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FEEBLY LIFTING MODULES

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Abstract. In this article feebly lifting modules are defined. A module W is called feebly lifting provided, for each fully idempotent $N \leq W$ there exists a direct summand $D \leq W$ providing $D \leq N$ and $\frac{N}{D} \ll \frac{W}{D}$. The basic properties and ring characterizations of these modules are investigated.

Keywords: Lifting module, Fully idempotent submodule, Weak lifting module, Feebly lifting module.

2020 MSC: Primary 16D10, 16D40, 16D70, 16D99.

1. Introduction

The concept of lifting is now a well-recognized aspect of module theory in associative unital rings, providing practical applications in specific ring structures ([11], [13] and [7]). In the present paper, we obtain some possible results on lifting modules by defining the notion of feebly lifting.

Before we move forward, it is essential to refresh our memory on common terminology and elementary principles. They pertain to small submodules, lifting and weak lifting modules, idempotent submodules, fully idempotent modules, etc. The rings and modules discussed in this paper are all assumed to be associative with identity and unital on the left, and will be signed as $\mathfrak{R}$ and ${}_{\mathfrak{R}}W$, respectively. $N \leq W$ and $N \ll W$ mean N is a submodule and N is small in W, respectively. We will refer to [7, 10] for all undefined notions used in the text.

$N \leq W$ is called *idempotent* if $N = Hom(W,N)N = \sum\{\phi(N) : \phi : W \to N\}$ (see [7, Page 32]). Equivalently, $N \leq W$ is idempotent if, for each $x \in N$, a positive integer k, homomorphisms $\phi_i : W \to N$ and elements $x_i \in N$ $(1 \leq i \leq k)$ exist providing $x = \phi_1(x_1) + ... + \phi_k(x_k)$. It is known that, if A is a left ideal of $\mathfrak{R}$, then ${}_{\mathfrak{R}}A$ is idempotent iff $A = A^2$. Also, every direct summand of a module is an idempotent submodule. In [12], in the context of a general ring $\mathfrak{R}$, the term *fully idempotent* is used by the authors to describe an $\mathfrak{R}$-module W in which every submodule of W is idempotent.

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$N \leq W$ is called *small*, if $W \neq N + K$ for any proper submodule K of W and written $N \ll W$. A module W is called *lifting* if for every submodule $N \leq W$, a direct summand $K \leq W$ exists providing $K \leq N$ and $N/K \ll W/K$. Equivalently, W is lifting if, for every $N \leq W$, a decomposition $W = A \oplus B$ exists providing $A \leq N$ and $N \cap B \ll W$. Following [14], a module W is called *weak lifting* provided, for each semisimple submodule $N \leq W$, a direct summand $K \leq W$ exists providing $K \leq N$ and $N/K \ll W/K$. Despite this deep-rooted definition, it is still possible to find current studies in the literature [2, 3, 16].

In this note, we introduce feebly lifting modules as follows. A module W is called *feebly lifting* provided, for each fully idempotent submodule N of W, a direct summand $K \leq W$ exists providing $K \leq N$ and $N/K \ll W/K$. So, it can be verified that every lifting module is feebly lifting and every feebly lifting module is weak lifting. We present some examples showing that the relations are not reversible in general. In addition to these, we examine the cases when the converse implications are true. It is proved that a feebly lifting fully idempotent module is lifting. And the cases of being feebly lifting and weak lifting of an $\mathfrak{R}$-module W coincide over a commutative Noetherian ring $\mathfrak{R}$. We also show that any direct summand of a feebly lifting module is feebly lifting and any fully idempotent supplement submodule of a feebly lifting module is a direct summand. We investigate the suitable conditions for a factor module of a feebly lifting module to be feebly lifting. If W is a projective multiplication module and N is an idempotent submodule of W, then $\frac{W}{N}$ is feebly lifting whenever W is feebly lifting. Moreover, for a multiplication module $W = \oplus_{i=1}^{n} W_i$ if each W_i is feebly lifting, then so is W. In particular, we give a ring characterization theorem for feebly lifting modules. A fully idempotent ring $\mathfrak{R}$ is semiperfect if and only if every finitely generated projective left $\mathfrak{R}$-module is feebly lifting if and only if ${}_{\mathfrak{R}}\mathfrak{R}$ is feebly lifting.

2. FEEBLY LIFTING MODULES

For completeness let us recall the definitions of semisimple and hollow modules. A module is semisimple if it is a sum of simple modules. This is equivalent to the fact that every submodule of a module is a direct sum term. And a module whose proper submodules are all small, is called hollow [20].

Definition 2.1. Let W be a module. We call W feebly lifting provided, for each fully idempotent submodule T of W, a direct summand $D \leq W$ exists providing $D \leq T$ and $\frac{T}{D} \ll \frac{W}{D}$.

(1) Semisimple modules, hollow modules and lifting modules are feebly lifting.
(2) For any prime integer p, $\frac{\mathbb{Z}}{p\mathbb{Z}}$ is feebly lifting since the only submodules of the $\mathbb{Z}$-module $\frac{\mathbb{Z}}{p\mathbb{Z}}$ are trivial ones.

The following lemma gives some equivalent conditions for a module W to be feebly lifting.

Lemma 2.2. *The following statements are equivalent for a module W.*

(1) W is feebly lifting.
(2) For every fully idempotent $T \leq W$, a decomposition $W = K \oplus L$ exists providing $K \leq T$ and $T \cap L \ll L$.
(3) Every fully idempotent submodule T of W is of the form $T = A \oplus S$ with $A \leq_{\oplus} W$ and $S \ll W$.

Proof. $1 \Rightarrow 2$: Let W be feebly lifting and $T \leq W$ fully idempotent. Since W is feebly lifting, a direct summand D of W exists providing $D \leq T$ and $\frac{T}{D} \ll \frac{W}{D}$. As D is a direct summand, $L \leq W$ exists providing $W = D \oplus L$. Consider the isomorphism $\varphi : \frac{W}{D} \longrightarrow L$ with $\varphi(\frac{T}{D}) = T \cap L$. Since small modules are preserved under isomorphism and $\frac{T}{D} \ll \frac{W}{D}$, then $\varphi(\frac{T}{D}) = T \cap L \ll L$.

$2 \Rightarrow 3$: Let $T \leq W$ be fully idempotent. Then, a decomposition $W = K \oplus L$ exists providing $K \leq T$ and $T \cap L \ll L$. By modularity, $T = K \oplus (L \cap T)$, as required.

$3 \Rightarrow 1$: Let $T \leq W$ be fully idempotent. Then, $T = A \oplus S$ providing $A \leq_{\oplus} W$ and $S \ll W$. Consider the natural epimorphism $\pi : W \longrightarrow \frac{W}{A}$. Since $S \ll W$, $\pi(S) = \frac{S+A}{A} = \frac{T}{A} \ll \frac{W}{A}$. Hence, W is feebly lifting □

Recall from [14] that a module W is called weak lifting provided, for each semisimple submodule T of W a direct summand $D \leq W$ exists providing $D \leq T$ and $\frac{T}{D} \ll \frac{W}{D}$.

Proposition 2.3. *Let W be a feebly lifting module. If W is indecomposable, then every proper fully idempotent submodule T is small in W.*

Proof. Let $T \lneqq W$ be fully idempotent. Then, $W = A \oplus B$ providing $A \leq T$ and $T \cap B \ll B$. As W is indecomposable and T is proper, $A = 0$ and $B = W$. Then $T \ll W$. □

Lemma 2.4. *Every feebly lifting module is weak lifting.*

Proof. Let W be a feebly lifting module. Suppose that $T \leq W$ is a semisimple submodule. Then T is fully idempotent. Since W is feebly lifting, then a direct summand D of W exists providing $D \leq T$ and $\frac{T}{D} \ll \frac{W}{D}$. □

So we have the following implications:

$$\textit{Lifting} \Longrightarrow \textit{Feebly lifting} \Longrightarrow \textit{Weak lifting}$$

The examples given below show that the arrows are not reversible.

(1) A feebly lifting module need not be lifting. For instance, it is known that $\mathbb{Z}$-module $\mathbb{Z}$ is not lifting. However, since the only fully idempotent submodule of $\mathbb{Z}$ is zero, then ${}_{\mathbb{Z}}\mathbb{Z}$ is feebly lifting.

(2) Since the only fully idempotent submodule of the $\mathbb{Z}$-module $\mathbb{Q}$ is zero, then the indecomposable module $\mathbb{Q}$ is a feebly lifting module which is not lifting.
(3) For any prime integer p, let us consider the $\mathbb{Z}$-module $W = \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^3\mathbb{Z}}$. W is a weak lifting module by [14, Example 2.7] which is not lifting [11, Example 10].

Lemma 2.5. *The following statements hold for a feebly lifting module W.*
(1) *Any direct summand of W is feebly lifting.*
(2) *Any supplement fully idempotent submodule of W is a direct summand of W.*

Proof. (1) Let $T \leq_{\oplus} W$ and $X \leq T$ be fully idempotent. By, [12, Lemma 2.2 (ii)], X is fully idempotent in W. So, a decomposition $W = A \oplus B$ exists providing $A \leq X$ and $\frac{X}{A} \ll \frac{W}{A}$. Since T is a direct summand, it is coclosed in W. Then $\frac{T}{A}$ is coclosed in $\frac{W}{A}$ and hence, by [14, Lemma 1.1] $\frac{X}{A} \ll \frac{T}{A}$. Thus, T is feebly lifting.
(2) Let $T \leq W$ be a fully idempotent supplement submodule of W. Then there exists $K \leq W$ such that T is minimal with the property $W = T + K$. By Lemma 2.2, $T = A \oplus S$ with $A \leq_{\oplus} W$ and $S \ll W$. Thus, $W = A + K$. By the minimality of T, $T = A$.
□

Now we investigate the conditions for a feebly lifting module W to be lifting. But firstly let us recall some basic concepts from [4], [8] and [5].

For a module W, $T \leq W$ is called *nilpotent* if $T^k = 0$ for some $k \in \mathbb{Z}^+$. W is called a *multiplication* (*comultiplication*) $\mathfrak{R}$-module provided, for every $T \leq W$, there exists an ideal $I \leq \mathfrak{R}$ providing $T = IW$ ($T = (0 :_W I)$).

Following [1], a submodule T of an R-module W is called a *nilpotent* submodule if $(T : W)^k T = 0$ for some positive integer k, where $(T : W) = \{r \in R : rW \subseteq T\}$.

Proposition 2.6. *Let W be a fully idempotent module. Then, W is lifting iff W is feebly lifting.*

Proof. The necessity is clear. Conversely, since every submodule of a fully idempotent module is fully idempotent by [4, Proposition 2.7(b)], then W is lifting whenever W is feebly lifting. □

Corollary 2.7. *Let W be a multiplication and comultiplication module providing W does not have any non-zero nilpotent submodule. Then, W is lifting if and only if W is feebly lifting.*

Proof. By [4, Theorem 2.10(a)], W is fully idempotent. So, the result follows from Proposition 2.6. □

A module W is called *cosemisimple* if all simple modules in $\sigma(W)$ are W-injective [7]. By [7, 3.8 and 4.17], the following result can be given easily.

Corollary 2.8. *Let W be a π-projective cosemisimple module. Then W is lifting iff W is feebly lifting.*

Proposition 2.9. *For a semisimple ring $\mathfrak{R}$ the following statements are equivalent for an $\mathfrak{R}$-module W:*

(1) W is lifting.
(2) W is feebly lifting.
(3) W is weak lifting.

Proof. $1 \Rightarrow 2$ and $2 \Rightarrow 3$ are clear.

$3 \Rightarrow 1$: Let W be a weak lifting module. Since R is semisimple, by [17, Proposition 3.7] every R-module is semisimple. Then W is lifting. □

A ring $\mathfrak{R}$ is called fully idempotent if it is fully idempotent as an $\mathfrak{R}$-module [9].

Proposition 2.10. *For a fully idempotent ring $\mathfrak{R}$ the conditions given below are equivalent.*

(1) $\mathfrak{R}$ *is left semiperfect.*
(2) *Every finitely generated projective left* $\mathfrak{R}$*-module is feebly lifting.*
(3) ${}_{\mathfrak{R}}\mathfrak{R}$ *is feebly lifting.*
(4) ${}_{\mathfrak{R}}\mathfrak{R}$ *is lifting.*

Proof. $1 \Rightarrow 2$: By [7, 27.21], every finitely generated projective left $\mathfrak{R}$-module is lifting and so it is feebly lifting.

$2 \Rightarrow 3$: Clear.

$3 \Rightarrow 4$: It is clear that $\mathfrak{R}$ is fully idempotent.

$4 \Rightarrow 1$: Since ${}_{\mathfrak{R}}\mathfrak{R}$ is lifting, it is supplemented. Hence $\mathfrak{R}$ is semiperfect by [20, 42.6].

□

The Dorroh extension is typically applied to embed a ring without unity into a ring containing unity. For a ring $\mathfrak{R}$ and for the set of the integers $\mathbb{Z}$, a common method for embedding $\mathfrak{R}$ into a ring with identity is via the Dorroh extension. On the underlying set $\mathbb{Z} \times \mathfrak{R}$, define addition and multiplication by

$$(z_1, r_1) + (z_2, r_2) = (z_1 + z_2, r_1 + r_2)$$
$$(z_1, r_1) * (z_2, r_2) = (z_1 z_2, z_1 r_2 + z_2 r_1 + r_1 r_2).$$

Then, $(\mathbb{Z} \times \mathfrak{R}, +, *)$ is a ring with identity $(1, 0)$ [6]. Motivated by [9, Example 2.6] we give an example that shows an application of Proposition 2.10.

Example 2.11. *Let $\mathbb{F}$ be a countable field. By [18, Theorem 6.6] there exists a simple nil algebra A over $\mathbb{F}$ providing $A^2 = A$. Assume $\mathfrak{R}$ be the Dorroh extension of A by $\mathbb{F}$. Here $\mathfrak{R}$ is both a local ring (and so semiperfect) with $Rad(\mathfrak{R}) = A \oplus 0 \cong A$ and also a fully idempotent ring. Hence, every finitely generated projective $\mathfrak{R}$-module is feebly lifting, particularly so is ${}_{\mathfrak{R}}\mathfrak{R}$ by Proposition 2.10.*

The following example shows that the class of feebly lifting modules is not closed under factor modules.

Example 2.12. *Let $\mathfrak{R} = \mathbb{Z}_8$. Then $0 \subset 4\mathfrak{R} \subset 2\mathfrak{R} \subset \mathfrak{R}$ is the only composition series of $\mathfrak{R}$. Note that since $\mathfrak{R}$ is perfect, every free $\mathfrak{R}$-module and so ${}_{\mathfrak{R}}(\mathfrak{R}^2)$ is feebly lifting by Proposition 2.10.*

However, ${}_{\mathfrak{R}}\mathfrak{W} = (2\mathfrak{R}/4\mathfrak{R}) \oplus \mathfrak{R} = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not lifting by [11, Example 10] and so it is not feebly lifting by Proposition 2.10. Hence $\mathfrak{W}$ indicates an example as a factor module of ${}_{\mathfrak{R}}(\mathfrak{R}^2)$ which is not feebly lifting.

Proposition 2.13. *Let W be a projective multiplication module and $T \leq W$ be fully idempotent. If W is feebly lifting, then so is $\frac{W}{T}$.*

Proof. Let $\frac{X}{T} \leq \frac{W}{T}$ be fully idempotent and so X is fully idempotent by [9, Theorem 2.2] because T is fully idempotent. A decomposition $W = A \oplus B$ exists providing $\frac{X}{A} \ll \frac{W}{A}$. Let us consider the projection $p : W \longrightarrow B$ and the injection $i : B \longrightarrow W$. For the composition $ip = h : W \longrightarrow W$ we have $h(X) \leq X$ and $h(T) \leq T$ as W is a multiplication module by [19, Lemma 1.2]. Thus, it can be seen that $h^{-1}(X) = X$ and so $h^{-1}(T) \leq X = h^{-1}(X)$. It must be verified that $\frac{X}{h^{-1}(T)} \ll \frac{W}{h^{-1}(T)}$ and $\frac{h^{-1}(T)}{T} \leq_{\oplus} \frac{W}{T}$. Let $\frac{X}{h^{-1}(T)} + \frac{T'}{h^{-1}(T)} = \frac{W}{h^{-1}(T)}$ for some $T' \leq W$ where $h^{-1}(T) \subseteq T'$. Thus, $X + T' = W$ and so $T' = W$ is got as $\frac{X}{A} \ll \frac{W}{A}$. Therefore, $\frac{\frac{X}{T}}{\frac{h^{-1}(T)}{T}} \ll \frac{\frac{W}{T}}{\frac{h^{-1}(T)}{T}}$. Moreover, $W = h^{-1}(T) + B$ is obtained as $W = A \oplus B$. Clearly, $\frac{h^{-1}(T)}{T} + \frac{B+T}{T} = \frac{W}{T}$. Hence, $\frac{h^{-1}(T)}{T} \leq_{\oplus} \frac{W}{T}$ as $\frac{h^{-1}(T)}{T} \cap \frac{B+T}{T} = \frac{(h^{-1}(T) \cap B)+T}{T} = \frac{T}{T} = 0$. □

Corollary 2.14. *Let W be a projective fully idempotent module. If W is feebly lifting, then so is every factor module [4, Proposition 2.7].*

Theorem 2.15. *Let $W = \oplus_{i=1}^{n} W_i$ be a multiplication module. If each W_i is feebly lifting, then W is feebly lifting.*

Proof. Let $T \leq W$ be fully idempotent. Then by [4, Proposition 2.7] $T \cap W_i$ is fully idempotent. By hypothesis, there exists $D_i \leq_{\oplus} W_i$ for each i providing $D_i \leq T \cap W_i$ and $\frac{T \cap W_i}{D_i} \ll \frac{W_i}{D_i}$. It follows that $D = \oplus_{i=1}^{n} D_i \leq_{\oplus} W$ and $D \leq \oplus_{i=1}^{n}(T \cap W_i) = T$ by [19, Lemma 1.3]. Take the homomorphism $f : \oplus_{i=1}^{n} \frac{W_i}{D_i} \longrightarrow \frac{W}{D} = \frac{\oplus_{i=1}^{n} W_i}{D}$ via $f(m_1 + D_1, ..., m_n + D_n) = (\sum_{i=1}^{n} m_i) + D_i$ where $m_i \in W_i$ for each $i = 1, 2, ..., n$. Then $f(\oplus_{i=1}^{n} \frac{T \cap W_i}{D_i}) = \frac{\oplus_{i=1}^{n} T \cap W_i}{D} = \frac{T}{D}$. Since $\frac{T \cap W_i}{D_i} \ll \frac{W_i}{D_i}$ for each $i = 1, 2, ..., n$, then $\oplus_{i=1}^{n} \frac{T \cap W_i}{D_i} \ll \oplus_{i=1}^{n} \frac{W_i}{D_i}$ by [7, 2.2(4)]. Hence, $f(\oplus_{i=1}^{n} \frac{T \cap W_i}{D_i}) = \frac{T}{D} \ll f(\oplus_{i=1}^{n} \frac{W_i}{D_i})$ by [7, 2.2(5)] and so $\frac{T}{D} \ll \frac{W}{D}$ is obtained. □

Corollary 2.16. *If a multiplication module W is a finite direct sum of hollow modules, then W is feebly lifting.*

Corollary 2.17. *Let $\mathfrak{R}$ be a commutative Noetherian ring and W be an $\mathfrak{R}$-module. W is feebly lifting iff W is weak lifting.*

Proof. Since $\mathfrak{R}$ is a commutative Noetherian ring, fully idempotent and semisimple submodules of W coincide by [12, Theorem 3.5]. □

Example 2.18. *Let $\mathfrak{R} = \mathbb{Z}$ and ${}_{\mathfrak{R}}W = \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{8\mathbb{Z}}$. Since $\mathfrak{R}$ is commutative Noetherian, then any fully idempotent and semisimple submodule of W coincides by [12, Theorem 3.5]. Hence W is feebly lifting as it is also weak lifting by [14, Example 2.7]. Besides, each direct summand of W is feebly lifting as a hollow module.*

Example 2.19. *Consider the $\mathbb{Z}$-module $W = \mathbb{Z}\oplus\mathbb{Z}$ which is not multiplication. Clearly, each direct summand of W is feebly lifting. However, W is not feebly lifting by [14, Theorem 2.2] and Corollary 2.17, as ${}_{\mathbb{Z}}\mathbb{Z}$ is not injective.*

Example 2.20. *Let $\mathfrak{R}$ be a discrete valuation ring, $\mathfrak{K}$ be the field of fractions of $\mathfrak{R}$ and ${}_{\mathfrak{R}}\mathfrak{W} = \frac{\mathfrak{K}}{\mathfrak{R}} \oplus \frac{\mathfrak{R}}{\mathfrak{P}}$ where $\mathfrak{P}$ is the unique maximal ideal of $\mathfrak{R}$. Here as $\frac{\mathfrak{K}}{\mathfrak{R}}$ and $\frac{\mathfrak{R}}{\mathfrak{P}}$ are both hollow modules, they are lifting and so weak lifting modules, in particular $\frac{\mathfrak{R}}{\mathfrak{P}}$ is simple. Hence, $\mathfrak{W}$ is weak lifting by [14, Theorem 2.5]. So by Corollary 2.17 $\mathfrak{W}$ is a feebly lifting module which is not lifting from [15, Proposition A.7].*

Example 2.21. *Let ${}_{\mathbb{Z}}W = \mathbb{Q} \oplus \mathbb{Z}_2$. Here, W is not a lifting module as it is not amply supplemented. On the other side, since $\mathbb{Z}$-module $\mathbb{Q}$ is injective weak lifting and $\mathbb{Z}$-module $\mathbb{Z}_2$ is weak lifting as a hollow module which does not contain a non-zero injective submodule, then W is weak lifting by [14, Theorem 2.2] and so W is feebly lifting by Corollary 2.17.*

3. Conclusion

Feebly lifting modules are defined in this study. Examples that embody the theory are given. Conditions that could make feebly lifting modules lifting have been investigated. Conditions equivalent to a module being feebly lifting have been identified. It has been investigated whether the factor modules and finite direct sums of a feebly lifting module also meet this condition. And finally, ring characterizations of these modules are included.

4. Author Contributions

For research articles with several authors, a short paragraph specifying their individual contributions must be provided. The following statements should be used "Conceptualization, C.A.; methodology, C.A., E.Ö.S; software, C.A., E.Ö.S ; validation, C.A., E.Ö.S; investigation, C.A., E.Ö.S; resources, E.Ö.S; writing—original draft preparation, E.Ö.S; writing—review and editing, C.A., E.Ö.S; visualization, C.A., E.Ö.S; supervision, C.A.; All authors have read and agreed to the published version of the manuscript."

5. Data Availability Statement

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References

[1] Ali, M. M. (2008). Idempotent and nilpotent submodules of multiplication modules. Comm. Algebra, 36(12), 4620-4642. https://doi.org/10.1080/00927870802186805

[2] Amouzegar, T. and Moniri Hamzekolaee, A. R. (2020). Lifting modules with respect to images of a fully invariant submodule, Novi Sad J. Math. 50(2), 41-50. https://doi.org/10.30755/NSJOM.09413

[3] Amouzegar, T., Moniri Hamzekolaee, A. R. and Tercan, A. (2023) Fully invariant submodules for constructing to dual Rickart modules and dual Baer modules, Bulletin Matematique, 66(114), 295-306.

[4] Ansari-Toroghy, H. & Farshadifar, F. (2012). Fully idempotent and coidempotent modules. Bulletin of the Iranian Mathematical Society, 38(4), 987-1005.

[5] Barnard, A. (1981). Multiplication modules. J. Algebra, 71(1), 174-178.

[6] Cannon, G.A., Neuerburg, K.M. (2008). Ideals in Dorroh extensions of rings. Missouri Journal of Mathematical Sciences, 20(3), 165-168.

[7] Clark, J., Lomp,C., Vanaja, N. & Wisbauer, R. (2006). Lifting modules, Birkhauser.

[8] El-Bast, Z. A. & Smith, P. F. (1988). Multiplication modules, Comm. Algebra, 16, 775-779. https://doi.org/10.1080/00927878808823601

[9] Jeon, Y. C., Kim, N. K. & Lee, Y. (2010). On fully idempotent rings. Bull. Korean Math. Soc., 47(4), 715-726. 10.4134/BKMS.2010.47.4.715

[10] Kasch, F. (1982). Modules and rings. New York, London Mathematical Society, Academic Press Inc.

[11] Keskin, D. (1998). Finite direct sum of $(D1)$-modules. Turkish Journal of Mathematics, 22, 85-91. https://journals.tubitak.gov.tr/math/vol22/iss1/9

[12] Keskin Tütüncü, D., Orhan Ertaş, N., Tribak, R. & Smith, P. (2011). On fully idempotent modules. Communications in Algebra, 39, 2707-2722. https://doi.org/10.1080/00927872.2010.489916

[13] Keskin, D. (2000). On lifting modules. Comm. Algebra, 28(7), 3427–3440. https://doi.org/10.1080/00927870008827034

[14] Keskin Tütüncü, D. & Orhan Ertaş, N. (2003). $CCSR$-modules and weak lifting modules. East-West Journal of Mathematics, 5(1), 89-96.

[15] Mohammed, S. H. and Muller, B. J. (1990). Continuous and discrete modules. London Math. Soc. LNS 147, Cambridge University Press, Cambridge.
[16] Moniri Hamzekolaee, A. R. and Amouzegar, T. (2023). I-lifting modules and noncosingularity, Annali Dell'Universita' Di Ferrara, 69, 363-374. https://doi.org/10.1007/s11565-022-00432-7
[17] Sharpe, D. W. & Vamos, P. (1972). Injective modules. Cambridge University Press, Cambridge.
[18] Smoktunowicz, A. (2002). A Simple nil ring exists. Comm. Algebra, 30(1), 27-59. https://doi.org/10.1081/AGB-120006478
[19] Tuganbaev, A. (2003). Multiplication modules over non-commutative rings. Sbornik Mathematics, 194(12), 1837-1864. 10.1070/SM2003v194n12ABEH000788
[20] Wisbauer, R. (1991). Foundations of module and ring theory. Gordon and Breach, Reading.

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