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DERIVATIONS ON THE MATRIX SEMIRINGS OF MAX-PLUS ALGEBRA

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Abstract. Let $(S, \oplus, \otimes)$ be a matrix semiring of max-plus algebra with the addition operation $\oplus$ and the multiplication operation $\otimes$, where the set S consists of matrices constructed from real numbers together with the element negative infinity. A derivation on the semiring S is an additive mapping δ from S to itself that satisfies the axiom $\delta(x \otimes y) = (\delta(x) \otimes y) \oplus (x \otimes \delta(y))$, for every $x, y \in S$. From S we construct all of semiring derivations of S are denoted by D. On the set D, we defined two binary operations, i.e., addition "$\dotplus$" and composition "$\circ$". We want to investigate the structure of D over "$\dotplus$" and "$\circ$" operations. We show that D is not a semiring, but there exists a sub-semiring $H \subseteq D$. Here, triple $(H, \oplus, \circ)$ is a semiring which is constructed from max-plus algebra.

Keywords: semirings, matrix semiring, derivation, max-plus algebra
2020 MSC: 16Y60.

1. Introduction

In ring theory, the study of derivation has an important role in the properties of algebraic systems, analysis, and algebraic geometry. In [3], the authors introduced the concept of derivations in rings and their applications. The paper's title is "On Derivations In Rings and Their Application". Especially in analysis, it is already common to treat the derivation of one algebra into a larger algebra (into a bimodule). Several recent studies have also explored the role of derivations in various algebraic structures. Ansari et al. investigated generalized differential identities on prime rings and algebras [1], while Jordan *-derivations on standard operator algebras have been analyzed in another work [2].

A ring is an algebraic structure with two binary operations. The ring $(R, +, \cdot)$ is a non-empty set equipped with an addition operation "$+$" and a multiplication operation "$\cdot$" that satisfy some axioms, such that $(R, +)$ is a commutative group, $(R, \cdot)$ is a semigroup, and $(R, +, \cdot)$ satisfies the left and right distributive laws [6]. In 1934, H.S. Vandiver introduced a structure known as a semiring [12]. The nontrivial example of semiring first appeared in Dedekind's

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work in 1894, when he worked on a study of the algebra of ideals of a commutative ring. Despite the great efforts of some mathematicians on semiring theory in the 1940s, 1950s, and early 1960s, they did not get the attention of the mathematical community to consider semiring theory as a research area of mathematical importance. It was not until the late 1960s that semiring theory was considered a more important topic for some researches.

In this paper, we also construct a matrix semiring in Max-Plus Algebra. Many references discuss this material, such as [4], [8], [9], [10], and [11]. Max-plus algebra is the set of all real numbers $\mathbb{R}_{max} := \mathbb{R} \cup \{-\infty\}$ equipped with two binary operations, namely the addition operation $\oplus$ defined by $x \oplus y = max\{x, y\}$ and the multiplication operation $\otimes$ defined by $x \otimes y = x + y$ for every $x, y \in \mathbb{R} \cup \{-\infty\}$. The neutral element of the addition operation of the semiring structure of max-plus algebra is ε with $\varepsilon = -\infty$ and the unit element in the multiplication operation is e with $e = 0$. We construct that S is the matrix semiring from $\mathbb{R}_{max} := \mathbb{R} \cup \{-\infty\}$ with binary operations $\oplus$ and $\otimes$.

Semiring $(S, \oplus, \otimes)$ is a matrix semiring where S is a non-empty set equipped with an addition operation "$\oplus$" and a multiplication operation "$\otimes$" that satisfies some axioms. These mean S with addition operation is a commutative monoid, S with multiplication operation is a semigroup, and S satisfies the left distributive and right distributive law [7]. Furthermore, we want to apply the derivation concepts to the semiring S of $\mathbb{R}_{max} := \mathbb{R} \cup \{-\infty\}$.

A derivation on the semiring S is a linear mapping δ from S to itself that satisfies the axiom $\delta(x \otimes y) = (\delta(x) \otimes y) \oplus (x \otimes \delta(y))$, for every $x, y \in S$ [5]. In the context of max-plus algebra, we explore the set D of all derivations on the semiring S. This paper discusses definitions and examples related to derivations on the semiring of max-plus algebra, including the semiring construction of these derivations. By analyzing these derivations, we aim to deepen the understanding of max-plus algebra, which could provide insight into its applications in optimization and control theory, where it is widely used. Our research provides new insights into the role and the behaviors of derivations in semiring, contributing to the larger field of algebraic structures.

2. Methods

We study some theories related to max-plus algebra and derivation theory. The authors defined the matrix semiring algebra structure of max-plus algebra, namely $(S, \oplus, \otimes)$. The characteristics that appear due to the definition will be analyzed. Given some examples of derivations on semiring $(S, \oplus, \otimes)$.

From semiring S, we construct the set D as the all of semiring derivations from S to itself. In D, we define binary operations $\dotplus$ and composition $\circ$. By using the set of D, we want to explain that there exists H $\subseteq$ D such that $(H, \dotplus, \circ)$ is a semiring.

3. Result

We start our discussion by giving the definition of semiring and max-plus algebra theories. For the following definition, the authors cited [?].

Definition 3.1. [7] Let S be a non-empty set with two binary operations $+$ and $\cdot$. The triple $(S, +, \cdot)$ is called a semiring if it satisfies the following axioms:

(1) $(S, +)$ is a commutative monoid,
(2) $(S, \cdot)$ is a semigroup,
(3) $(S, +, \cdot)$ is satisfying the distributive laws.
 a. For each $a, b, c \in S, a \cdot (b + c) = a \cdot b + a \cdot c$ (left distributive law)
 b. For each $a, b, c \in S, (a + b) \cdot c = a \cdot c + b \cdot c$ (right distributive law)

After defining the definition of semiring, I will define max-plus algebra to understand the concept of derivation on the matrix semiring of max-plus algebra in this paper.

Definition 3.2. [8] Let $\mathbb{R} \cup \{-\infty\}$ with $\varepsilon = -\infty$. Defined the operations $\oplus$ and $\otimes$ i.e. for any $a, b \in \mathbb{R} \cup \{-\infty\}$, $a \oplus b = max\{a, b\}$ and $a \otimes b = a + b$. The triple $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is Max-Plus Algebra, we denoted by $(\mathbb{R}_{max}, \oplus, \otimes)$.

Example 3.3. *Triple $(\mathbb{R}_{max}, \oplus, \otimes)$ is a semiring with $\varepsilon = -\infty$ is a neutral element of $\oplus$ and $e = 0$ is a neutral element of $\otimes$.*

(1) *Tuple $(\mathbb{R}_{max}, \oplus)$ is a commutative monoid. For any $x, y, z, \varepsilon \in \mathbb{R}_{max}$, we have:*
 a. The operation $\oplus$ is closed because $x \oplus y = max\{x, y\} \in \mathbb{R}_{max}$.
 b. The operation $\oplus$ is well-defined because for every $x, y \in \mathbb{R}_{max}$, the value of $x \oplus y = max\{x, y\}$ is unique.
 c. Associative property, $(x \oplus y) \oplus z = max(max(x, y), z) = max(x, y, z) = max(x, max(y, z)) = x \oplus (y \oplus z)$.
 d. There exist $\varepsilon = -\infty$ for any $x \in \mathbb{R}_{max}$, $x \oplus \varepsilon = max(x, -\infty) = x = max(-\infty, x) = \varepsilon \oplus x$.
 e. For any $x, y \in \mathbb{R}_{max}$, $x \oplus y = max(x, y) = max(y, x) = y \oplus x$.
(2) *Tuple $(\mathbb{R}_{max}, \otimes)$ is a semigroup. For any $x, y, z, \varepsilon \in \mathbb{R}_{max}$, we have:*
 a. The operation $\otimes$ is closed because $x \otimes y = x + y \in \mathbb{R}_{max}$.
 b. The operation $\otimes$ is well-defined because for every $x, y \in \mathbb{R}_{max}$, the value of $x \otimes y = x + y$ is unique.
 c. Associative property, $(x \otimes y) \otimes z = (x + y) + z = x + (y + z) = x \otimes (y \otimes z)$.
(3) *Triple $(\mathbb{R}_{max}, \oplus, \times)$ satisfies the distributive law. For every $x, y, z \in \mathbb{R}_{max}$ below.*
 a. $(x \oplus y) \otimes z = max(x, y) + z = max(x + z, y + z) = (x \otimes z) \oplus (y \otimes z)$.
 b. $x \otimes (y \oplus z) = x + max(y, z) = max(x + y, x + z) = (x \otimes y) \oplus (x \otimes z)$.

The triple $(\mathbb{R}_{max}, \oplus, \otimes)$ is semiring of max-plus algebra.

Next, we can continue to define the derivation on semiring.

Definition 3.4. [5] Let $(S, +, \cdot)$ is a semiring. A map $\delta : S \to S$ is said to be a derivation on S, if

(1) $\delta(x+y) = \delta(x) + \delta(y), \forall x, y \in S$
(2) $\delta(x \cdot y) = \delta(x) \cdot y + x \cdot \delta(y), \forall x, y \in S$

Based on the definition, the following derivation on matrix semiring of max-plus algebra is defined.

Remark 3.5. Let $(S, \oplus, \otimes)$ be a matrix semiring of max-plus algebra. A map $\delta : S \to S$ is said to be a derivation on S, if

(1) $\delta(x \oplus y) = \delta(x) \oplus \delta(y), \forall x, y \in S$
(2) $\delta(x \otimes y) = \delta(x) \otimes y \oplus x \otimes \delta(y), \forall x, y \in S$

Next, we give an example of derivation on the matrix semiring of max-plus algebra.

Example 3.6. *Given a semiring $(S, \oplus, \otimes)$, where S is the set of matrices from $\mathbb{R}_{max} := \mathbb{R} \cup \{-\infty\}$, $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbb{R}_{max} \right\}$. Defined a relation $\delta : S \to S$ where $\delta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}$, for each $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S$. It will be proved that the relation δ is a function as below.*

1. For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S$, we have $\delta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}$, since $\varepsilon \in \mathbb{R}_{max}$, then $\begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \in S$.

2. For any $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in S$ where $a_1 = a_2, b_1 = b_2, c_1 = c_2$, and $d_1 = d_2$. Then, $\delta\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) = \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} = \delta\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)$.

Thus, δ is a mapping. Now, we prove that δ is a derivation on S. For any $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in S$, we have:

$$\begin{aligned} \delta\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \oplus \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= \delta\left(\begin{bmatrix} \max(a_1, a_2) & \max(b_1, b_2) \\ \max(c_1, c_2) & \max(d_1, d_2) \end{bmatrix}\right) \\ &= \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \end{aligned}$$

Since $max(a_1, a_2), max(b_1, b_2), max(c_1, c_2), max(d_1, d_2) \in \mathbb{R}_{max}$

$$\begin{aligned}
\delta\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \otimes \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= \delta\left(\begin{bmatrix} \max(a_1 \otimes a_2, b_1 \otimes c_2) & \max(a_1 \otimes b_2, b_1 \otimes d_2) \\ \max(c_1 \otimes a_2, d_1 \otimes c_2) & \max(c_1 \otimes b_2, d_1 \otimes d_2) \end{bmatrix}\right) \\
&= \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \oplus \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \\
&= \left(\begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \oplus \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \otimes \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}\right) \\
&= \delta\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) \otimes \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \oplus \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \otimes \delta\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right)
\end{aligned}$$

Thus, $\delta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}$ *for each* $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S$ *is a derivation of* S*.*

Example 3.7. *Let* $(S, \oplus, \otimes)$*, where* S *is the set of the matrix from* $\mathbb{R}_{max} := \mathbb{R} \cup \{-\infty\}$ *and* $S = \left\{ \begin{bmatrix} a & b \\ \varepsilon & a \end{bmatrix} \middle| a, b, \varepsilon \in \mathbb{R}_{max} \right\}$*. Defined a relation* $\delta : S \to S$ *where* $\delta\left(\begin{bmatrix} a & b \\ \varepsilon & a \end{bmatrix}\right) = \begin{bmatrix} \varepsilon & b \\ \varepsilon & \varepsilon \end{bmatrix}$ *for each* $\begin{bmatrix} a & b \\ \varepsilon & a \end{bmatrix} \in S$*. It will be proved that the relation* δ *is a function as below.*

1. *For any* $\begin{bmatrix} a & b \\ \varepsilon & a \end{bmatrix} \in S$*, we have* $\delta\left(\begin{bmatrix} a & b \\ \varepsilon & a \end{bmatrix}\right) = \begin{bmatrix} \varepsilon & b \\ \varepsilon & \varepsilon \end{bmatrix}$*, since* $b, \varepsilon \in \mathbb{R}_{max}$*, then* $\begin{bmatrix} \varepsilon & b \\ \varepsilon & \varepsilon \end{bmatrix} \in S$*.*
2. *For any* $\begin{bmatrix} a & b \\ \varepsilon & a \end{bmatrix}, \begin{bmatrix} c & d \\ \varepsilon & c \end{bmatrix} \in S$ *where* $a = c$ *and* $b = d$*. Then,*

$$\begin{aligned}
\delta\left(\begin{bmatrix} a & b \\ \varepsilon & a \end{bmatrix}\right) &= \begin{bmatrix} \varepsilon & b \\ \varepsilon & b \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon & d \\ \varepsilon & d \end{bmatrix} \\
&= \delta\left(\begin{bmatrix} c & d \\ \varepsilon & c \end{bmatrix}\right)
\end{aligned}$$

Thus, δ *is a mapping. Now, we prove that* δ *is a derivation on* S*. For any* $\begin{bmatrix} a & b \\ \varepsilon & a \end{bmatrix}, \begin{bmatrix} c & d \\ \varepsilon & c \end{bmatrix} \in S$*, we have :*

$$\begin{aligned}\delta\left(\begin{bmatrix}a & b\\ \varepsilon & a\end{bmatrix}\oplus\begin{bmatrix}c & d\\ \varepsilon & c\end{bmatrix}\right) &= \delta\left(\begin{bmatrix}\max(a,c) & \max(b,d)\\ \varepsilon & \max(a,c)\end{bmatrix}\right)\\ &= \begin{bmatrix}\varepsilon & \max(b,d)\\ \varepsilon & \varepsilon\end{bmatrix}\\ &= \begin{bmatrix}\varepsilon & b\\ \varepsilon & \varepsilon\end{bmatrix}\oplus\begin{bmatrix}\varepsilon & d\\ \varepsilon & \varepsilon\end{bmatrix}\\ &= \delta\left(\begin{bmatrix}a & b\\ \varepsilon & a\end{bmatrix}\right)\oplus\delta\left(\begin{bmatrix}c & d\\ \varepsilon & c\end{bmatrix}\right)\end{aligned}$$

$$\begin{aligned}\delta\left(\begin{bmatrix}a & b\\ \varepsilon & a\end{bmatrix}\otimes\begin{bmatrix}c & d\\ \varepsilon & c\end{bmatrix}\right) &= \delta\left(\begin{bmatrix}\max(a\otimes c,\varepsilon) & \max(a\otimes d, b\otimes c)\\ \varepsilon & a\otimes c\end{bmatrix}\right)\\ &= \delta\left(\begin{bmatrix}a\otimes c & \max(a\otimes d, b\otimes c)\\ \varepsilon & a\otimes c\end{bmatrix}\right)\\ &= \begin{bmatrix}\varepsilon & \max(a\otimes d, b\otimes c)\\ \varepsilon & \varepsilon\end{bmatrix}\\ &= \begin{bmatrix}\varepsilon & a\otimes d\\ \varepsilon & \varepsilon\end{bmatrix}\oplus\begin{bmatrix}\varepsilon & b\otimes c\\ \varepsilon & \varepsilon\end{bmatrix}\\ &= \begin{bmatrix}\varepsilon & b\otimes c\\ \varepsilon & \varepsilon\end{bmatrix}\oplus\begin{bmatrix}\varepsilon & a\otimes d\\ \varepsilon & \varepsilon\end{bmatrix}\\ &= \left(\begin{bmatrix}\varepsilon & b\\ \varepsilon & \varepsilon\end{bmatrix}\otimes\begin{bmatrix}c & d\\ \varepsilon & c\end{bmatrix}\right)\oplus\left(\begin{bmatrix}a & b\\ \varepsilon & a\end{bmatrix}\otimes\begin{bmatrix}\varepsilon & \varepsilon\\ \varepsilon & \varepsilon\end{bmatrix}\right)\\ &= \delta\left(\begin{bmatrix}a & b\\ \varepsilon & a\end{bmatrix}\right)\otimes\begin{bmatrix}c & d\\ \varepsilon & c\end{bmatrix}\oplus\begin{bmatrix}a & b\\ \varepsilon & a\end{bmatrix}\otimes\delta\left(\begin{bmatrix}c & d\\ \varepsilon & c\end{bmatrix}\right)\end{aligned}$$

Thus, $\delta\left(\begin{bmatrix}a & b\\ \varepsilon & a\end{bmatrix}\right) = \begin{bmatrix}\varepsilon & b\\ \varepsilon & \varepsilon\end{bmatrix}$ *for each* $\begin{bmatrix}a & b\\ \varepsilon & a\end{bmatrix}\in S$ *is a derivation of* S*.*

Next, we will construct the set of all derivations on the semiring S, denoted by D, as bellow:

$D = \{\delta : S \longrightarrow S \mid\ \delta$ is a derivation on matrix semiring of max-plus algebra$\}$.

On the D, given two operations " $\dot{+}$ " and " $\circ$ " where " $\dot{+}$ " operation is the addition function and " $\circ$ " is the composition function. From this, we want to investigate the structure of D over " $\dot{+}$ " and " $\circ$ ". Take any δ, δ' ∈ D, where

(1) $(\delta \dot{+} \delta')(x) = \delta(x) \oplus \delta'(x), \forall x \in S$,
(2) $(\delta \circ \delta')(x) = \delta(\delta'(x)), \forall x \in S$.

Furthermore, we want to check for every δ, δ' ∈ D, such that $\delta \dot{+} \delta'$ and $\delta \circ \delta'$ are functions.

(1) Take any δ, $\delta' \in$ D and $x \in S$. It will be proved that " $\dot{+}$ " is closed on the set D. We have :

$$(\delta \dot{+} \delta')(x) = \delta(x) \oplus \delta'(x), \forall x \in S.$$

Since δ: $S \to S$ and δ': $S \to S$ are functions, then $\delta(x) \oplus \delta'(x) \in D$. Next, take any $\delta, \delta', \delta'', \delta''' \in D$ and $x \in S$ with $\delta = \delta''$ and $\delta' = \delta'''$ such that

$$\begin{aligned}(\delta \dot{+} \delta')(x) &= \delta(x) \oplus \delta'(x)\\ &= \delta''(x) \oplus \delta'''(x)\\ &= (\delta'' \dot{+} \delta''')(x).\end{aligned}$$

From this, it can be understood that " $\dot{+}$ " on the set D is well-defined. Since it is proved to be closed and well-defined, then $(\delta \dot{+} \delta')(x)$ is a function.

(2) Take any δ, $\delta' \in$ D and $x \in S$. It will be proved that " $\circ$ " is closed on the set D. We have:

$$(\delta \circ \delta')(x) = \delta(\delta'(x)), \forall x \in S.$$

Since δ: $S \to S$ and δ': $S \to S$ are functions, then $\delta(x) \circ \delta'(x) \in D$. Next, take any $\delta, \delta', \delta'', \delta''' \in D$ and $x \in S$ with $\delta = \delta''$ and $\delta' = \delta'''$ such that

$$\begin{aligned}(\delta \circ \delta')(x) &= \delta(\delta'(x))\\ &= \delta''(\delta'''(x))\\ &= (\delta'' \circ \delta''')(x).\end{aligned}$$

From this, it can be understood that " $\circ$ " on the set D is well-defined. Since it is proved to be closed and well-defined, then $(\delta \circ \delta')(x)$ is a function.

We will construct the set of derivations on the semiring, for which $\delta \dot{+} \delta'$ and $\delta \circ \delta'$ are functions, it will be shown that $\delta \dot{+} \delta'$ and $\delta \circ \delta'$ are derivations with the following proposition.

Proposition 3.8. *If $\delta, \delta' \in D$ are derivations on the matrix semiring of max-plus algebra, then $\delta \dot{+} \delta'$ is a derivation.*

Proof. Take any $\delta, \delta' \in D$ and $x, y \in S$. It will be proved that $\delta \dot{+} \delta'$ is a derivation. Then obtained:

$$\begin{aligned}(\delta \dotplus \delta')(x \oplus y) &= \delta(x \oplus y) \oplus \delta'(x \oplus y)\\ &= (\delta(x) \oplus \delta(y)) \oplus (\delta'(x) \oplus \delta'(y))\\ &= (\delta(x) \oplus \delta'(x)) \oplus (\delta(y) \oplus \delta'(y))\\ &= (\delta \dotplus \delta')(x) \oplus (\delta \dotplus \delta')(y)\end{aligned}$$

$$\begin{aligned}(\delta \dotplus \delta')(x \otimes y) &= \delta(x \otimes y) \oplus \delta'(x \otimes y)\\ &= ((\delta(x) \otimes y) \oplus (x \otimes \delta'(y))) \oplus ((\delta'(x) \otimes y) \oplus (x \otimes \delta(y)))\\ &= ((y \otimes \delta(x)) \oplus (x \otimes \delta(y))) \oplus ((y \otimes \delta'(x)) \oplus (x \otimes \delta'(y)))\\ &= ((x \otimes \delta(y)) \oplus (x \otimes \delta'(y))) \oplus ((y \otimes \delta(x)) \oplus (y \otimes \delta'(x)))\\ &= (x \otimes (\delta(y) \oplus \delta'(y))) \oplus (y \otimes (\delta(x) \oplus \delta'(x)))\\ &= (x \otimes (\delta \dotplus \delta')(y)) \oplus (y \otimes (\delta \dotplus \delta')(x))\\ &= ((\delta \dotplus \delta')(x) \otimes y) \oplus (x \otimes (\delta \dotplus \delta')(y))\\ &= ((\delta \dotplus \delta')(x) \otimes y) \oplus (x \otimes (\delta \dotplus \delta')(y))\end{aligned}$$

Thus, we prove that if $\delta, \delta' \in D$ are derivations on the matrix semiring of max-plus algebra, then $\delta \dotplus \delta'$ is a derivation.

□

Proposition 3.9. *Given $\delta, \delta' \in D$ are derivations on the matrix semiring of max-plus algebra. If $((\delta(x) \otimes \delta'(y)) \oplus (\delta'(x) \otimes \delta(y)) = \varepsilon_S$ where $\varepsilon_S = \begin{bmatrix}\varepsilon & \varepsilon\\ \varepsilon & \varepsilon\end{bmatrix}$ for all $x, y, \varepsilon_S \in S$ then $\delta \circ \delta'$ is a derivation.*

Proof. Take any $\delta, \delta' \in D$. It will be proved that $\delta \circ \delta'$ is not a derivation for every $\delta, \delta' \in D$. For any $x, y \in S$ we have:

$$\begin{aligned}(\delta \circ \delta')(x \oplus y) &= \delta(\delta'(x \oplus y))\\ &= \delta(\delta'(x) \oplus \delta'(y))\\ &= \delta((\delta'(x)) \oplus \delta(\delta'(y))\\ &= (\delta \circ \delta')(x) \oplus (\delta \circ \delta')(y)\end{aligned}$$

$$\begin{aligned}(\delta \circ \delta')(x \otimes y) &= \delta(\delta'(x \otimes y))\\ &= \delta((\delta'(x) \otimes y) \oplus (x \otimes \delta'(y)))\\ &= \delta((\delta'(x) \otimes y) \oplus (x \otimes \delta'(y)))\\ &= ((\delta(\delta'(x)) \otimes y) \oplus (\delta'(x) \otimes \delta(y))) \oplus ((\delta(x) \otimes \delta'(y)) \oplus (x \otimes \delta(\delta'(y))))\\ &= ((\delta(\delta'(x)) \otimes y) \oplus (x \otimes \delta(\delta'(y)))) \oplus ((\delta(x) \otimes \delta'(y)) \oplus (\delta'(x) \otimes \delta(y)))\\ &= ((\delta \circ \delta')(x) \otimes y) \oplus (x \otimes (\delta \circ \delta')(y)) \oplus ((\delta(x) \otimes \delta'(y)) \oplus (\delta'(x) \otimes \delta(y))).\end{aligned}$$

We can take

$$\varepsilon_S = ((\delta(x) \otimes \delta'(y)) \oplus (\delta'(x) \otimes \delta(y)).$$

We have that,

$$\begin{aligned}(\delta \circ \delta')(x \otimes y) &= ((\delta \circ \delta')(x) \otimes y) \oplus (x \otimes (\delta \circ \delta')(y)) \oplus ((\delta(x) \otimes \delta'(y)) \oplus (\delta'(x) \otimes \delta(y))) \\ &= max\{((\delta \circ \delta')(x) \otimes y) \oplus (x \otimes (\delta \circ \delta')(y)), \varepsilon_S\} \\ &= ((\delta \circ \delta')(x) \otimes y) \oplus (x \otimes (\delta \circ \delta')(y)).\end{aligned}$$

It is proven that if $((\delta(x) \otimes \delta'(y)) \oplus (\delta'(x) \otimes \delta(y)) = \varepsilon_S$ for each $x, y, \varepsilon_S \in S$ then $\delta \circ \delta'$ is a derivation.

□

Based on Proposition 3.8 and Proposition 3.9, we can construct the set of derivations on the matrix semiring structure of max-plus algebra and attempt to make it semiring.

Theorem 3.10. *Given a set $D = \{\delta : S \to S \mid \delta$ is a derivation on the matrix semiring of max-plus algebra$\}$. Defined two binary operations on D namely addition operation " $\dot{+}$ " and composition operation " $\circ$ ". Next, we have the set $H \subseteq D$ such that*

$$H = \{\delta : S \to S \mid \forall \delta' \in D, ((\delta(x) \otimes \delta'(y)) \oplus (\delta'(x) \otimes \delta(y)) = \varepsilon_S, \forall x, y, \varepsilon_S \in S\}.$$

Then triple $(H, \dot{+}, \circ)$ is a semiring.

Proof. First, we need to ensure the existence of H. We know that $\delta(x) = x$ with $x \neq \varepsilon_S$ and $\delta'(x) = \varepsilon_S$ satisfy $((\delta(x) \otimes \delta'(y)) \oplus (\delta'(x) \otimes \delta(y)) = \varepsilon_S$ for every $x, y, \varepsilon_S \in S$ and $\delta, \delta' \in H$. By Proposition 3.8 and Proposition 3.9, it has been shown that the addition operation " $\dot{+}$ " and composition " $\circ$ " are functions and derivations on the set D. In the next step, it is shown that the triple $(H, \dot{+}, \circ)$ is a semiring.
1. It will be proved that $(H, \dot{+})$ is a commutative monoid.
a. Take any δ, $\delta' \in$ H and $x \in S$, let it be proved whether the addition operation " $\dot{+}$ " is closed on the set H. Then obtained:

$$(\delta \dot{+} \delta')(x) = \delta(x) \oplus \delta'(x), \forall x \in S.$$

Since δ: $S \to S$ and δ': $S \to S$ are functions, then $\delta(x) \oplus \delta'(x) \in S$.
b. Take any $\delta, \delta', \delta'', \delta''' \in D$ and $x \in S$ with $\delta = \delta''$ and $\delta' = \delta'''$ such that it

$$\begin{aligned}(\delta \dot{+} \delta')(x) &= \delta(x) \oplus \delta'(x) \\ &= \delta''(x) \oplus \delta'''(x) \\ &= (\delta'' \dot{+} \delta''')(x).\end{aligned}$$

Since δ: $S \to S$ and δ': $S \to S$ are functions, then $\delta \dot{+} \delta'$ is unique. So, $\delta \dot{+} \delta'$ proved to be well-defined.

c. Take any $\delta, \delta', \delta'' \in H$ for each $x \in S$. Then obtained:

$$\begin{aligned}(\delta \dotplus (\delta' \dotplus \delta''))(x) &= \delta(x) \oplus (\delta'(x) \oplus \delta''(x)) \\ &= (\delta(x) \oplus \delta'(x)) \oplus \delta''(x) \\ &= ((\delta \dotplus \delta') \dotplus \delta''')(x).\end{aligned}$$

Thus, the operation $\dotplus$ proved to be associative.
d. There is $O(x) = \varepsilon_S$. Based on Example 3.6 it has been shown that O is a derivation on the matrix semiring S with $O(x) = \varepsilon_S$. For each $\delta \in H$ and $x \in S$, such that

$$\begin{aligned}(\delta \dotplus O)(x) &= \delta(x) \oplus O(x) \\ &= max\{\delta(x), \varepsilon_S\} \\ &= \delta(x)\end{aligned}$$

$$\begin{aligned}(O \dotplus \delta)(x) &= O(x) \oplus \delta(x) \\ &= max\{\varepsilon_S, \delta(x)\} \\ &= \delta(x)\end{aligned}$$

Thus, it is proved that there exists a neutral element.
e. Take any $\delta, \delta' \in H$ for each $x \in S$, such that

$$\delta(x) \dotplus \delta'(x) = \delta'(x) \dotplus \delta(x).$$

Thus, $\delta_1 \dotplus \delta_2$ proved to be commutative.

2. It will be proved that $(H, \circ)$ is semigroup.
a. Take any δ, $\delta' \in$ H, and $x \in S$. We want to show that the composition operation " $\circ$ " is closed on the set H. Then obtained:

$$(\delta \circ \delta')(x) = \delta(\delta'(x)), \forall x \in S.$$

Since δ: $S \to S$ and δ': $S \to S$ are functions. Thus, $\delta \circ \delta' \in H$.
b. Take any $\delta, \delta', \delta'', \delta''' \in H$ and $x \in S$ with $\delta = \delta''$ and $\delta' = \delta'''$ such that

$$\begin{aligned}(\delta \circ \delta')(x) &= \delta(\delta'(x)) \\ &= \delta''(\delta'''(x)) \\ &= (\delta'' \circ \delta''')(x).\end{aligned}$$

From this, it can be understood that the operation " $\circ$ " is well-defined.
c. Take any $\delta, \delta', \delta'' \in H$ for each $x \in S$. Take any $\delta, \delta', \delta'' \in H$ for each $x \in S$. Then obtained:

$$\begin{aligned}\delta(x) \circ ((\delta' \circ \delta'')(x)) &= \delta(x) \circ (\delta'(\delta''(x))) \\ &= \delta(\delta'(\delta''(x))) \\ &= (\delta(\delta'(\delta''(x)))) \\ &= ((\delta \circ \delta')(x)) \circ \delta''(x)\end{aligned}$$

Thus, it can be proved that operation $\circ$ satisfies the associative property.
3. It will be proved that $(H, \dot{+}, \circ)$ is distributive.
Take any $\delta, \delta', \delta'' \in H$ for each $x \in S$. Then obtained:

$$\begin{aligned}\delta(x) \circ ((\delta' \dot{+} \delta'')(x)) &= (\delta((\delta' \dot{+} \delta'')(x))) \\ &= (\delta(\delta'(x))) \dot{+} (\delta(\delta''(x))) \\ &= (\delta \circ \delta')(x) \dot{+} (\delta \circ \delta'')(x)\end{aligned}$$

$$\begin{aligned}((\delta' \dot{+} \delta'')(x)) \circ \delta(x) &= ((\delta' \dot{+} \delta'')(x)) \circ \delta(x) \\ &= ((\delta' \dot{+} \delta'')(\delta(x))) \\ &= ((\delta'(\delta(x))) \dot{+} ((\delta''(\delta(x))) \\ &= (\delta' \circ \delta)(x) \dot{+} (\delta'' \circ \delta)(x)\end{aligned}$$

Thus, $(H, \dot{+}, \circ)$ proved to be distributive.

So, it is proven that the triple $(H, \dot{+}, \circ)$ is a semiring.

□

4. Conclusion

Semiring is a non-empty set S with addition and multiplication operations to satisfy several axioms. Let $(S, \oplus, \otimes)$ be a matrix semiring of max-plus algebra with the addition operation $\oplus$ and the multiplication operation $\otimes$. The derivation on the semiring S is a mapping of δ to itself and satisfies $\delta(x \otimes y) = (\delta(x) \otimes y) \oplus (x \otimes \delta(y)), \forall x, y \in S$, for every $x, y \in S$.

In the semiring, S, a set D is constructed for all semiring derivations of S. Furthermore, on the set D, binary operations are defined, namely the addition operation "$\dot{+}$" and the composition operation "$\circ$". Moreover, we want to investigate D over "$\dot{+}$" and "$\circ$." We show that D is not a semiring but we have that there exists a sub-semiring $H \subseteq D$ where for any $x, y, \varepsilon_S \in S$, the mappings $\delta, \delta' : S \to S$ are elements of H such that $(\delta(x) \otimes \delta'(y)) \oplus (\delta'(x) \otimes \delta(y)) = \varepsilon_S$. So, the triple $(H, \dot{+}, \circ)$ is a semiring algebra structure of max-plus algebra. Future work could focus on a deeper algebraic analysis of H, including its internal structure and properties. Further exploration could also involve extending these results to other algebraic structures, such as different classes of semirings or algebras, to understand their derivation theories and potential applications.

5. Author Contributions

Suffi Nuralesa contributed to conceptualizing the study, designing the methodology, and interpreting the results. Nikken Prima Puspita as my teacher and colleague, helped guide me in the preparation of the article, provided critical revisions, and approved the final version for publication. All authors reviewed the article.

6. Data Availability Statement

Not applicable.

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8. Conflict of Interest

The authors declare no conflict of interest.

References

[1] Ansari, A. Z., Shujat, F., & Fallatah, A. (2023). Generalized differential identities on prime rings and algebras. AIMS Mathematics, 8(10), 22758-22765. https://doi.org/10.3934/math.20231159

[2] Ansari, A. Z., & Shujat, F. (2023). Jordan *-derivations on standard operator algebras. Filomat, 37(1), 37-41. https://doi.org/10.2298/FIL2301037A

[3] Ashraf, M., Ali, S., & Haetinger, C. (2006). On derivations in rings and their applications. The Aligarh Bulletin of Mathematics, 25(2), 79-107.

[4] Baccelli, F., Cohen, G., Olsder, G. L., & Quadrat, J. P. (1994). Synchronization and linearity: An algebra for discrete event systems. Journal of the Operational Research Society, 45(1), 15. https://doi.org/10.1057/JORS.1994.15

[5] Dimitrov, S. (2017). Derivations on semirings. AIP Conference Proceedings, 1910(1), 060011. https://doi.org/10.1063/1.5014005

[6] Fraleigh, J. B. (1999). A first course in abstract algebra (7th ed.). Addison Wesley Publishing Company.

[7] Golan, J. S. (1999). Semirings and their applications. Springer Dordrecht. https://doi.org/10.1007/978-94-015-9333-5

[8] Heidergott, B., Olsder, G. J., & van der Woude, J. (2006). Max plus at work: Modeling and analysis of synchronized systems: A course on max-plus algebra and its applications. Princeton University Press. https://doi.org/10.1515/9781400865239

[9] Mceneaney, W. (2006). Max-plus methods for nonlinear control and estimation. Boston: Birkhäuser. https://doi.org/10.1007/0-8176-4453-9

[10] Olsder, G. J., & Roos, C. (1988). Cramer and Cayley-Hamilton in the max algebra. Linear Algebra and its Applications, 101, 87-108. https://doi.org/10.1016/0024-3795(88)90145-0

[11] Schutter, B. D., & Moor, B. D. (1997). A note on the characteristic equation in the max-plus algebra. Linear Algebra and its Applications, 261(1), 237-250. https://doi.org/10.1016/S0024-3795(96)00407-7

[12] Vandiver, H. S. (1934). Note on a simple type of algebra in which the cancellation law of addition does not hold. Bulletin of the American Mathematical Society, 40(12), 914-920. https://doi.org/10.1090/S0002-9904-1934-06003-8

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