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ON THE ZHANG-POWER SERIES DISTRIBUTIONS WITH APPLICATION TO LIFETIME MODELING

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Abstract. Many recent probability distributions are introduced by compounding the well-known continuous distributions with the power series distribution. In this paper, we provide a general closed-form expression for the cumulative distribution function of this class. Then, we introduce a new four parameter lifetime distribution called Zhang-power series (ZPS) distribution. This distribution is very useful in the lifetime, reliability and extreme-value data analysis. The distribution properties including survival function, hazard function and limiting behavior of the probability density and hazard functions are studied. The method of maximum likelihood estimation is used to estimate the model parameters. Applications to real data sets are given to show the flexibility and potentiality of the proposed model.

Keywords: Maximum likelihood estimation, Monte Carlo simulation, Power series distribution, Zhang distribution.
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1. Introduction

In many fields such as business, environment, actuarial science, biomedical science, survival analysis and industrial reliability, we need to find an appropriate model for fitting the lifetime data. On the other hand, classical probability distributions do not provide adequate fits to real data. Therefore, several methods for generating new probability distributions by adding one or more parameters have been presented in the literature. Recently, some new families of distributions have been introduced by compounding useful lifetime distributions. In a known approach, one may consider a series (parallel) system with N component lifetimes $X_1, \ldots, X_N$, where N is a random variable, and then study the lifetime of the system, i.e. $T = \min\{X_1, X_2, \ldots, X_N\}$ $(T = \max\{X_1, X_2, \ldots, X_N\})$. After considering different discrete distributions for N and continuous distributions for the lifetimes of the components, we can find some new distributions with flexible hazards. Exponential-geometric (EG),

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exponential-Poisson (EP) and exponential-logarithmic (EL) distributions proposed by Adamidis and Loukas [1], Kus [15] and Tahmasbi and Rezaei [30], respectively, are some examples of distributions which are produced in this direction. Chahkandi and Ganjali [8] introduced exponential-power series (EPS) distributions which contain as special cases the EP, EG and EL distributions. Silva et al. [26] studied the extended Weibull power series (EWPS) family, which includes as special models the EPS and Weibull power series (WPS) family introduced by Morais and Barreto-Souza [21]. Some of other new works have included: exponentiated generalized gamma distribution by Cordeiro et al. [11]; exponentiated Weibull-Poisson distribution by Mahmoudi and Sepahdar [18]; exponentiated Weibull-logarithmic distribution by Mahmoudi et al. [19]; a general family of continuous lifetime distributions by Asgharzadeh et al. [6]; Weibull Rayleigh by Faton and Elbatal [12]; the generalized gamma power series distribution by Silva et al. [27]; generalized modified Weibull power series (GMWPS) by Bagheri et al. [7], and a new four-parameter distribution by Alizadeh et al. [4]. Some other recent works have studied by Ahmad et al. [2], Roozegar et al. [24], Alizadeh [5], Moakofi [20], Hassan et al. [14], Shakhatreh [25] and Soliman Hassan [29]. In the sequel, we present a general form for finding new distributions in this manner. Some authors considered a parallel system instead of the series system and followed the same approach, see for example Alizadeh et al. [3] and Mahmoodian [17]. Transmutation of the baseline distribution is also another approach for constructing the new distributions, we refer to Rahman et al. [23] and Pandey et al. [22] for more details.

In this paper, we are motivated to introduce a new distribution for fitting the lifetime data. We first consider a general case in which N is a discrete random variable from a power series distribution truncated at zero, and the component lifetimes follow from a general distribution F. Then, we obtain the probability density function (pdf), cumulative distribution function (cdf) and hazard function of the lifetime of a series system. In the sequel, we will consider $F(x) = (1 - e^{\theta(1-e^{x^\beta})})^\alpha$ known as Zhang distribution, introduced by Zhang (2004), and define the new class of distributions known as Zhang-power series (ZPS) distribution. The Zhang distribution, just like exponentiated-Weibull distribution, is very useful in the lifetime, reliability and extreme-value data analysis. It is known that the lifetime distribution of many electronic and mechanical products usually has non-monotone failure rate. In some reliability situations such as life-cycle of the products because of aging and wearout, we have high initial and eventual failure rates that conclude bathtub shape failure rate. The high initial failure rate may be because of undiscover defects and starting to wear-out the items after some period of time causes the high eventual failure rate. The ZPS distribution includes decreasing and different shapes of the bathtub failure rates and is useful for modeling these cases. Some properties of the new proposed distribution such as the density function, the hazard rate and some properties of the k^{th} order statistic are given in Section

3. The Zhang logarithmic distribution as a especial case of the new introduced family of distribution is discussed in Section 4. In Section 5, we consider some real data sets to show the flexibility and potentiality of the new distribution in finding the appropriate model.

2. A general class

. Let $X_1, ..., X_N$ be a random sample from a continuous distribution F, where N is a discrete random variable from a power series distribution (truncated at zero) with probability function $P(N = n) = \frac{a_n \lambda^n}{C(\lambda)}$, $n = 1, 2, \ldots$, where $C(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$, $a_n > 0$ and $\lambda > 0$. The power series family of distributions includes Binomial, Poisson, Geometric and Logarithmic distributions. Table 5 in the Appendix contains some useful quantities for power series family of distributions. Let $T = min(X_1, ..., X_N)$, and N is independent of X_i, $i = 1, 2, \ldots$, then the conditional cdf of $T|N = n$ is given by

$$F_{T|N=n}(x) = P\left(T \le x | N = n\right) = 1 - \left(\bar{F}(x)\right)^n;\ x > 0, \tag{1}$$

where $\bar{F}(.)$ is the survival function of F. The marginal cdf of T can be found as

$$\begin{aligned} G(x) &= P(T \le x) \\ &= \sum_{n=1}^{\infty} P(T \le x, N = n) \\ &= \sum_{n=1}^{\infty} P(T \le x | N = n) P(N = n) \\ &= \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{C(\lambda)} \left(1 - \left(\bar{F}(x)\right)^n\right) \\ &= 1 - \sum_{n=1}^{\infty} \frac{a_n \left(\lambda \bar{F}(x)\right)^n}{C(\lambda)} \\ &= 1 - \frac{C\left(\lambda \bar{F}(x)\right)}{C(\lambda)};\quad x > 0,\ \lambda > 0. \qquad (2) \end{aligned}$$

The respective pdf, hazard and reversed hazard functions of this general class of distributions are as follows:

$$g(x) = G'(x) = \frac{\lambda f(x) C'\left(\lambda \bar{F}(x)\right)}{C(\lambda)}, \tag{3}$$

$$h(x) = \frac{g(x)}{\bar{G}(x)} = \frac{\lambda f(x) C'(\lambda \bar{F}(x))}{C\left(\lambda \bar{F}(x)\right)}, \tag{4}$$

and

$$r(x) = \frac{\lambda f(x) C'(\lambda \bar{F}(x))}{C(\lambda) - C(\lambda \bar{F}(x))}. \tag{5}$$

These representations help us to find some new distributions by compounding the well-known continuous distributions and power series family. For example, the EG, EP, EL, EPS, EWPS, GMWPS and some other distributions are produced based on this idea.

3. The class of ZPS distributions

. Chen [9] proposed a two-parameter lifetime distribution with bathtub shape or increasing hazard function as

$$F_c(x;\beta,\theta) = 1 - e^{\theta\left(1-e^{x^\beta}\right)}, \quad x > 0. \tag{6}$$

Zhang [33] introduced a new three-parameter distribution by adding a shape parameter on the Chen distribution given by

$$F_Z(x;\alpha,\beta,\theta) = \left(1 - e^{\theta\left(1-e^{x^\beta}\right)}\right)^\alpha, \quad x > 0, \tag{7}$$

where $\alpha > 0$, $\beta > 0$ and $\theta > 0$.
The pdf and hazard function of Zhang distribution are

$$f_Z(x;\alpha,\beta,\theta) = \alpha\beta\theta x^{\beta-1} e^{x^\beta} (\Psi(x;\beta,\theta))^{\alpha-1} (1 - \Psi(x;\beta,\theta)), \tag{8}$$

and

$$h_Z(x;\alpha,\beta,\theta) = \frac{\alpha\beta\theta x^{\beta-1} e^{x^\beta} (\Psi(x;\beta,\theta))^{\alpha-1} (1 - \Psi(x;\beta,\theta))}{1 - (\Psi(x;\beta,\theta))^\alpha},$$

respectively, where $\Psi(x;\beta,\theta) = 1 - e^{\theta(1-e^{x^\beta})}$.
Figure 1 shows the flexibility of $h_Z(x;\alpha,\beta,\theta)$ for different values of parameters. Thus, we consider Zhang distribution for the component lifetimes and power series distribution (truncated at zero) for N, and then study the distribution of $T = \min\{X_1, \ldots, X_N\}$. From Eq. (2), the cdf of the new family of distribution, Zhang-power series (ZPS) distribution, is given by

$$F_{ZPS}(x;\alpha,\beta,\theta,\lambda) = 1 - \frac{C\left(\lambda \bar{F}(x)\right)}{C(\lambda)} = 1 - \frac{C\left(\lambda\left(1 - (\Psi(x;\beta,\theta))^\alpha\right)\right)}{C(\lambda)}. \tag{9}$$

The pdf and hazard function of the ZPS distribution are given, respectively, as

$$\begin{aligned} f_{ZPS}(x;\alpha,\beta,\theta,\lambda) =& \lambda\alpha\theta\beta x^{\beta-1} e^{x^\beta} \left(\Psi(x;\beta,\theta)\right)^{\alpha-1} \left(1 - \Psi(x;\beta,\theta)\right) \\ & \times \frac{C'\left(\lambda\left(1 - (\Psi(x;\beta,\theta))^\alpha\right)\right)}{C(\lambda)}, \end{aligned} \tag{10}$$

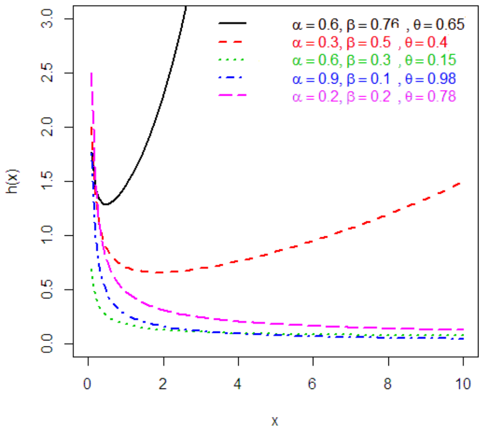

FIGURE 1. Plot of the hazard function for different values of parameters.

and

$$h_{ZPS}(x;\alpha,\beta,\theta,\lambda) = \lambda\alpha\theta\beta x^{\beta-1}e^{x^\beta}\left(\Psi(x;\beta,\theta)\right)^{\alpha-1}\left(1-\Psi(x;\beta,\theta)\right) \times \frac{C'\left(\lambda\left(1-\left(\Psi(x;\beta,\theta)\right)^{\alpha}\right)\right)}{C\left(\lambda\left(1-\left(\Psi(x;\beta,\theta)\right)^{\alpha}\right)\right)}. \tag{11}$$

The following propositions present the limiting behavior of the cdf and hazard function of the ZPS distribution.

Proposition 3.1. *The Zhang distribution is a limiting distribution of the ZPS when* $\lambda \to 0^+$.

Proof.

$$\begin{aligned}
\lim_{\lambda\to 0^+} F_{ZPS}(x;\alpha,\beta,\theta,\lambda) &= 1 - \lim_{\lambda\to 0^+}\frac{C\left(\lambda\left(1-\left(\Psi(x;\beta,\theta)\right)^{\alpha}\right)\right)}{C(\lambda)} \\
&= 1 - \lim_{\lambda\to 0^+}\frac{\sum_{n=1}^{\infty} a_n(\lambda(1-(\Psi(x;\beta,\theta))^{\alpha}))^n}{\sum_{n=1}^{\infty} a_n\lambda^n} \\
&= 1 - \lim_{\lambda\to 0^+}\frac{a_1(\lambda(1-(\Psi(x;\beta,\theta))^{\alpha})) + \sum_{n=2}^{\infty} a_n(\lambda(1-(\Psi(x;\beta,\theta))^{\alpha}))^n}{a_1\lambda + \sum_{n=2}^{\infty} a_n\lambda^n} \\
&= 1 - (1-(\Psi(x;\beta,\theta))^{\alpha}) \\
&= (1-e^{\theta(1-e^{x^\beta})})^{\alpha},
\end{aligned} \tag{12}$$

where the fourth equality is derived from the Hopital's rule and replacing λ with zero. $\square$

Proposition 3.2. *The behavior of the hazard function when $x \to 0^+$ and $x \to \infty$ is as follows:*

$$\lim_{x\to 0^+} h_{ZPS}(x;\alpha,\beta,\theta,\lambda) = 0, \quad \textit{and} \quad \lim_{x\to\infty} h_{ZPS}(x;\alpha,\beta,\theta,\lambda) = \infty.$$

Order statistics play an important role in several areas of statistical inference. They are also important in studying the lifetime of k-out-of-n systems in reliability analysis. Therefore, we now discuss some properties of the order statistic from the class of ZPS distributions.

Let $X_{1:n} < \ldots < X_{n:n}$ denote the order statistics from a random sample $X_1, ..., X_n$ with distribution F. It is not difficult to verify that the cdf of k^{th} order statistic can be expressed as

$$F_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\frac{\binom{n-k}{l}(-1)^l}{k+l}(F(x))^{k+l}. \tag{13}$$

Thus, the cdf and pdf of k^{th} order statistic of ZPS distribution are as

(14)

$$F_{ZPS(k)}(x;\alpha,\beta,\theta,\lambda) = k\binom{n}{k}\sum_{l=0}^{n-k}\frac{\binom{n-k}{l}(-1)^l}{k+l}\left(1-\frac{C\left(\lambda\left(1-\left(\Psi(x;\beta,\theta)\right)^{\alpha}\right)\right)}{C(\lambda)}\right)^{k+l}.$$

and

$$\begin{aligned} f_{ZPS(k)}(x;\alpha,\beta,\theta,\lambda) = {} & k\binom{n}{k}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\lambda\alpha\theta\beta x^{\beta-1}e^{x^\beta}\left(\Psi(x;\beta,\theta)\right)^{\alpha-1} \\ & \times\left(1-\Psi(x;\beta,\theta)\right)\frac{C'\left(\lambda\left(1-\left(\Psi(x;\beta,\theta)\right)^{\alpha}\right)\right)}{C(\lambda)} \\ & \times\left(1-\frac{C\left(\lambda\left(1-\left(\Psi(x;\beta,\theta)\right)^{\alpha}\right)\right)}{C(\lambda)}\right)^{k+l-1}, \end{aligned} \tag{15}$$

respectively.

3.1. **Maximum likelihood estimation.**

In this section, we obtain the estimation of parameters by using the maximum likelihood (ML) method. Let $x_1, ..., x_n$ be n observations of a random sample from ZPS distribution and our unknown parameter vector be $\boldsymbol{\theta} = (\alpha, \beta, \theta, \lambda)$. The log-likelihood function of

ZPS distribution is given by

$$\begin{aligned}\ell(\boldsymbol{\theta}) \;=\;& n\left(\log\lambda+\log\alpha+\log\theta+\log\beta\right)+(\beta-1)\sum_{i=1}^{n}\log x_i\\ &+(\alpha-1)\sum_{i=1}^{n}\log\left(\Psi(x_i;\beta,\theta)\right)+\sum_{i=1}^{n}(x_i^{\beta}+\theta(1-e^{x_i^{\beta}}))\\ &-n\log\left(C\left(\lambda\right)\right)+\sum_{i=1}^{n}\log(C'(\lambda(1-(\Psi(x_i;\beta,\theta))^{\alpha}))).\end{aligned}\tag{16}$$

The associated score function is $U_n(\boldsymbol{\theta})=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\theta},\frac{\partial\ell}{\partial\lambda}\right)$, where $\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\theta}$ and $\frac{\partial\ell}{\partial\lambda}$ are the partial derivatives of the log-likelihood function. Thus, we have

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha}=&\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(\Psi(x_i;\beta,\theta)\right)-\lambda\sum_{i=1}^{n}\Big\{\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\log\left(\Psi(x_i;\beta,\theta)\right)\\ &\times\frac{C''\left(\lambda\left(1-\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\right)\right)}{C'\left(\lambda\left(1-\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\right)\right)}\Big\},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\beta}=&\frac{n}{\beta}+\sum_{i=1}^{n}\log x_i+(\alpha-1)\sum_{i=1}^{n}\frac{\theta e^{x_i^{\beta}}x_i^{\beta}\log x_i\;(1-\Psi(x_i;\beta,\theta))}{\Psi(x_i;\beta,\theta)}\\ &+\sum_{i=1}^{n}x_i^{\beta}\log x_i-\sum_{i=1}^{n}\theta x_i^{\beta}\log x_i\;e^{x_i^{\beta}}-\lambda\alpha\sum_{i=1}^{n}\Big\{\left(\Psi(x_i;\beta,\theta)\right)^{\alpha-1}\theta x_i^{\beta}\log x_i\;e^{x_i^{\beta}}\\ &\times(1-\Psi(x_i;\beta,\theta))\frac{C''\left(\lambda\left(1-\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\right)\right)}{C'\left(\lambda\left(1-\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\right)\right)}\Big\},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell}{\partial\theta}=&\frac{n}{\theta}-(\alpha-1)\sum_{i=1}^{n}\frac{(1-e^{x_i^{\beta}})(1-\Psi(x_i;\beta,\theta))}{\Psi(x_i;\beta,\theta)}+\sum_{i=1}^{n}(1-e^{x_i^{\beta}})\\ &+\lambda\alpha\sum_{i=1}^{n}\Big\{(1-e^{x_i^{\beta}})\left(\Psi(x_i;\beta,\theta)\right)^{\alpha-1}\left(1-\Psi(x_i;\beta,\theta)\right)\\ &\times\frac{C''\left(\lambda\left(1-\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\right)\right)}{C'\left(\lambda\left(1-\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\right)\right)}\Big\},\end{aligned}$$

and

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}-\frac{nC'(\lambda)}{C(\lambda)}+\sum_{i=1}^{n}\Big(1-\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\Big)\frac{C''\left(\lambda\left(1-\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\right)\right)}{C'\left(\lambda\left(1-\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\right)\right)}.$$

The maximum likelihood estimations of the parameters may be found by solving the above system of nonlinear equations with respect to α, β, θ and λ. Since the system has no closed form solution in α, β, θ and λ, we have to use a numerical method such as the Newton-Raphson procedure to find the solution.

As a special case of the ZPS class, we study Zhang-Logarithmic distribution in the next section.

4. **Zhang logarithmic distribution**

The Zhang logarithmic (ZL) distribution is a special case of ZPS class obtained from compounding the logarithmic distribution and Zhang distribution. The Logarithmic distribution is a member of power series distributions with $a_n = \dfrac{1}{n}$ and $C(\lambda) = -\log(1-\lambda)$. From Eqs. (9) and (10), the cdf and pdf of ZL distribution are given by

$$F_{ZL}(x;\alpha,\beta,\theta,\lambda) = 1 - \frac{\log\left(1-\lambda\left(1-(\Psi(x;\beta,\theta))^{\alpha}\right)\right)}{\log\left(1-\lambda\right)}, \tag{17}$$

and

$$f_{ZL}(x;\alpha,\beta,\theta,\lambda) = \frac{\lambda\alpha\theta\beta x^{\beta-1}\; e^{x^{\beta}}\left(\Psi(x;\beta,\theta)\right)^{\alpha-1}\left(1-\Psi(x;\beta,\theta)\right)}{\log\left(1-\lambda\right)\left(\lambda\left(1-(\Psi(x;\beta,\theta))^{\alpha}\right)-1\right)}, \tag{18}$$

where $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $0 < \lambda < 1$. Figure 2 shows the plots of the pdf of ZL distribution for several combinations of the parameters. We find that the density function is unimodal or decreasing when α or β are greater or less than 1. The hazard function of ZL distribution can be found as

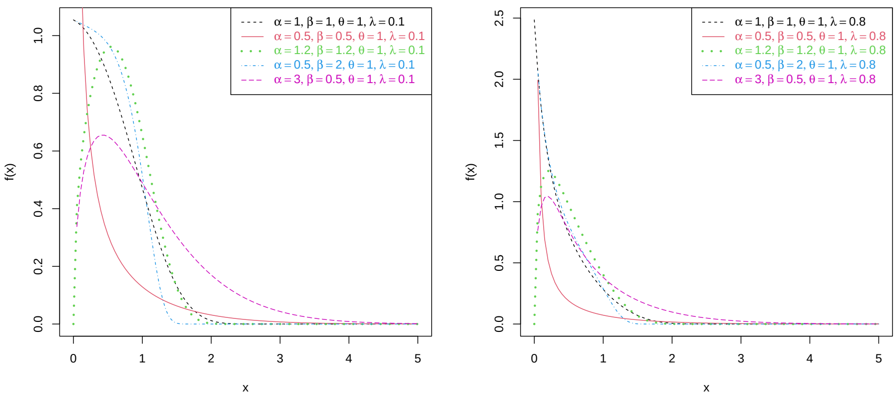

FIGURE 2. Plots of the pdf of ZL distribution for different values of parameters.

$$h_{ZL}(x;\alpha,\beta,\theta,\lambda) = \frac{\lambda\alpha\beta\theta x^{\beta-1}\; e^{x^{\beta}}\left(\Psi(x;\beta,\theta)\right)^{\alpha-1}\left(1-\Psi(x;\beta,\theta)\right)}{\log\Big(1-\lambda\left(1-(\Psi(x;\beta,\theta))^{\alpha}\right)\Big)\Big(\lambda\left(1-(\Psi(x;\beta,\theta))^{\alpha}\right)-1\Big)}.$$

Plots of the hazard function of ZL distribution for some values of the parameters α, β, θ and λ are given in Figure 3. Figure 3 shows that the hazard function

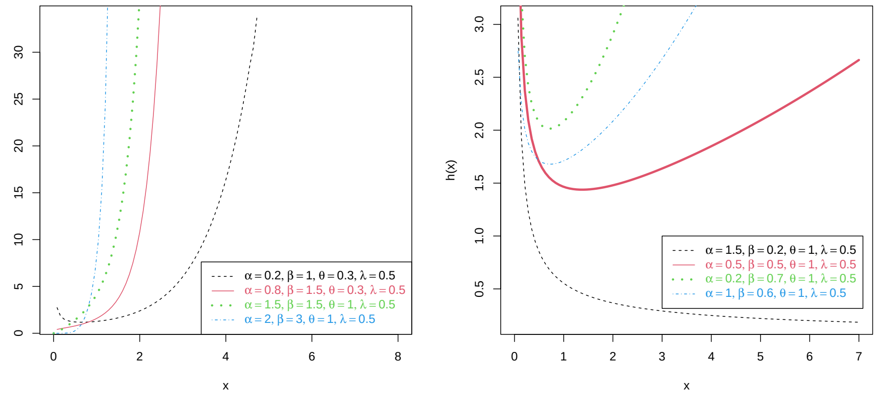

FIGURE 3. Plots of the hazard function of the ZL distribution for several values of parameters.

of ZL has different shapes including decreasing and bathtub shape. This implies that ZL distribution is suitable for monotonic and non-monotonic hazard behaviors which are more likely to be encountered in real life.

Now, we find the maximum likelihood estimators of the parameters of ZL distribution. Let $x_1, ..., x_n$ be n observations of a random sample from ZL distribution with unknown parameter vector $\boldsymbol{\theta} = (\alpha, \beta, \theta, \lambda)$. From Eqs. (16) and (18), we have

$$\begin{aligned}\ell(\boldsymbol{\theta}) =& n\big(\log\lambda + \log\alpha + \log\theta + \log\beta\big) + (\beta-1)\sum_{i=1}^{n}\log x_i \\ &+ (\alpha-1)\sum_{i=1}^{n}\log\left(\Psi(x_i;\beta,\theta)\right) + \sum_{i=1}^{n}\left(x_i^{\beta} + \theta(1-e^{x_i^{\beta}})\right) \\ &- n\log\big(-\log(1-\lambda)\big) - \sum_{i=1}^{n}\log\left(1-\lambda\left(1-(\Psi(x_i;\beta,\theta))^{\alpha}\right)\right).\end{aligned} \tag{19}$$

The score function is $U_n(\boldsymbol{\theta}) = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\lambda}\right)$, and the maximum likelihood estimate of $\boldsymbol{\theta}$ can be obtained by solving $U_n(\boldsymbol{\theta}) = 0$. The normal equations are as

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(\Psi(x_i;\beta,\theta)\right) - \sum_{i=1}^{n}\frac{\lambda\left(\Psi(x_i;\beta,\theta)\right)^{\alpha}\log\left(\Psi(x_i;\beta,\theta)\right)}{\left(1-\lambda\left(1-(\Psi(x_i;\beta,\theta))^{\alpha}\right)\right)} = 0,$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} =& \frac{n}{\beta} + \sum_{i=1}^{n} \log x_i + (\alpha-1)\sum_{i=1}^{n} \frac{\theta \log x_i \; x_i^{\beta} e^{x_i^{\beta}} (1-\Psi(x_i;\beta,\theta))}{\Psi(x_i;\beta,\theta)} \\ &+ \sum_{i=1}^{n} x_i^{\beta}\log x_i - \sum_{i=1}^{n} \theta \log x_i \; x_i^{\beta} e^{x_i^{\beta}} - \theta\lambda\alpha \sum_{i=1}^{n} \Big\{ \log x_i \; x_i^{\beta} e^{x_i^{\beta}} \\ &\times \frac{(1-\Psi(x_i;\beta,\theta))\,(\Psi(x_i;\beta,\theta))^{\alpha-1}}{(1-\lambda\,(1-(\Psi(x_i;\beta,\theta))^{\alpha}))} \Big\} = 0,\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \theta} =& \frac{n}{\theta} - (\alpha-1)\sum_{i=1}^{n} \frac{(1-e^{x_i^{\beta}})(1-\Psi(x_i;\beta,\theta))}{\Psi(x_i;\beta,\theta)} + \sum_{i=1}^{n}\left(1-e^{x_i^{\beta}}\right) + \sum_{i=1}^{n}\Big\{\lambda\alpha\left(1-e^{x_i^{\beta}}\right) \\ &\times \frac{(\Psi(x_i;\beta,\theta))^{\alpha-1}\,(1-\Psi(x_i;\beta,\theta))}{(1-\lambda\,(1-(\Psi(x_i;\beta,\theta))^{\alpha}))}\Big\} = 0,\end{aligned}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} + \frac{n}{(1-\lambda)\log(1-\lambda)} + \sum_{i=1}^{n} \frac{(1-(\Psi(x_i;\beta,\theta))^{\alpha})}{(1-\lambda\,(1-(\Psi(x_i;\beta,\theta))^{\alpha}))} = 0.$$

These nonlinear system can be numerically solved by MATLAB or R software. We will use the 'bbmle' package in R to obtain the estimations of the parameters.

4.1. **Monte Carlo simulation study.** In this section, we use the simulation study to check the performance and accuracy of the maximum likelihood estimates of the ZL model parameters. The following algorithm can be used to generate random data from ZL distribution using the inverse cdf of ZL distribution.

Algorithm I (Generating random data from ZL distribution)

1. Generate $U_i \sim Uniform(0,1);\;\; i = 1, ..., n.$
2. Set $X_i = \left(\log\left\{1 - \frac{1}{\theta}\log\left\{1 - \left(1 - \frac{1}{\lambda}\left(1 - e^{(1-U_i)\log(1-\lambda)}\right)\right)^{\frac{1}{\alpha}}\right\}\right\}\right)^{\frac{1}{\beta}}.$

The simulation study is repeated $N = 10000$ times that the size of the samples are $n = 25, 50, 75, 100, 200, 400, 600$ and the parameter values are $I : \alpha = 0.2, \beta = 0.7, \theta = 0.4, \lambda = 0.5$ and $II : \alpha = 0.6, \beta = 0.4, \theta = 0.3, \lambda = 0.1$.
We compute four quantities in this simulation study to verify the performance of the estimators.

1. Relative average bias (RAB) of the ML estimator of θ: $\frac{1}{N\theta}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)$.
2. Relative root mean squared error (RRMSE) of ML estimator of θ: $(\frac{1}{N\theta^2}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2)^{0.5}$.

3. Coverage probability (CP) of 0.95 confidence intervals of θ.
4. Average width (AW) of 0.95 confidence intervals of the parameter θ.

Table 1 shows these items for different sample sizes. From the results of Table 1, we find that as n increases, the average widths decrease and the RMSEs and average biases tend to zero. The CP of the confidence intervals are also quite close to 95%. The approximate confidence intervals for the parameters is derived using the asymptotic distribution of the MLEs. Under certain regularity conditions, the limiting distribution of the vector $\sqrt{n}(\hat{\alpha}-\alpha, \hat{\beta}-\beta, \hat{\theta}-\theta, \hat{\lambda}-\lambda)$, as $n \to \infty$, is a four-dimensional normal distribution with mean zero and covariance matrix Σ^{-1}, where Σ_{ij} is the (i,j)-th entry of the Fisher information matrix, given by

$$\Sigma_{ij} = -E(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log(f_{ZL}(\boldsymbol{x};\boldsymbol{\theta}))).$$

The asymptotic behavior remains valid if Σ is replaced by $\hat{\Sigma}^{-1}$, where $\hat{\Sigma}$ is the Fisher information matrix with $(\alpha,\beta,\theta,\lambda)$ replaced by $(\hat{\alpha},\hat{\beta},\hat{\theta},\hat{\lambda})$. Using this approximation, $100(1-\alpha)$ percent confidence intervals for α, β, θ, and λ are as follows:

$$\hat{\alpha} \pm Z_{1-\frac{\alpha}{2}}\sqrt{\hat{\Sigma}^{-1}_{11}},\ \hat{\beta} \pm Z_{1-\frac{\alpha}{2}}\sqrt{\hat{\Sigma}^{-1}_{22}},\ \hat{\theta} \pm Z_{1-\frac{\alpha}{2}}\sqrt{\hat{\Sigma}^{-1}_{33}},\ \hat{\lambda} \pm Z_{1-\frac{\alpha}{2}}\sqrt{\hat{\Sigma}^{-1}_{44}},$$

where Z_α is the lower α-th percentile of the standard normal distribution.

Table 1. Monte Carlo simulation results

		I				*II*			
Parameter	n	*RAB*	*RRMSE*	*CP*	*AW*	*RAB*	*RRMSE*	*CP*	*AW*
α	25	2.700	64.258	0.949	16.988	0.242	0.626	0.999	1.287
	50	0.580	2.795	0.942	2.908	−0.002	0.766	0.876	1.747
	75	−0.220	1.985	0.931	2.058	−0.040	0.536	0.902	1.311
	100	−0.260	1.715	0.923	1.698	−0.003	0.458	0.916	1.122
	200	−0.305	1.255	0.898	1.109	0.020	0.320	0.934	0.766
	400	−0.215	0.950	0.897	0.783	0.003	0.216	0.945	0.523
	600	−0.180	0.765	0.917	0.639	0.006	0.173	0.954	0.422
β	25	0.325	0.721	0.986	4.183	−0.452	0.885	0.945	2.363
	50	0.287	0.555	0.994	2.644	0.095	0.347	0.992	0.848
	75	0.268	0.485	0.997	2.015	0.052	0.265	0.992	0.669
	100	0.230	0.420	0.996	1.625	0.030	0.230	0.991	0.575
	200	0.182	0.330	0.998	1.022	−0.002	0.165	0.993	0.395
	400	0.121	0.238	0.993	0.656	−0.010	0.122	0.993	0.273
	600	0.087	0.184	0.987	0.510	−0.010	0.107	0.992	0.219
θ	25	1.227	6.562	0.992	9.201	1.370	2.310	0.962	2.845
	50	0.262	2.052	0.994	4.455	0.127	0.683	0.961	1.751
	75	0.005	1.452	0.992	3.272	1.157	0.570	0.972	1.407
	100	−0.087	1.172	0.989	12.738	−0.127	0.520	0.977	1.241
	200	−0.235	0.840	0.977	1.858	0.120	0.417	0.982	0.865
	400	−0.202	0.655	0.9569	1.334	0.097	0.337	0.989	0.588
	600	−0.165	0.567	0.951	1.106	−0.077	0.297	0.988	0.464
λ	25	0.130	0.602	0.944	6.006	−2.090	3.360	0.979	3.856
	50	0.194	0.576	0.950	4.091	−1.640	3.060	0.974	4.439
	75	0.224	0.568	0.925	3.284	−1.440	2.830	0.984	3.538
	100	0.212	0.558	0.895	2.878	−1.410	2.780	0.986	3.133
	200	0.228	0.536	0.829	2.031	−1.210	2.520	0.994	2.133
	400	0.178	0.506	0.807	1.575	−0.990	2.280	0.998	1.405
	600	0.128	0.464	0.823	1.375	−0.790	2.030	0.999	1.053

The rth moment of a random variable X from the ZL distribution, denoted by μ'_r, is given by

$$\mu'_r = E(X^r) = \int_0^\infty x^r f_{ZL}(x)dx = \sum_{n=1}^{\infty} P(N=n)E(X^r_{(1)}),$$

where $E(X^r_{(1)})$ is the rth moment of the first order statistics of the Zhang distribution. Due to the complexity of the distribution, it is not possible to provide a closed-form expression for the expected value of $E(X^r)$. Therefore, the moments of the distribution are computed numerically and reported in Table 2. We have also presented the related measures such as coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) of the ZL distribution in Table 2. These measures are calculated as follows:

$$\begin{aligned} CV &= \frac{\sqrt{E(X-\mu'_1)^2}}{\mu'_1} = \sqrt{\frac{\mu'_2}{\mu'_1} - 1}, \\ CS &= \frac{E(X-\mu'_1)^3}{\sqrt{(E(X-\mu'_1)^2)^3}} = \frac{\mu'_3 - 3\mu'_1\mu'_2 + 2\mu'^3_1}{(\mu'_2 - \mu'^2_1)^{3/2}}, \\ CK &= \frac{E(X-\mu'_1)^4}{(E(X-\mu'_1)^2)^2} = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6\mu'^2_1\mu'_2 - 3\mu'^4_1}{(\mu'_2 - \mu'^2_1)^2}. \end{aligned}$$

The Lorenz and Bonferroni curves are widely used tools for analyzing inequality and distribution in various domains. In medicine, these curves help assess the distribution of healthcare resources or treatment outcomes across populations. In the context of income and poverty, they are instrumental in measuring economic disparities and understanding wealth distribution. Similarly, in reliability and insurance, these curves are used to model risk distribution and to make informed decisions regarding policy pricing and resource allocation. The Lorenz and Bonferroni curves for the ZL distribution are given by

$$\begin{aligned} L(x) &= \frac{1}{\mu_1{}'} \int_0^{F^{-1}_{ZL}(x)} y f_{ZL}(y)dy = \frac{1}{\mu_1{}'} \int_0^x F^{-1}_{ZL}(y)dy, \\ &\textit{and} \\ B(x) &= \frac{1}{x\mu_1{}'} \int_0^{F^{-1}_{ZL}(x)} y f_{ZL}(y)dy, \end{aligned}$$

respectively. Figure 4 illustrates the Lorenz curves for the ZL distribution corresponding to selected parameters, which is calculated numerically.

TABLE 2. Moments and some related measures for selected values of the parameters.

Moments	$(\alpha = 1, \beta = 1, \theta = 1, \lambda = 0.8)$	$(\alpha = 2, \beta = 1, \theta = 1, \lambda = 0.2)$	$(\alpha = 0.5, \beta = 1, \theta = 0.3, \lambda = 0.5)$
μ'_1	0.4267	0.8060	0.6968
μ'_2	0.3276	0.8094	0.9425
μ'_3	0.3307	0.9385	1.5959
μ'_4	0.3931	1.2085	3.0668
CV	0.8939	0.4958	0.9700
CS	1.2011	0.4468	0.9789
CK	4.1110	2.8054	3.1472

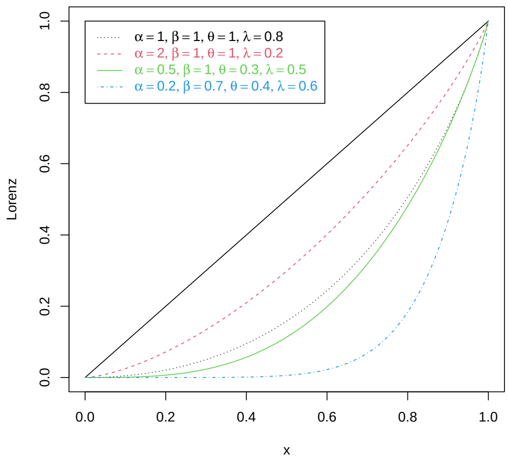

FIGURE 4. Lorenz curve for the ZL distribution.

5. Real data applications

. In this section, we analyze two real data sets to demonstrate the performance of the ZPS distribution in practice. We illustrate the superiority of the ZL distribution from the new class as compared to some of the previous models. We compare the fit of the ZL distribution with nine distributions as given in the Appendix. The following criteria are used to compare the performance of the models.

1. Akaike Information Criterion (AIC):

$$AIC = 2k - 2\log(L).$$

2. Corrected Akaike Information Criterion (AICC):

$$AICC = AIC + \frac{2k(k+1)}{n-k-1}.$$

3. Bayesian Information Criterion (BIC):

$$BIC = k\log(n) - 2\log(L).$$

4. Sum of Squares (SS) from the probability plots:

$$SS = \sum_{i=1}^{n}\left(p_{(i)} - \frac{i-0.375}{n+0.25}\right)^2,$$

where $L = L(\hat{\boldsymbol{\theta}})$ is the value of the likelihood function calculated at the parameter estimates, n is the number of observations, k is the number of estimated parameters and where $p_{(i)} = F_{\hat{\boldsymbol{\theta}}}(x_{(i)})$ is the value of the distribution function calculated at the parameter estimates.

To verify which distribution provides the best fit for the data, we calculate the p-values of Kolmogorov-Smirnov test and the goodness-of-fit test statistics W^* and A^* as follows.

1. Cramer von Mises statistic:

$$W^* = \frac{1}{12n} + \sum_{i=1}^{n}\left(\frac{2i-1}{2n} - p_{(i)}\right)^2,$$

2. Anderson-Darling statistic:

$$A^* = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log(p_{(i)}) + \log(1-p_{(n-i+1)})\right\}.$$

In general, the smaller the values of W^* and A^*, the better the fit. For more details of these statistics, see Chen and Balakrishnan [10].

Now, we consider two real data sets to verify the performance of the new class of distributions in practice.

5.1. **Plasma concentrations data.**
. The first data set contains 66 observations of plasma concentrations of indomethicin (mcg/ml). Warahena-Liyanage and Pararai [31] show that Lindly logarithmic distribution provides the best fit for the data. Table 3 lists the MLEs of the model parameters and the values of model selection criteria. The values of AIC, AICC, BIC and SS show that the ZL distribution performs the best for the data. Also, based on the goodness-of-fit statistics W^* and A^*, we find that the ZL distribution is better than the LL distribution and is the best fit for the plasma concentrations data.

Figure 5 shows the fitted densities, histogram and observed probability versus predicted probability of plasma concentrations data.

5.2. **Glass fibres data set.**

TABLE 3. Estimates of models for plasma concentration data

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-2\log L$	AIC	$AICC$	BIC	SS	$KS(p.value)$	W^*	A^*
ZL	10.271	0.820	0.430	0.999	**37.37**	**45.36**	**46.02**	**54.13**	**0.151**	0.192	**0.149**	**0.865**
Z	1157	0.091	5.126	-	55.51	63.51	64.165	72.27	0.235	0.199	0.233	1.432
LL	-	1.122	-	0.754	61.09	65.09	65.28	69.47	0.160	0.370	0.162	1.205
LP	-	1.660	-	1.403	61.81	65.09	65.28	69.47	0.200	0.210	0.203	1.371
LG	-	1.577	-	0.553	61.19	65.09	65.28	69.47	0.165	0.352	0.166	1.220
EL	-	1.331	-	0.617	61.27	65.27	65.28	69.64	0.161	**0.379**	0.162	1.206
EG	-	1.300	-	0.402	61.32	65.32	65.28	69.70	0.164	0.359	0.165	1.215
EP	-	1.321	-	0.966	61.51	65.32	65.28	69.70	0.178	0.287	0.178	1.263
L	-	2.213	-	-	64.35	66.35	66.41	68.54	0.369	0.050	0.365	2.182
W	-	0.957	-	1.684	62.64	65.32	65.28	69.70	0.228	0.167	0.228	1.478
G	-	1.659	-	0.982	62.84	66.84	67.03	71.22	0.271	0.107	0.270	1.675

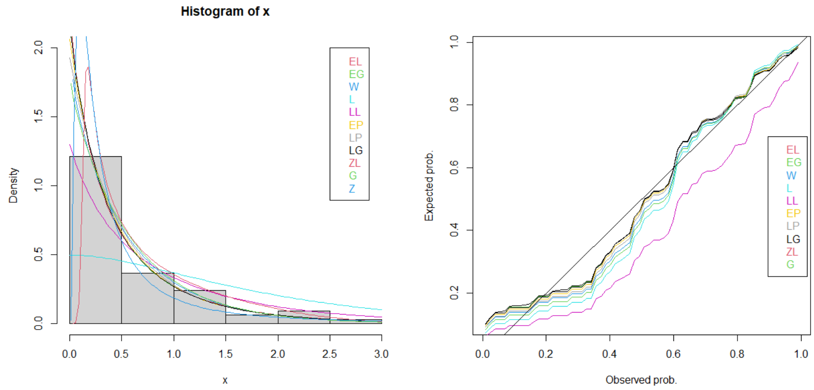

FIGURE 5. Histogram, fitted density and probability plots for plasma concentration data.

. The second data set is 63 observations obtained from Smith and Naylor [28] represent the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Results of fitting the ZL and other mentioned distributions are given in Table 4. These results show that the ZL and Weibull distributions have better fit than the other proposed models.

TABLE 4. Estimates of models for glass fibres data

Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$-2\log L$	AIC	$AICC$	BIC	SS	$KS(p.value)$	W^*	A^*
ZL	1.949	1.683	0.173	0.0007	**28.547**	36.547	37.237	45.120	**0.160**	**0.194**	**0.165**	**0.933**
LL	-	0.996	-	0.0001	162.558	166.558	166.758	170.845	3.302	$5.6 \times e^{-9}$	3.332	16.245
LP	-	0.996	-	0.0001	162.559	166.559	166.759	170.845	3.301	$5.6 \times e^{-9}$	3.332	16.245
LG	-	0.996	-	$5.25 \times e^{-5}$	162.558	166.558	166.758	170.845	3.302	$5.6 \times e^{-9}$	3.332	16.245
EL	-	0.664		$1.68 \times e^{-11}$	177.660	181.660	181.860	185.947	7.374	$1.7 \times e^{-10}$	7.406	34.389
EG	-	0.664	-	$1.39 \times e^{-11}$	177.661	181.660	181.860	185.947	3.831	$1.7 \times e^{-10}$	3.862	18.426
EP	-	0.664	-	$6.14 \times e^{-6}$	177.661	181.661	181.861	185.947	3.831	$1.7 \times e^{-10}$	3.862	18.426
L	-	0.996	-	-	162.557	164.557	164.623	166.700	3.302	$5.6 \times e^{-9}$	3.332	16.245
W	-	5.781	-	0.0600	30.414	**34.414**	**34.614**	**38.700**	0.211	0.097	0.215	1.241
G	-	11.574	-	17.4400	47.903	51.903	52.103	56.189	0.559	0.004	0.566	3.087

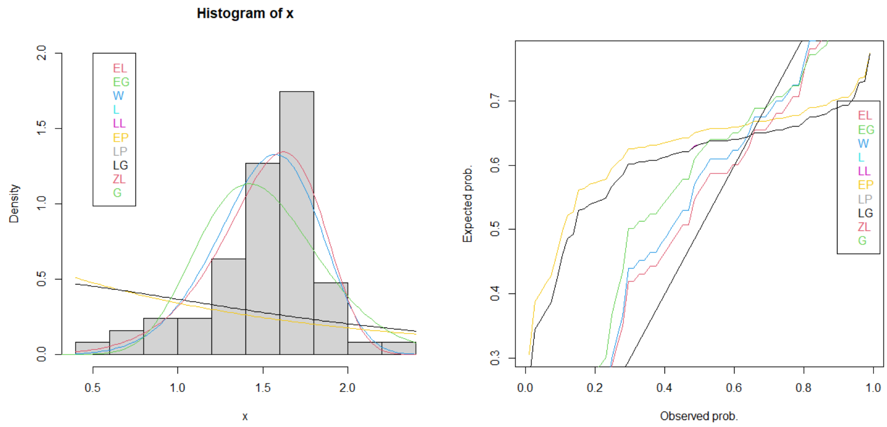

FIGURE 6. Histogram, fitted density and probability plots for plasma concentration data.

6. Conclusion

In this paper, we provided a general closed-form expression for the cumulative distribution function of a class of distributions that can be found by compounding the well-known continuous distributions with the power series distribution. Then, we proposed a new class of four-parameter lifetime distributions called the ZPS distribution. Some mathematical properties of the cumulative distribution, probability density and failure rate functions of the ZPS distribution are examined. In the sequel, we considered the ZL distribution as a special case of the new proposed distribution and presented a simulation study to exhibit the performance and accuracy of the maximum likelihood estimators of the parameters. Real data applications are also presented to illustrate the usefulness and applicability of the ZL distribution. The results of goodness-of-fit tests showed that the ZL distribution has better fit than some other recent introduced models.

7. Author contributions

Conceptualization by M.C. and J.E.; Software by J.E. and H.M.; Writing and original draft by H.M.; Writing, review, and editing by M.C, J.E. and H.M.

8. Data Availability

The authors confirm that the data supporting the findings of this study are available within the papers which is addressed in the manuscript.

9. Aknowledgement

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10. Ethical statements

The submitted manuscript is original and has not been published elsewhere in any form or language.

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12. Conflicts of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

13. Appendix A

Some useful quantities for power series family of distributions are given in Table 5.

Table 5. Useful quantities of some members of power series family.

Dis.	$C(\lambda)$	$C'(\lambda)$	$C''(\lambda)$	$C'''(\lambda)$	$C^{-1}(\lambda)$	a_n	Par. Space
Poisson	$e^{\lambda}-1$	e^{λ}	e^{λ}	e^{λ}	$\log(1+\lambda)$	$(n!)^{-1}$	$(0,\infty)$
Geometric	$\lambda(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$2(1-\lambda)^{-3}$	$6(1-\lambda)^{-4}$	$\lambda(1+\lambda)^{-1}$	1	$(0,1)$
Logarithmic	$-\log(1-\lambda)$	$(1-\lambda)^{-1}$	$(1-\lambda)^{-2}$	$2(1-\lambda)^{-3}$	$1-e^{-\lambda}$	n^{-1}	$(0,1)$
Binomial	$(1+\lambda)^m-1$	$\frac{m}{(1+\lambda)^{1-m}}$	$\frac{m(m-1)}{(1+\lambda)^{2-m}}$	$\frac{m(m-1)(m-2)}{(1+\lambda)^{3-m}}$	$(1+\lambda)^{\frac{1}{m}}-1$	$\binom{m}{n}$	$(0,1)$

The list of nine distributions which are compared with ZL distribution in Section 4.

1. Lindley- logarithmic (LL) distribution, introduced by Warahena-Lianage and Pararai [31], with given pdf as

$$f_{LL}(x;\beta,\lambda)=\frac{\lambda\beta^2(1+x)e^{-\beta x}}{(\beta+1)\log(1-\lambda)\left(\lambda\left(\frac{1+\beta+\beta x}{\beta+1}\right)e^{-\beta x}-1\right)}, \tag{20}$$

where $x>0,\beta>0,0<\lambda<1$.

2. The Lindley geometric (LG) distribution, introduced and studied by Zakerzadeh and Mahmoudi [32], with density function

$$f_{LG}(x;\beta,\lambda)=\frac{\beta^2(1-\lambda)(1+x)e^{-\beta x}}{(\beta+1)[1-\lambda(\frac{1+\beta+\beta x}{\beta+1})e^{-\beta x}]^2},\quad x>0,\beta>0,0<\lambda<1.$$

3. The Lindley Poisson (LP) distribution, introduced and studied by Gui et al. [13], given by

$$f_{LP}(x;\beta,\lambda) = \frac{\lambda\beta^2(1+x)e^{\lambda(\frac{1+\beta+\beta x}{\beta+1})e^{-\beta x}-\beta x}}{(\beta+1)(e^{\lambda}-1)}, \quad x>0, \beta>0, \lambda>0.$$

4. The pdf of exponential-logarithmic (EL) distribution, introduced by Tahmasbi and Rezaei [30], given by

$$f_{EL}(x;\beta,\lambda) = \frac{\beta\lambda\ e^{-\beta x}}{\log(1-\lambda)\,(\lambda\ e^{-\beta x}-1)}, \quad x>0, \beta>0, 0<\lambda<1,$$

5. Exponential-geometric (EG) distribution, introduced by Adamidis and Loukas [1], which is given by

$$f_{EG}(x;\beta,\lambda) = \frac{\beta(1-\lambda)e^{-\beta x}}{(1-\lambda e^{-\beta x})^2}, \quad x>0, \beta>0, 0<\lambda<1,$$

6. Exponential-Poisson (EP) distribution, introduced by Kus [15], which is expressed as

$$f_{EP}(x;\beta,\lambda) = \frac{\lambda\beta e^{-\beta x}e^{\lambda e^{-\beta x}}}{e^{\lambda}-1}, \quad x>0, \beta>0, \lambda>0,$$

7. Lindley (L) distribution, introduced by Lindley [16], with the following pdf

$$f_L(x) = \frac{\beta^2}{\beta+1}(1+x)e^{-\beta x}, \quad x>0, \beta>0, \tag{21}$$

8. Weibull (W) distribution with pdf

$$f_W(x;\beta,\lambda) = \lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}, \quad x>0, \beta>0, \lambda>0,$$

9. Gamma (G) distribution with pdf

$$f_G(x;\beta,\lambda) = \frac{\beta^{\lambda}x^{\lambda-1}e^{-\beta x}}{\Gamma(\lambda)}, \quad x>0, \beta>0, \lambda>0.$$

References

[1] Adamidis, K., & Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics and Probability Letters*, 39, 35-42. https://doi.org/10.1016/S0167-7152(98)00012-1

[2] Ahmad, Z., Elgarhy, M. & Nasir Abbas, (2020). A new extended alpha power transformed family of distributions: properties and applications, *Journal of Statistical Modelling: Theory and Applications*, 1, 13-27. https://doi.org/10.22034/jsmta.2020.1706

[3] Alizadeh, M., Bagheri, S. F., Alizadeh, M., & Nadarajah, S. (2017). A new four-parameter lifetime distribution. *Journal of Applied Statistics*, 44(5), 767-797. https://doi.org/10.1080/02664763.2016.1182137

[4] Alizadeh, M., Bagheri, S. F., Samani, E. B., Ghobadi, S., & Nadarajah, S. (2018). Exponentiated power Lindley power series class of distributions: Theory and applications. *Communications in Statistics: Simulation and Computation*, 47, 2499-2531. https://doi.org/10.1080/03610918.2017.1350270

[5] Alizadeh, M., Tahmasebi, S., & Haghbin, H. (2020). The exponentiated odd log-logistic family of distributions: properties and applications. *Journal of Statistical Modelling: Theory and Applications*, 1, 29-52. https://doi.org/10.22034/jsmta.2020.1707

[6] Asgharzadeh, A., Bakouch, H. S., Nadarajah, S., & Esmaeili, L. (2014). A new family of compound lifetime distributions. *Kybernetika*, 50, 142-169. https://doi.org/10.14736/kyb-2014-1-0142

[7] Bagheri, S. F., Samani, E. B., & Ganjali. M. (2016). The generalized modified Weibull power series distribution: Theory and applications. *Computational Statistics and Data Analysis*, 94, 136-160.

[8] Chahkandi, M., & Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. *Computational Statistics and Data Analysis*, 53, 4433-4440. https://doi.org/10.1016/j.csda.2009.06.016

[9] Chen, Z. (2000). A new two-parameter Lifetime distribution with bathtub shape or increasing failure rate function. *Statistics and Probability Letters*, 49, 155-161. https://doi.org/10.1016/S0167-7152(00)00044-4

[10] Chen, G., & Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, 27, 154-161. https://doi.org/10.1080/00224065.1995.11979578

[11] Cordeiro, G. M., Ortega, E. M. M., & Silva, G. O. (2011). The exponentiated generalized gamma distribution with application to lifetime data. *Journal of Statistical Computation and Simulation*, 81, 827-842. https://doi.org/10.1080/00949650903517874

[12] Faton, M., & Elbatal, I. (2015). Weibull Rayleigh distribution: Theory and applications. *Applied Mathematics and Information Sciences*, 5, 1-11. https://dx.doi.org/10.12785/amis/paper

[13] Gui, W., Zhang S., & Lu, X. (2014). The Lindley-Poisson distribution in lifetime analysis and its properties. *Journal of Mathematics and Statistics*, 10-63. https://doi.org/10.15672/HJMS.201427453

[14] Hassan, A., Akhtar, N., Rashid, A., & Iqbal, A. (2021). A New Compound Lifetime Distribution: Ishita Power Series Distribution with Properties and Applications. *Journal of Statistics Applications and Probability*, 10(3), 883-896. http://dx.doi.org/10.18576/jsap/100324

[15] Kus, C. (2007). A new lifetime distribution. *Computational Statistics and Data Analysis*, 51, 4497-4509. https://doi.org/10.1016/j.csda.2006.07.017

[16] Lindley, D.V. (1958). Fiducial distributions and Bayes' theorem. *Journal of the Royal Statistical Society, Series B (Methodological)*, 20, 102-107. https://www.jstor.org/stable/2983909

[17] Mahmoodian, H. (2020). Power Normal-Geometric Distribution: Model, Properties and Applications. *Journal of Statistical Research of Iran*, 17(1), 95-111. https://doi.org/10.52547/jsri.17.1.95

[18] Mahmoudi, E., & Sepahdar, A. (2013). Exponentiated Weibull-Poisson distribution: Model, properties and applications. *Mathematics and Computers in Simulation*, 92, 76-97. https://doi.org/10.1016/j.matcom.2013.05.005

[19] Mahmoudi, E., Sepahdar, A., & Lemonte, A. (2014). Exponentiated Weibull-logarithmic distribution: model, properties and applications. https://doi.org/10.48550/arXiv.1402.5264

[20] Moakofi, T., Oluyede, B., Chipepa, F., & Makubate, B. (2021). Odd power generalized Weibull-G family of distributions: Model, properties and applications. *Journal of Statistical Modelling: Theory and Applications*, 2, 121-142. https://doi.org/10.22034/jsmta.2021.2333

[21] Morais, A. L., & Barreto-Souza, W. (2011). A compound class of Weibull and power series distributions. *Computational Statistics and Data Analysis*, 55, 1410-1425. https://doi.org/10.1016/j.csda.2010.09.030

[22] Pandey, A. K., Ali, A., & Pathak, A. K. (2024). On generalized transmuted lifetime distribution. https://doi.org/10.48550/arXiv.2405.11624

[23] Rahman, M. M., Al-Zahrani, B., & Shahbaz, M. Q. (2018). A general transmuted family of distributions. *Pakistan Journal of Statistics and Operation Research*, 14(2), 451-469. https://doi.org/10.18187/pjsor.v14i2.2334

[24] Roozegar, R., Hamedani, G. G., Amiri, L., & Esfandiyari, F. (2020). A New Family of Lifetime Distributions: Theory, Application and Characterizations. *Annals of Data Science*, 7, 109-138. https://doi.org/10.1007/s40745-019-00216-5

[25] Shakhatreh, M.K., Dey, S., & Kumar, D. (2022). Inverse Lindley power series distributions: a new compounding family and regression model with censored data. *Journal of Applied Statistics*, 49, 3451-3476. https://doi.org/10.1080/02664763.2021.1951683

[26] Silva, R. B., Bourguignon, M., Dias, C. R. B., & Cordeiro, G. M. (2013). The compound class of extended Weibull power series distributions. *Computational Statistics and Data Analysis*, 58, 352-367. https://doi.org/10.1016/j.csda.2012.09.009

[27] Silva, R. B., Bourguignon, M., & Cordeiro, G. M. (2016). A new compounding family of distributions: The generalized gamma power series distributions. *Journal of Computational and Applied Mathematics*, 303, 119-139. https://doi.org/10.1016/j.cam.2016.02.033

[28] Smith, R. L., & Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three parameter Weibull distribution. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 36, 358-369. https://doi.org/10.2307/2347795

[29] Soliman Hassan, a. & Abd-Allah, M. (2023). Power Quasi Lindley Power Series Class of Distributions:Theory and Applications. *Thailand Statistician*, 21(2), 314-336. http://dx.doi.org/10.1016/j.jare.2014.08.005

[30] Tahmasbi, R., & Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate. *Computational Statistics and Data Analysis*, 52, 3889-3901. https://doi.org/10.1016/j.csda.2007.12.002

[31] Warahena-Liyanag, G., & Pararai, M. (2015). The Lindley power series class of distributions: model, properties and applications. *Journal of Computations and Modeling*, 5, 35-80. https://doi.org/10.9734/BJMCS/2016/21903

[32] Zakerzadeh, H., & Mahmoudi, E. (2012). A new two parameter lifetime distribution: model and properties, arXiv preprint arXiv:1204.4248. https://doi.org/10.48550/arXiv.1204.4248

[33] Zhang, R. (2004). A new three-parameter lifetime distribution with bathtub shape or increasing failure rate function and a flood data application. Concordia University, Canada.

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