independent and identical distributed random variables that follow the normal distribution, $N(0, \sigma^2)$. In this section, we estimate unknown parameters based on the method of moment and modified maximum likelihood method.

2.1. Parameter Estimation based on the Method of Moments.

One of the oldest method of parameter estimation is the method of moments, MM. Although MMEs may not be the best estimators, they almost always produce some asymptotically unbiased and consistent estimators. Consider the true model (1) and let $\mu_{y,j} = E\left\{Y^j\right\}$ be the jth moment and let $\tilde{\mu}_{y,j} = \frac{1}{n}\sum_{t=1}^{n} y_t^j$ be the jth sample moment, which is an unbiased estimator of $\mu_{y,j}$. We consider the usual moment conditions as in the case of the Yule-Walker estimators and write:

$$
\begin{aligned}
E\{Y_tY_{t-k}\} &= \alpha_0 E\{Y_{t-k}\} + \alpha_1 E\{Y_{t-1}Y_{t-k}\} + \beta_0 E\{Z_{t-d}Y_{t-k}\} \\
&\quad + \beta_1 E\{Y_{t-1}Z_{t-d}Y_{t-k}\} + E\{\epsilon_t Y_{t-k}\}
\end{aligned} \tag{2}
$$

We linearize the function z_{t-d} using the Taylor expansion of z_{t-d} about $\gamma = 0$. So we have,

$$
z_{t-d} \cong \frac{1}{2} + \frac{\gamma}{4}(y_{t-d} - c), \tag{3}
$$

where $\cong$ indicates asymptotic equivalence. Substituted (3) in (2) and obtain $E\{Y_tY_{t-k}\}$ as follows

$$
\sigma(k) = \alpha_0\mu_y + \alpha_1\sigma(k-1) + \beta_0\left(\frac{\mu_y}{2} + \frac{\gamma}{4}\Gamma(d-k)\right) + \beta_1\left(\frac{\sigma(k-1)}{2} + \frac{\gamma}{4}E\{Y_{t-1}(Y_{t-d}-c)Y_{t-k}\}\right)
$$

where $\sigma(k) = E\{Y_tY_{t-k}\}$ and $\Gamma(k) = E\{(Y_t - \mu_y)(Y_{t-k} - \mu_y)\}$. For $k = 1, \cdots, 4$, the system of equations can be calculated as follows

$$
\begin{aligned}
\sigma(1) &= \alpha_0\mu_y + \alpha_1\sigma(0) + \beta_0\left(\frac{\mu_y}{2} + \frac{\gamma}{4}\Gamma(d-1)\right) + \beta_1\left(\frac{\sigma(0)}{2} + \frac{\gamma}{4}E\{Y_{t-1}(Y_{t-d}-c)Y_{t-1}\}\right) \\
\sigma(2) &= \alpha_0\mu_y + \alpha_1\sigma(1) + \beta_0\left(\frac{\mu_y}{2} + \frac{\gamma}{4}\Gamma(d-2)\right) + \beta_1\left(\frac{\sigma(1)}{2} + \frac{\gamma}{4}E\{Y_{t-1}(Y_{t-d}-c)Y_{t-2}\}\right) \\
\sigma(3) &= \alpha_0\mu_y + \alpha_1\sigma(2) + \beta_0\left(\frac{\mu_y}{2} + \frac{\gamma}{4}\Gamma(d-3)\right) + \beta_1\left(\frac{\sigma(2)}{2} + \frac{\gamma}{4}E\{Y_{t-1}(Y_{t-d}-c)Y_{t-3}\}\right) \\
\sigma(4) &= \alpha_0\mu_y + \alpha_1\sigma(3) + \beta_0\left(\frac{\mu_y}{2} + \frac{\gamma}{4}\Gamma(d-4)\right) + \beta_1\left(\frac{\sigma(3)}{2} + \frac{\gamma}{4}E\{Y_{t-1}(Y_{t-d}-c)Y_{t-4}\}\right).
\end{aligned}
$$

This system of equations can be rewritten in matrix notation $\sigma = \Sigma\theta$, where

$$
\sigma = \begin{pmatrix} \sigma(1) \\ \sigma(2) \\ \sigma(3) \\ \sigma(4) \end{pmatrix}, \Sigma = \begin{pmatrix}
\mu_y & \sigma(0) & \frac{\mu_y}{2} + \frac{\gamma}{4}\Gamma(d-1) & \frac{\sigma(0)}{2} + \frac{\gamma}{4}E\{Y_{t-1}(Y_{t-d}-c)Y_{t-1}\} \\
\mu_y & \sigma(1) & \frac{\mu_y}{2} + \frac{\gamma}{4}\Gamma(d-2) & \frac{\sigma(1)}{2} + \frac{\gamma}{4}E\{Y_{t-1}(Y_{t-d}-c)Y_{t-2}\} \\
\mu_y & \sigma(2) & \frac{\mu_y}{2} + \frac{\gamma}{4}\Gamma(d-3) & \frac{\sigma(2)}{2} + \frac{\gamma}{4}E\{Y_{t-1}(Y_{t-d}-c)Y_{t-3}\} \\
\mu_y & \sigma(3) & \frac{\mu_y}{2} + \frac{\gamma}{4}\Gamma(d-4) & \frac{\sigma(3)}{2} + \frac{\gamma}{4}E\{Y_{t-1}(Y_{t-d}-c)Y_{t-4}\}
\end{pmatrix}
$$

$$
\theta = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix}.
$$

So $\tilde{\theta} = \tilde{\Sigma}^{-1}\tilde{\sigma}$, where $\tilde{\theta}$, $\tilde{\Sigma}$ and $\tilde{\sigma}$ are MM estimators of θ, Σ and σ, respectively. Similarly, the relationship between the variance of y_t and the variance of the residuals is

$$
Var(Y_t) = Var(\theta'\mathbf{Y}) + Var(\epsilon_t) = \theta' Var(\mathbf{Y})\theta + Var(\epsilon_t) = \theta'\Sigma_{\mathbf{y}}\theta + Var(\epsilon_t).
$$

Therefore

$$
\tilde{\sigma}^2_\epsilon = \tilde{\sigma}^2_{\mathbf{y}} - \tilde{\theta}'\tilde{\Sigma}_{\mathbf{y}}\tilde{\theta},
$$

where $\mathbf{y} = \begin{pmatrix} 1 \\ y_{t-1} \\ z_{t-d} \\ y_{t-1}z_{t-d} \end{pmatrix}$ and $\Sigma_{\mathbf{y}}$ is the variance and covariance matrix of $\mathbf{y}$. The estimator of parameters c and γ can be calculated using the Taylor expansion (3) as follows:

$$E(Z_{t-d}) \cong \frac{1}{2} + \frac{\gamma}{4}E(Y_{t-d} - c),$$

so

$$\bar{y} = \tilde{E}\left\{Y_{t-d}\right\} = \tilde{c}.$$

Also,

$$Var(Z_{t-d}) = \frac{\gamma^2}{16}Var(Y_{t-d}),$$

so

$$\tilde{\gamma} = \frac{2}{s\sqrt{3}}$$

where $\bar{y}$ is the sample mean and s^2 is the sample variance of Y_{t-d}. Note that, Z_{t-d} denotes the logistic distribution, so it has a uniform distribution $U(0,1)$ based on the probability integral transformation.

2.2. **Modified Maximum Likelihood Estimator of the LSTAR Model.** Consider the true model (1), where ϵ_t's are independent and identical distributed random variables that follow the normal distribution, $N(0, \sigma^2)$. According to the condition of independence of the residuals, the joint probability density function, pdf, can be written as the product of the marginal density function. Thus the joint pdf of $Y_{d+1}, ..., Y_n$ given $Y_1 = y_1, \cdots, Y_d = y_d$ is

$$f^{\theta}(y_{d+1}, ..., y_n | Y_1 = y_1, \cdots, Y_d = y_d) =$$
$$\left(2\pi\sigma^2\right)^{-\frac{n-d}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{t=d+1}^{n}\left(y_t - \alpha_0 - \alpha_1 y_{t-1} - (\beta_0 + \beta_1 y_{t-1})\frac{1}{1+\exp\left\{-\gamma(y_{t-d} - c)\right\}}\right)^2\right\},$$

where $\theta = \left(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma, c, \sigma^2\right)$. To calculate the maximum likelihood estimates, we obtain the log-likelihood of $Y_{d+1}, \cdots, y_n$ conditional on $Y_1 = y_1, \cdots, Y_d = y_d$ as:

$$l(\theta) = -\frac{(n-d)}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{t=d+1}^{n}\left(y_t - \alpha_0 - \alpha_1 y_{t-1} - (\beta_0 + \beta_1 y_{t-1})\frac{1}{1+\exp\left\{-\gamma(y_{t-d} - c)\right\}}\right)^2.$$

The maximum likelihood estimators are obtained by solving the estimating equations

$$\frac{\partial l(\theta)}{\partial \sigma^2} = -\frac{(n-d)}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=d+1}^{n}\left(y_t - \alpha_0 - \alpha_1 y_{t-1} - (\beta_0 + \beta_1 y_{t-1})\frac{1}{1+\exp\left\{-\gamma(y_{t-d} - c)\right\}}\right)^2 = 0$$

$$\frac{\partial l(\theta)}{\partial \alpha_0} = \frac{1}{\sigma^2}\sum_{t=d+1}^{n}\left(y_t - \alpha_0 - \alpha_1 y_{t-1} - (\beta_0 + \beta_1 y_{t-1})\frac{1}{1+\exp\left\{-\gamma(y_{t-d} - c)\right\}}\right) = 0$$

$$\frac{\partial l(\theta)}{\partial \alpha_1} = \frac{1}{\sigma^2}\sum_{t=d+1}^{n} y_{t-1}\left(y_t - \alpha_0 - \alpha_1 y_{t-1} - (\beta_0 + \beta_1 y_{t-1})\frac{1}{1+\exp\left\{-\gamma(y_{t-d} - c)\right\}}\right) = 0$$

$$\frac{\partial l(\theta)}{\partial \beta_0} = \frac{1}{\sigma^2}\sum_{t=d+1}^{n}\left(\frac{1}{1+\exp\{-\gamma(y_{t-d}-c)\}}\right)\left(y_t-\alpha_0-\alpha_1 y_{t-1}-(\beta_0+\beta_1 y_{t-1})\frac{1}{1+\exp\{-\gamma(y_{t-d}-c)\}}\right)=0$$

$$\frac{\partial l(\theta)}{\partial \beta_1} = \frac{1}{\sigma^2}\sum_{t=d+1}^{n}y_{t-1}\left(\frac{1}{1+\exp\{-\gamma(y_{t-d}-c)\}}\right)\left(y_t-\alpha_0-\alpha_1 y_{t-1}-(\beta_0+\beta_1 y_{t-1})\frac{1}{1+\exp\{-\gamma(y_{t-d}-c)\}}\right)=0$$

$$\begin{aligned}\frac{\partial l(\theta)}{\partial \gamma} &= -\frac{1}{\sigma^2}\sum_{t=d+1}^{n}(\beta_0+\beta_1 y_{t-1})\,(y_{t-d}-c)(y_t-\alpha_0-\alpha_1 y_{t-1})\exp\{-\gamma(y_{t-d}-c)\}\\ &\times(1+\exp\{-\gamma(y_{t-d}-c)\})^{-2}\\ &+\frac{1}{\sigma^2}\sum_{t=d+1}^{n}(\beta_0+\beta_1 y_{t-1})^2\,(y_{t-d}-c)\exp\{-\gamma(y_{t-d}-c)\}\,(1+\exp\{-\gamma(y_{t-d}-c)\})^{-3}\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\partial l(\theta)}{\partial c} &= \frac{\gamma}{\sigma^2}\sum_{t=d+1}^{n}(\beta_0+\beta_1 y_{t-1})\,(y_t-\alpha_0-\alpha_1 y_{t-1})\exp\{-\gamma(y_{t-d}-c)\}\,(1+\exp\{-\gamma(y_{t-d}-c)\})^{-2}\\ &-\frac{\gamma}{\sigma^2}\sum_{t=d+1}^{n}(\beta_0+\beta_1 y_{t-1})^2\exp\{-\gamma(y_{t-d}-c)\}\,(1+\exp\{-\gamma(y_{t-d}-c)\})^{-3}.\end{aligned} \tag{5}$$

Since explicit solutions from the likelihood Equations (4 and 5) cannot be obtained, so the modified maximum likelihood method used to estimate unknown parameters. Define

$$F_1(\gamma,c,x_{t-d})=\exp\{-\gamma x_{t-d}\}\,(1+\exp\{-\gamma x_{t-d}\})^{-2}$$

and

$$F_2(\gamma,c,x_{t-d})=\exp\{-\gamma x_{t-d}\}\,(1+\exp\{-\gamma x_{t-d}\})^{-3}$$

where $x_{t-d}=y_{t-d}-c$. The function $F_1(\gamma,c,x_{t-d})$ can be written as a linear function using Taylor's expansion of $F_1(\gamma,c,x_{t-d})$ about $\gamma=0$, in other words

$$\begin{aligned}F_1(\gamma,c,x_{t-d}) &\cong F_1(\gamma,c,x_{t-d})\mid_{\gamma=0}+F_1^{'}(\gamma,c,x_{t-d})\mid_{\gamma=0}(\gamma-0)\\ &=\exp\{-\gamma x_{t-d}\}\,(1+\exp\{-\gamma x_{t-d}\})^{-2}\mid_{\gamma=0}\\ &-\left(x_{t-d}\exp\{-\gamma x_t\}\,(1+\exp\{-\gamma x_{t-d}\})^{-2}+2x_{t-d}\exp\{-2\gamma x_{t-d}\}\,(1+\exp\{-\gamma x_{t-d}\})^{-3}\right)\mid_{\gamma=0}(\gamma-0)\\ &=\frac{1}{4}-\left(\frac{x_{t-d}}{4}-\frac{2x_{t-d}}{8}\right)\gamma=\frac{1}{4}\end{aligned} \tag{6}$$

and similarly

$$\begin{aligned}F_2(\gamma,c,x_{t-d}) &\cong F_2(\gamma,c,x_{t-d})\mid_{\gamma=0}+F_2^{'}(\gamma,c,x_{t-d})\mid_{\gamma=0}(\gamma-0)\\ &=\exp\{-\gamma x_{t-d}\}\,(1+\exp\{-\gamma x_{t-d}\})^{-3}\mid_{\gamma=0}\\ &-\left(x_{t-d}\exp\{-\gamma x_{t-d}\}\,(1+\exp\{-\gamma x_{t-d}\})^{-3}-3x_{t-d}\exp\{-2\gamma x_{t-d}\}\,(1+\exp\{-\gamma x_{t-d}\})^{-4}\right)\mid_{\gamma=0}(\gamma-0)\\ &=\frac{1}{8}-\left(\frac{x_{t-d}}{8}-\frac{3x_{t-d}}{16}\right)(\gamma-0)=\frac{1}{8}+\frac{\gamma}{16}x_{t-d}.\end{aligned} \tag{7}$$

The modified maximum likelihood estimators are calculated by substituting Equations (6 and 7) in Equations (4 and 5) as follows

$$\hat{\sigma}^2 = \frac{1}{n-d}\sum_{t=d+1}^{n}\left(y_t - \hat{\alpha}_0 - \hat{\alpha}_1 y_{t-1} - (\hat{\beta}_0 + \hat{\beta}_1 y_{t-1})\frac{1}{1+\exp\{-\hat{\gamma}(y_{t-d}-\hat{c})\}}\right)^2$$

$$\hat{\alpha}_0 = \frac{1}{n-d}\sum_{t=d+1}^{n}\left(y_t - \hat{\alpha}_1 y_{t-1} - (\hat{\beta}_0 + \hat{\beta}_1 y_{t-1})\frac{1}{1+\exp\{-\hat{\gamma}(y_{t-d}-\hat{c})\}}\right)$$

$$\hat{\alpha}_1 = \frac{\sum_{t=d+1}^{n} y_{t-1}\left(y_t - \hat{\alpha}_0 - (\hat{\beta}_0 + \hat{\beta}_1 y_{t-1})\frac{1}{1+\exp\{-\hat{\gamma}(y_{t-d}-\hat{c})\}}\right)}{\sum_{t=d+1}^{n} y_{t-1}^2}$$

$$\hat{\beta}_0 = \frac{\sum_{t=d+1}^{n}\left(1+\exp\{-\hat{\gamma}(y_{t-d}-\hat{c})\}\right)^{-1}\left(y_t - \hat{\alpha}_0 - \hat{\alpha}_1 y_{t-1} - \frac{\hat{\beta}_1 y_{t-1}}{1+\exp\{-\hat{\gamma}(y_{t-d}-\hat{c})\}}\right)}{\sum_{t=d+1}^{n}\left(1+\exp\{-\hat{\gamma}(y_{t-d}-\hat{c})\}\right)^{-2}}$$

$$\hat{\beta}_1 = \frac{\sum_{t=d+1}^{n}\left(1+\exp\{-\hat{\gamma}(y_{t-d}-\hat{c})\}\right)^{-1} y_{t-1}\left(y_t - \hat{\alpha}_0 - \hat{\alpha}_1 y_{t-1} - \frac{\hat{\beta}_0}{1+\exp\{-\hat{\gamma}(y_{t-d}-\hat{c})\}}\right)}{\sum_{t=d+1}^{n}\left(1+\exp\{-\hat{\gamma}(y_{t-d}-\hat{c})\}\right)^{-2} y_{t-1}^2}$$

$$\hat{\gamma} = \frac{\frac{1}{8}\sum_{t=d+1}^{n}(\hat{\beta}_0+\hat{\beta}_1 y_{t-1})^2(y_{t-d}-\hat{c}) - \frac{1}{4}\sum_{t=d+1}^{n}(y_t-\hat{\alpha}_0-\hat{\alpha}_1 y_{t-1})(\hat{\beta}_0+\hat{\beta}_1 y_{t-1})(y_{t-d}-\hat{c})}{\frac{1}{16}\sum_{t=d+1}^{n}(\hat{\beta}_0+\hat{\beta}_1 y_{t-1})^2(y_{t-d}-\hat{c})^2}$$

$$\hat{c} = \frac{\frac{1}{4}\sum_{t=d+1}^{n}(\hat{\beta}_0+\hat{\beta}_1 y_{t-1})(y_t-\hat{\alpha}_0-\hat{\alpha}_1 y_{t-1}) - \frac{1}{16}\sum_{t=d+1}^{n}(\hat{\beta}_0+\hat{\beta}_1 y_{t-1})^2(2+\hat{\gamma} y_{t-d})}{\frac{\hat{\gamma}}{16}\sum_{t=d+1}^{n}(\hat{\beta}_0+\hat{\beta}_1 y_{t-1})^2 y_{t-d}}$$

The modified maximum likelihood estimators have asymptotic properties of maximum likelihood estimators (such as asymptotic normal distribution), see Zamani and Sayyareh [15].

3. Simulation Analysis

Consider

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + (\beta_0 + \beta_1 y_{t-1})\frac{1}{1+\exp\{-\gamma(y_{t-1}-c)\}} + \epsilon_t$$

as a true model, where ϵ_t's are independent and identical distributed random variables that follow the normal distribution, $N(0,\sigma^2)$. The observations are generated from the LSTAR model with $\alpha_0 = 0.4, \alpha_1 = 0.3, \beta_0 = 0.4, \beta_1 = -0.5, \gamma = 2, c = 0.4$, $\sigma^2 = 1$ and d=1. It is assumed that the true model is known and only the parameters are estimated based on the method of moment (MM), modified maximum likelihood (MML) and maximum likelihood (ML) methods. We do $m = 10^4$ replications. The results of the mean of the estimator values and their bias, $\frac{1}{m}\sum_{j=1}^{m}(\hat{\theta}^{(j)} - \theta_0)$, and mean square error, $\frac{1}{m}\sum_{j=1}^{m}(\hat{\theta}^{(j)} - \theta_0)^2$, are given for different sample sizes, of n=25, 50, 100, 200,

300, 500, 1000 and are summarized in Tables 1-3, where $\hat{\theta}^{(j)}$ denotes the estimated parameter in the jth iteration and θ_0 is a true parameter.
In Table 1, we presented the average, across replications, estimates of the parameters. For all estimators, as the sample size increases, the value of the estimators is close to the true parameter. It can also be seen that MMLE converges to the true parameter. For small sample size, maximum likelihood estimators does not perform well in estimating parameters, especially α_1, β_0 and γ. The maximum likelihood estimators of these parameters have the largest MSE values. The results in Tables 1-3 show that the mean squared error and bias of the mentioned estimators decreases, when the sample size increases. We also observe that the maximum likelihood and modified maximum likelihood estimators usually have the highest and lowest mean square error and bias respectively. Although the modified maximum likelihood estimators for smaller sample sizes are better than the moment estimators, but as the sample size increases the obtained values of mean squared error and bias for both methods become closer. The bias of all estimators converges to zero. In other words, they are asymptotically unbiased estimators.

4. **A Real Data Example**

In this section, we consider the finance rate on consumer installment loans at commercial banks, new autos 48 month loan, and estimate LSTAR model as proposed model. According to the United States Federal Reserve, the finance rate on consumer installment loans at commercial banks was 5.15% in May of 2022. This dataset consists of the monthly returns with the sample extending from February 1992 to May 2022 for a total of n = 202 observations. This data can be found at https://fred.stlouisfed.org/series/TERMCBAUTO48NS. We denote the standard form of this dataset by $y_t, t = 1, ..., 202$. Descriptive statistics of y_t is presented in Table 4. It shows that the record high of 17.36 and record low of 4.00 of the finance rate on consumer installment loans at commercial banks is reached in November 1981 and November 2015, respectively. The unconditional mean is not statistically different from zero and it has positive skewness and negative the sample excess kurtosis.

The time series plot, qqnorm, the sample autocorrelation function and partial autocorrelation function of y_t dataset are given in Figure 1. It shows that this dataset follows Normal distribution and there is a correlation between y_t and y_{t-1}. We consider the LSTAR model and estimate parameters of model based on the proposed methods. The sample is partitioned into two subsamples $y_1, \ldots, y_m$ and $y_{m+1}, \ldots, y_n$ where m is 198. The subsample $y_1, \ldots, y_m$ is used as training data and the subsample $y_{m+1}, \ldots, y_n$ is retained as the validation data for testing the model. The model is initially fitted on the training dataset based on the proposed methods of estimation. The fitted model is used to predict for the observations in a the validation dataset. Table 5 shows the

Table 1. The values of the estimated parameters of LSTAR model.

n	method	$\alpha_0 = 0.4$	$\alpha_1 = 0.3$	$\beta_0 = 0.4$	$\beta_1 = -0.5$	$\gamma = 2$	$c = 0.4$	$\sigma^2 = 1$
25	MM	0.41927	0.23852	0.40360	-0.48980	2.19258	0.47627	0.90507
	ML	0.30873	0.13932	0.64883	-0.43147	1.68558	0.48941	0.83998
	MML	0.42050	0.23397	0.40273	-0.48673	2.12176	0.47812	0.89929
50	MM	0.40648	0.27193	0.40341	-0.51602	1.79201	0.46850	0.95240
	ML	0.37664	0.24695	0.45217	-0.48292	1.70034	0.48859	0.92071
	MML	0.40431	0.28088	0.40240	-0.49376	1.92505	0.46737	0.92453
100	MM	0.39477	0.30439	0.40198	-0.51207	1.80235	0.42642	0.98030
	ML	0.42277	0.29368	0.41391	-0.51725	1.73469	0.47995	0.96163
	MML	0.40303	0.29777	0.40163	-0.51834	1.96683	0.44459	0.97813
200	MM	0.39896	0.30334	0.40122	-0.50992	1.81875	0.41792	0.99340
	ML	0.41322	0.30324	0.37167	-0.50989	1.79108	0.47265	0.98404
	MML	0.40267	0.29878	0.40118	-0.50859	1.97118	0.43148	0.98930
300	MM	0.39903	0.30265	0.40098	-0.50751	1.83208	0.41661	0.99490
	ML	0.40838	0.30224	0.37789	-0.50950	1.79236	0.47174	0.98530
	MML	0.40179	0.29968	0.40079	-0.50656	1.99414	0.42652	0.98981
500	MM	0.39923	0.30122	0.40074	-0.50416	1.86230	0.41079	0.99800
	ML	0.39838	0.30223	0.40642	-0.50904	1.88551	0.47117	0.99362
	MML	0.40004	0.30137	0.40028	-0.50557	2.08210	0.42188	0.99451
1000	MM	0.39926	0.30084	0.39970	-0.50245	1.97215	0.40063	0.99887
	ML	0.39865	0.30187	0.40109	-0.50511	1.93383	0.47117	0.99680
	MML	0.39991	0.30065	0.39987	-0.50519	2.05280	0.40353	0.99926

estimated value of parameters and their standard error and mean square error, MSE,

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2.$$

Furthermore, to check the normality of $\hat{e}_t = y_t - \hat{y}_t$, the Kolmogrov-Smirnov test is used, with a largely enough p-value, which confirms the normality assumption of the error terms. The p-value Kolmogrov-Smirnov test, $\mathcal{KS}$, is given in Table 5. Although all p-values are greater than 0.05, but the LSTAR model estimated by modified maximum likelihood method has the highest p-value. Also, the estimated LSTAR model based on the modified maximum likelihood method has the lowest value of MSE.

The value of prediction and mean square error of prediction, MSE_h,

$$MSE_h = \frac{1}{4}\sum_{i=1}^{4}(y_{m+i} - \hat{y}_{m+i})^2$$

Table 2. The values of the mean square error

n	method	α_0	α_1	β_0	β_1	γ	c	σ^2
25	MM	0.06581	0.08095	0.05153	0.11257	0.83894	0.05247	0.08129
	ML	1.99664	1.35399	13.42616	0.54670	12.09991	0.05382	0.09185
	MML	0.06982	0.08178	0.05125	0.11713	0.87334	0.05348	0.08476
50	MM	0.03006	0.03486	0.03429	0.05091	0.87768	0.02561	0.03823
	ML	0.67443	0.46151	4.63561	0.23861	1.06530	0.02679	0.04093
	MML	0.04729	0.04963	0.03176	0.07329	0.87420	0.02649	0.05743
100	MM	0.02327	0.01522	0.01124	0.02030	0.04828	0.01390	0.02153
	ML	0.29267	0.20351	2.03000	0.10221	0.05634	0.01730	0.02126
	MML	0.01340	0.01468	0.01065	0.01893	0.03129	0.01300	0.02052
200	MM	0.00541	0.00604	0.00328	0.00627	0.04743	0.01082	0.00964
	ML	0.14350	0.09959	0.99320	0.05285	0.05234	0.01120	0.00980
	MML	0.00509	0.00601	0.00324	0.00565	0.02022	0.01034	0.00958
300	MM	0.00346	0.00392	0.00165	0.00338	0.03893	0.00961	0.00667
	ML	0.09139	0.06330	0.65015	0.03530	0.04888	0.00977	0.00675
	MML	0.00310	0.00392	0.00161	0.00257	0.02195	0.00935	0.00662
500	MM	0.00202	0.00182	0.00066	0.00120	0.04932	0.00734	0.00384
	ML	0.05654	0.03842	0.40152	0.02169	0.04938	0.00744	0.00389
	MML	0.00197	0.00108	0.00063	0.00114	0.04921	0.00718	0.00363
1000	MM	0.00102	0.00090	0.00031	0.00055	0.02251	0.00639	0.00201
	ML	0.02714	0.01814	0.18812	0.01066	0.04853	0.00633	0.00201
	MML	0.00061	0.00089	0.00029	0.00038	0.01399	0.00590	0.00169

is given in Table 6. It shows that, although the prediction values based on the proposed methods are close to true values, the estimated LSTAR model based on the modified maximum likelihood method has the lowest value of MSE_h.

TABLE 3. The values of bias

n	method	α_0	α_1	β_0	β_1	γ	c	σ^2
25	MM	0.01927	-0.06148	0.00360	0.01920	0.19258	0.07627	-0.09493
	ML	-0.09127	-0.16068	0.24883	0.06853	-0.21442	0.07741	-0.16002
	MML	0.02050	-0.06603	0.00703	0.02327	0.22176	0.07812	-0.10071
50	MM	0.00648	-0.02807	-0.00259	-0.01602	-0.18799	0.06850	-0.04760
	ML	-0.02336	-0.05305	0.05217	0.01708	-0.20966	0.07659	-0.07929
	MML	0.00431	-0.04912	-0.00660	0.01624	-0.20495	0.06737	-0.07547
100	MM	-0.00529	0.00439	0.00198	-0.01207	0.16235	0.06642	-0.01969
	ML	0.01277	-0.00631	0.03391	-0.01705	-0.19530	0.06995	-0.03836
	MML	0.00303	0.00177	-0.00136	-0.01134	-0.13316	0.06459	-0.01186
200	MM	-0.00163	0.00134	0.00122	-0.00992	-0.15124	-0.06207	-0.00659
	ML	0.01222	0.00324	-0.02832	-0.00589	-0.16891	0.06265	-0.01595
	MML	-0.00032	-0.00121	-0.00121	-0.00859	-0.04881	0.06148	-0.00659
300	MM	-0.00096	0.00065	0.00098	-0.00751	-0.13791	-0.05138	-0.00609
	ML	0.00838	0.00024	-0.02210	-0.00150	-0.17763	0.05174	-0.01469
	MML	0.00019	-0.00031	-0.00090	-0.00556	-0.00585	0.05052	-0.00608
500	MM	-0.00076	0.00012	0.00074	-0.00416	-0.12769	-0.04920	-0.00199
	ML	-0.00161	-0.00016	0.00642	-0.00104	-0.17448	0.05010	-0.00637
	MML	0.00004	0.00007	-0.00071	-0.00357	-0.00489	-0.04811	-0.00148
1000	MM	-0.00074	0.00008	-0.00030	-0.00245	-0.10785	0.04163	-0.00113
	ML	0.00145	0.00011	-0.00189	-0.00101	-0.11617	0.04517	-0.00320
	MML	-0.00004	-0.00005	-0.00013	-0.00119	0.00280	0.03353	-0.00074

TABLE 4. Descriptive statistics for empirical series.

series	n	Min	Median	Max	$\bar{y}$	$\hat{\sigma}$	$\mathcal{S}$	$\mathcal{K}$
y_t	202	4.000	8.8150	17.360	0.9214e-16	0.9990	0.4186	-0.3840

Notes:
1. n denotes the number of observations and $\bar{y}$ denotes the sample mean.
2. $\hat{\sigma}$ denotes the sample standard deviation and $\mathcal{S}$ denotes the sample skewness.
3. $\mathcal{K}$ denotes the sample excess kurtosis.

Conclusion. In this paper, we considered the asymmetric smooth transition autoregressive model, LSTAR models. One of the problems in analyzing this

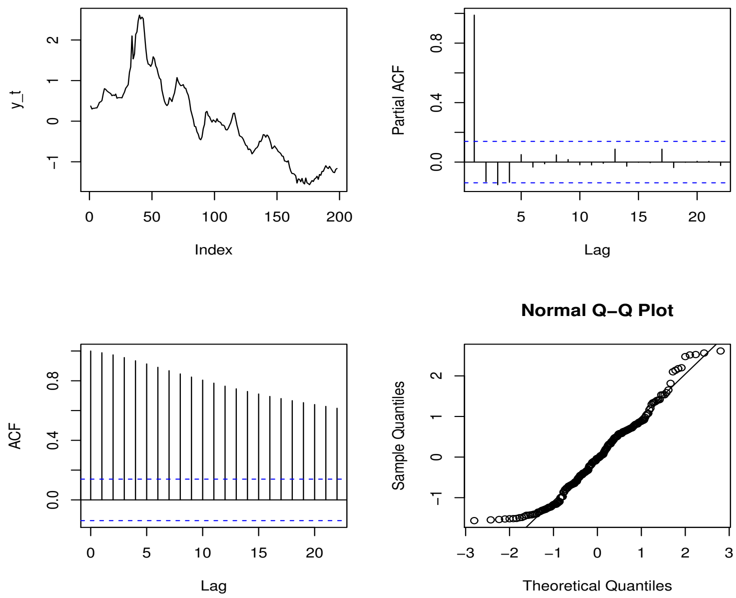

FIGURE 1. The time series plot, qqnorm, acf and pacf of the dataset.

TABLE 5. The value of estimated parameters based on the ML, MML and MM methods and their standard errors (in bracket).

Method	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\gamma}$	$\hat{c}$	$\mathcal{KS}$	MSE
ML	-0.0086	0.9976	0.3290	-0.1707	0.1054	1.3255	0.0882	0.0857
	(0.0008)	(0.1124)	(0.0456)	(0.0045)	(0.2885)	(0.0006)		
MML	-0.1926	0.9032	0.46307	0.0127	1.9525	0.0010	0.4201	0.0206
	(0.0007)	(0.1121)	(0.0449)	(0.0003)	(0.8845)	(0.0001)		
MM	-0.1452	0.9022	0.2733	0.0192	1.8307	0.0250	0.1405	0.0504
	(0.0008)	(0.1123)	(0.0512)	(0.0004)	(0.8862)	(0.0003)		

type of time series models is the estimation of the parameters. Since the likelihood equations do not have the explicit solutions, so the numerical methods are used to estimation of parameters. We proposed the modified maximum

TABLE 6. The values of prediction and MSE_h.

True values	-1.1768	-1.3509	-1.2514	-1.1737	MSE_h
MLE	-1.1451	-1.1510	-1.1570	-1.1629	0.0124
MMLE	-1.2239	-1.2535	-1.2803	-1.2045	0.0033
MM	-1.1672	-1.1729	-1.1782	-1.1833	0.0093

likelihood estimators which are asymptotically consistent and follow the normal distribution. Also we proposed method of moment that is unbiased and consistent. We examine by simulation, the performance of the proposed estimation methods and found that modified maximum likelihood estimator is the better one for logistic smooth transition autoregressive model. For large sample sizes, the modified maximum likelihood estimators usually have the lowest mean square error and bias.
We compare the performance of proposed estimation method for logistic smooth transition autoregressive model using monthly returns of finance rate on consumer installment loans at commercial banks index from February 1992 to May 2022. The result shows that the estimated LSTAR model based on the modified maximum likelihood method has the lowest value of MSE.

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