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MODULE STRUCTURES AND FILTERS ON SEMIHOOPS

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Abstract. In this paper, we study modules and filters on semihoops. Firstly, we introduce the definition of modules on semihoops and give some examples to illustrate it. Also, we get some significant results related to modules on semihoops. If the semihoop G can generate an Abelian group, then G is a module of any subalgebra S of the semihoop G. Then, we use modules and filters to investigate the relationship between modules and semihoops regarding quotient algebras. Secondly, by introducing the definitions of prime submodules and torsion free modules on semihoops, we explore the relationship among prime modules, filters, and torsion free modules. Moreover, we discuss the relationship between the images and inverse images under the homomorphism of semihoops and modules, respectively. Finally, we define multiplication modules and comultiplication modules on semihoops. We study the relationship among multiplication modules and submodules on semihoops and provide the condition for comultiplication modules to satisfy the descending chain condition.

Keywords: semihoop, filter, module, (prime) submodule, torsion free module, (co)multiplication module.
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1. Introduction

Non-classical logic has emerged as a valuable tool for computers to handle uncertain and fuzzy information. Logical algebras, such as BL-algebras, MV-algebras, Hoops, semihoops, and Residuated lattices etc, among others, have been introduced as the semantic systems of non-classical logic systems. Therefore, exploring algebras that display residuated law property is of considerable significance. Semihoops represent the most fundamental residuated structure, encompassing all logical algebras that satisfy the residuated law. In 1960, Bosbach [9, 10] first introduced the concept of semihoops. Semihoops are a generalization of hoop algebras, from a logical perspective, the semihoops algebra represents the algebraic semantics of a reasoning system that can be algebraized and is considered as a subreduction of basic logic. From an algebraic viewpoint, semihoops are essentially commutative integral semigroups of a natural order. In recent years, numerous scholars have conducted

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indepth studies on aspects such as semihoops, filters, states etc, obtaining significant conclusions. In 2015, Borzooei [5] introduced the concepts of local and perfect semihoops and proved some related results, and discovered a relation between local and perfect semihoops. In 2017, Aguzzoli [2] investigated the relationships between prelinear semihoops, Boolean algebras, and strong perfect MTL-algebras. In the same year, Aaly [16] investigated topological structures on a semihoop and under certain conditions showed that there exists a topology such that it can generate a topological semihoop. In 2018, Fu [13] studied the properties of internal mappings on semihoops and defined them as states. In the same year, Niu and Xin [18] examined the concept of tense operators on bounded semihoops and investigated related properties concerning tense filters. Xin [22] introduced the notions of derivations and differential filters on semihoops and investigated some related properties. In 2020, Niu et al. [19] studied different types of ideals on bounded semihoops and explored their relationships. In 2024, Xin [21] presented the concepts of L-fuzzy ideals, L-fuzzy prime ideals, and L-fuzzy maximal ideals on bounded semihoops and discussed some properties. In the same year, Wang [20] discussed the related properties of some particular derivations in semihoops and gave some characterizations of them. Consequently, semihoops have become a focal point in the study of logic algebras in the past decade.

Every module is an action of a ring on a certain group. This is a motivation for studying the action of semihoops on groups, that is modules of semihoops. Because the modules of semihoops have the partial order relation and Galois connection, this is a good perspective for studying the substructures of semihoops by using the module structure, and we can also explore the substructures of modules through the research of semihoops. Many scholars have a profound interest in modular structures in algebraic structures and have achieved numerous interesting results. In 1994, Aslam etal [1] introduced the concept of BCK-module as a function of BCK-algebra on commutative groups. In 2003, Nola [11] introduced the concept of MV-module as a function of PMV-algebra on MV-algebra. In 2011, Bakhshi [3] introduced the concepts of fuzzy BCK-submodule and fuzzy multiplication BCK-module, and used the concept of fuzzy residual quotient BCK-submodule to study their properties and provided some features. In 2014, Borzooei [6] defined the concept of extending the BCK-module, which is an extension of the BCK-module, and demonstrated some related results. In the same year, Motahari [17] studied some results of prime BCK-submodules under BCK-epimorphism. In 2015, Kashif [15] introduced the concept of BCK-topological modules in a natural way and established that each decreasing sequence of submodules on a bounded commutative BCK-algebra is indeed a BCK-topological module. In the same year, Borzooei [7] defined the concept of freely extending BCK-modules and free objects within the scope of extended BCK-modules, and proved the relationship between free modules and production modules. In 2024, Sabetkish [8] studied some important results on modules such as submodules and quotient structures, and investigated the relationships between ring modules and other modules on logical algebraic structures such as BCK-modules and MV-modules. In the same year, Borzooei [14] introduced the definitions of modules and prime submodules on L-algebras, exploring the relationship between prime ideals and module and prime submodule in L-representation. Since semihoops represent the most fundamental residuated structure, we hope to extend some of the results in this paper to other algebras, such as BL-algebras, MV-algebras, Hoops, and Residuated lattices etc.

The main purpose of this paper is to study modules and filters on semihoops. Firstly, we introduce the concept of modules on semihoops and give some examples to illustrate it. Moreover, we find that if the semihoop G can generate an Abelian group, then G is a module of any subalgebra S of the semihoop G. Afterwards, we use modules and filters to investigate the relationship between modules and semihoops regarding quotient algebras. Secondly, by introducing the definitions of prime submodules and torsion free modules on semihoops, we explore the relationship among prime modules, filters, and torsion free modules. We can conclude that let G be a module of the linear bounded semihoop S and H be a filter of the module G. If H is a prime submodule of G if and only if $C = (H : G) \cup \{1\}$ is a filter of S and $\frac{G}{H}$ is a torsion free module of $\frac{S}{C}$. Additionally, we discuss the relationship between the images and inverse images under the homomorphism of semihoops and modules, respectively. Finally, we define multiplication modules and comultiplication modules on semihoops. We study the relationship among multiplication modules and submodules on semihoops and provide the condition for comultiplication modules to satisfy the descending chain condition.

2. Preliminaries

Definition 2.1. [12] An algebra $(S, \odot, \to, \wedge, 1)$ of type $(2, 2, 2, 0)$ is called a semihoop if it satisfies:
$(S1)$ $(S, \wedge, 1)$ is a $\wedge$-semilattice and it has an upper bound 1;
$(S2)$ $(S, \odot, 1)$ is a commutative monoid;
$(S3)$ $(x \odot y) \to z = x \to (y \to z)$, for any $x, y, z \in S$.

In a semihoop $(S, \odot, \to, \wedge, 1)$, we define $x \leq y$ if and only if $x \to y = 1$, for any $x, y \in S$. It is easy to check that $\leq$ is a partial order relation on S and we get $x \leq 1$, for any $x \in S$.

Proposition 2.2. *[12] Let S be a semihoop. Then the following properties hold:*
(1) *$x \odot y \leq z$ if and only if $x \leq y \to z$, for every $x, y, z \in S$;*
(2) *$x \odot y \leq x, y$, for any $x, y \in S$;*
(3) *$1 \to x = x, x \to 1 = 1$, for all $x \in S$;*
(4) *$x \odot (x \to y) \leq y$, for any $x, y \in S$;*
(5) *$x \to (y \to z) = y \to (x \to z)$, for any $x, y, z \in S$.*

A semihoop $(S, \odot, \rightarrow, \wedge, 1)$ is called a bounded semihoop if there exists an element $0 \in S$ such that $0 \leq x$, for all $x \in S$. We denote a bounded semihoop $(S, \odot, \rightarrow, \wedge, 0, 1)$ by S. In a bounded semihoop S, we define the negation $'$: $x' = x \rightarrow 0$, for any $x \in S$.

Definition 2.3. [5] Let S be a semihoop. A nonempty subset F of S is called a filter of S, if it satisfies:
$(F1)$ $x, y \in F$ implies $x \odot y \in F$;
$(F2)$ $x \leq y$ and $x \in F$ imply $y \in F$, for any $x, y \in S$.

Definition 2.4. [21] Let S_1, S_2 be bounded semihoops. A mapping $f : S_1 \rightarrow S_2$ is called a homomorphism if for any $x, y \in S_1$, we have
(1) $f(x \odot_{S_1} y) = f(x) \odot_{S_2} f(y)$;
(2) $f(x \rightarrow_{S_1} y) = f(x) \rightarrow_{S_2} f(y)$;
(3) $f(x \wedge_{S_1} y) = f(x) \wedge_{S_2} f(y)$;
(4) $f(0) = 0_{S_2}$.

It is clear that $f(1) = 1_{S_2}$, since $f(1 \rightarrow_{S_1} 1) = f(1) \rightarrow_{S_2} f(1) = 1_{S_2}$.

Definition 2.5. [4] A group G is an algebra $(G, *, 1)$ with a binary, a unary, in which the following identities are true:
$(G1)$ $x * (y \cdot z) = (x * y) \cdot z$;
$(G2)$ $x * 1 = 1 * x = x$;
$(G3)$ $x * x^{-1} = x^{-1} * x = 1$.

We call 1 is an identity element and x^{-1} is an inverse element of x for any $x \in G$. The identity element 1 and inverse element of each element in the group G are unique.

Definition 2.6. [4] A group G is Abelian (or commutative) if the following identity is true: $x * y = y * x$.

3. Modules on the semihoops

In this section, we define the modules on semihoops and obtain some important results. Subsequently, we introduce the concepts of prime submodules, torsion free modules, multiplication modules, and comultiplication modules and explore the relationships among them through employing the concepts of homomorphism, filter, quotient structure.

Definition 3.1. Let $(S, \odot, \rightarrow, \wedge, 1)$ and $(G, \odot_G, \rightarrow_G, \wedge_G, 1)$ be semihoops. Then G is called a module of the semihoop S, if $(G, *, 1)$ is an Abelian group and there is an operation $\cdot : S \times G \rightarrow G$ defined by $(x, g) \mapsto x \cdot g$ such that for any $x, y \in S$ and $f, g \in G$, we have:
$(SH1)$ $1 \cdot g = g$;
$(SH2)$ $x \cdot (f * g) \leq x \cdot f * x \cdot g$;
$(SH3)$ $(x \odot y) \cdot g = x \cdot (y \cdot g)$;
$(SH4)$ $(x \wedge y) \cdot g = (x \cdot g) \wedge_G (y \cdot g)$;
$(SH5)$ $(x \rightarrow y) \cdot g = g * (x \cdot g \rightarrow_G y \cdot g)$.

When $g = 1$, we can get that $(x \to y) \cdot 1 = 1 * (x \cdot 1 \to_G y \cdot 1) = x \cdot 1 \to_G y \cdot 1$. Therefore, it can be found that when $g = 1$, the operation $\cdot$ is distributive with respect to the operation $\to$ of the semihoop. However, generally speaking, we can't prove that the operation $\cdot$ is distributive with respect to the operation $\to$ of the semihoop. The following example can illustrate that condition $(SH5)$ is appropriate.

Example 3.2. *Let $S = (\{0, a, b, 1\}, \odot, \to, \wedge, 0, 1)$ be a bounded semihoop. We define two operations on S as shown in Table 1 and Table 2.*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Table 2

Let $G = (G = \{0, \alpha, \beta, 1\}, \odot, \to, \wedge, 0, 1)$ be a bounded semihoop. We define two operations on G as shown in Table 3 and Table 4.

$\odot$	0	α	β	1
0	0	0	0	0
α	0	α	0	α
β	0	0	β	β
1	0	α	β	1

Table 3

$\to$	0	α	β	1
0	1	1	1	1
α	β	1	β	1
β	α	α	1	1
1	0	α	β	1

Table 4

We define $G = (\{0, \alpha, \beta, 1\}, *, 1)$ as an Abelian group and the operation $*$ on G as shown in Table 5:

$*$	0	α	β	1
0	1	β	α	0
α	β	1	0	α
β	α	0	1	β
1	0	α	β	1

Table 5

Set $\cdot : S \times G \to G$ by $x \cdot g = g$, for any $x \in S$ and $g \in G$. We have:

$(SH1)$: It is obvious that $1 \cdot g = g$, for any $g \in G$. Therefore, $(SH1)$ holds.

$(SH2)$: For any $x \in S$ and $f, g \in G$, we have $x \cdot (f * g) = f * g = x \cdot f * x \cdot g$. Therefore, $(SH2)$ holds.

$(SH3)$: For any $x, y \in S$ and $g \in G$, we have $(x \odot y) \cdot g = g = x \cdot g = x \cdot (y \cdot g)$. Therefore, $(SH3)$ holds.

$(SH4)$: For any $x, y \in S$ and $g \in G$, we have $(x \wedge y) \cdot g = g = g \wedge_G g = (x \cdot g) \wedge_G (y \cdot g)$. Therefore, $(SH4)$ holds.

$(SH5)$: For any $x, y \in S$ and $g \in G$, $(x \to y) \cdot g = g = g * 1 = g * (g \to_G g) = g * (x \cdot g \to_G y \cdot g)$. Therefore, $(SH5)$ holds.

Therefore, the group $G = (\{0, \alpha, \beta, 1\}, *, 1)$ is a module of S.

Example 3.3. *Let $S = (\{0, a, 1\}, \odot, \to, \wedge, 0, 1)$ be a bounded semihoop. We define two operations on S as shown in Table 6 and Table 7.*

$\odot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

Table 6

$\to$	0	a	1
0	1	1	1
a	0	1	1
1	0	a	1

Table 7

Let $G = (G = \{0, \alpha, 1\}, \odot, \to, \wedge, 0, 1)$ be a bounded semihoop. We define two operations on G as shown in Table 8 and Table 9.

$\odot$	0	α	1
0	0	0	0
α	0	0	α
1	0	α	1

Table 8

$\to$	0	α	1
0	1	1	1
α	α	1	1
1	0	α	1

Table 9

We define $G = (\{0, \alpha, 1\}, *, 1)$ as an Abelian group and the operation $*$ on G as shown in Table 10:

$*$	0	α	1
0	α	1	0
α	1	0	α
1	0	α	1

Table 10

Set $\cdot : S \times G \to G$ defined by $(x, g) \mapsto x \cdot g$, in which, $1 \cdot g = g$, $a \cdot g = g$, $0 \cdot 0 = 0$, $0 \cdot 1 = 1$, and $0 \cdot \alpha = 0$, for any $x \in S$ and $g \in G$. We have:

$(SH1)$: It is obvious that $1 \cdot g = g$, for any $g \in G$. Therefore, $(SH1)$ holds.

$(SH2)$: For any $x \in S$ and $f, g \in G$, we have $0 \cdot (0 * 0) = 0 \cdot \alpha = 0 \leq \alpha = 0 * 0 = 0 \cdot 0 * 0 \cdot 0$ and $0 \cdot (\alpha * 1) = 0 \cdot \alpha = 0 \leq 0 = 0 * 1 = 0 \cdot 0 * 0 \cdot 1$. We can use the same method to prove that $x \cdot (f * g) \leq x \cdot f * x \cdot g$. Therefore, $(SH2)$ holds.

$(SH3)$: For any $x, y \in S$ and $g \in G$, we have $(x \odot 0) \cdot \alpha = 0 \cdot \alpha = 0 = x \cdot 0 = x \cdot (0 \cdot \alpha)$ and $(a \odot a) \cdot \alpha = a \cdot \alpha = \alpha = a \cdot \alpha = a \cdot (a \cdot \alpha)$. We can use the same method to prove that $(x \odot y) \cdot g = x \cdot (y \cdot g)$. Therefore, $(SH3)$ holds.

$(SH4)$: For any $x, y \in S$ and $g \in G$, we can easily prove that $(x \wedge y) \cdot g = g = g \wedge_G g = (x \cdot g) \wedge_G (y \cdot g)$. Therefore, $(SH4)$ holds.

$(SH5)$: For any $x, y \in S$ and $g \in G$, we have $(1 \to 0) \cdot \alpha = 0 \cdot \alpha = 0 = \alpha * \alpha = \alpha * (\alpha \to_G 0) = \alpha * (1 \cdot \alpha \to_G 0 \cdot \alpha)$ and $(a \to 0) \cdot \alpha = 0 \cdot \alpha = 0 = \alpha * \alpha = \alpha * (\alpha \to_G 0) = \alpha * (a \cdot \alpha \to_G 0 \cdot \alpha)$. We can conclude that $(x \to y) \cdot g = g * (x \cdot g \to_G y \cdot g)$. Therefore, $(SH5)$ holds.

Therefore, the group $G = (\{0, \alpha, 1\}, *, 1)$ is a module of S.

Proposition 3.4. *Let $G = (G, \odot, \to, \wedge, 1)$ be a semihoop and $(S, \odot, \to, \wedge, 1)$ be a subalgebra of $(G, \odot, \to, \wedge, 1)$. If $(G, *, 1)$ is an Abelian group, then $(G, *, 1)$ is a module of the semihoop S.*

Proof. Let $G = (G, *, 1)$ be an Abelian group. We define the operation $\cdot : S \times G \to G$ by $x \cdot g = g$, for every $x \in S$ and $g \in G$. For any $x, y \in S$ and $f, g \in G$, we have:

$(SH1)$: It is obvious that $1 \cdot g = g$ for any $g \in G$. Therefore, $(SH1)$ holds.

$(SH2)$: For any $x \in S$ and $f, g \in G$, we have $x \cdot (g * f) = f * g = x \cdot g * x \cdot f$. Therefore, $(SH2)$ holds.

$(SH3)$: For any $x, y \in S$ and $g \in G$, we have $(x \odot y) \cdot g = g = y \cdot g = x \cdot (y \cdot g)$. Therefore, $(SH3)$ holds.

$(SH4)$: For any $x, y \in S$ and $g \in G$, we have $(x \wedge y) \cdot g = g = g \wedge g = (x \cdot g) \wedge (y \cdot g)$. Therefore, $(SH4)$ holds.

$(SH5)$: For any $x, y \in S$ and $g \in G$, $(x \to y) \cdot g = g = g * 1 = g * (g \to g) = g * (x \cdot g \to y \cdot g)$. Therefore, $(SH5)$ holds.

Therefore, the group $(G, *, 1)$ is a module of the semihoop S.

□

Example 3.5. *Let $G = (G = \{0, a, 1\}, \odot, \to, \wedge, 0, 1)$ be a bounded semihoop and $S = (\{0, 1\}, \odot, \to, \wedge, 0, 1)$ be a subalgebra of G. We define two operations on G as shown in Table 11 and Table 12.*

$\odot$	0	a	1
0	0	0	0
a	0	0	a
1	0	a	1

Table 11

$\to$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

Table 12

We define $(G = \{0, a, 1\}, *, 1)$ as an Abelian group and the operation $*$ on G as shown in Table 13.

$*$	0	a	1
0	a	1	0
a	1	0	a
1	0	a	1

Table 13

Set $\cdot : S \times G \to G$ by $x \cdot g = g$, for any $x \in S$ and $g \in G$. We can use the same method as Proposition 3.4 to prove the group $G = (\{0, a, 1\}, *, 1)$ is a module of S.

Corollary 3.6. *Let $S = (S, \odot, \to, \wedge, 1))$ be a semihoop. If $(S, *, 1)$ is an Abelian group, then $G = (S, *, 1)$ is a module of the semihoop S.*

Proof. Let $S = (S, \odot, \rightarrow, \wedge, 1)$ be a semihoop and $G = (S, *, 1)$ is an Abelian group. We use the same method as Proposition 3.4 can prove that the group $G = (S, *, 1)$ is a module of S. □

Example 3.7. *Let $S = (S = \{0, a, 1\}, \odot, \rightarrow, \wedge, 0, 1)$ be a bounded semihoop, where $0 < a < 1$. We define two operations on G as shown in Table 14 and Table 15.*

$\odot$	0	a	1
0	0	0	0
a	0	0	a
1	0	a	1

Table 14

$\rightarrow$	0	a	1
0	1	1	1
a	a	1	1
1	0	a	1

Table 15

We define $G = (S, *, 1)$ is an Abelian group and the operation $*$ on G as shown in Table 16.

$*$	0	a	1
0	a	1	0
a	1	0	a
1	0	a	1

Table 16

Set $\cdot : S \times G \rightarrow G$ by $x \cdot g = g$, for any $x \in S$ and $g \in G$. We use the same method as Proposition 3.4 can prove that the group $G = (S, *, 1)$ is a module of S.

Proposition 3.8. *Let G be a module of the semihoop S. Then for any $x, y \in S$ and $g, g_1 \in G$, the following statements hold:*

(i) If $x \cdot 1 = 1$ and $g * g_1 = 1$, then $x \cdot g * x \cdot g_1 = 1$;

(ii) If $x \leq_S y$, then $x \cdot g \leq_G y \cdot g$.

(iii) If G can generate a bounded semihoop, then $x \cdot 0 = 0$, where 0 is the smallest element of the bounded semihoop G.

Proof. (i) If $x \cdot 1 = 1$ and $g * g_1 = 1$, then $1 = x \cdot 1 = x \cdot (g * g_1) \leq x \cdot g * x \cdot g_1$. Therefore, $x \cdot g * x \cdot g_1 = 1$.

(ii) If $x \leq_S y$, then $g = 1 \cdot g = (x \rightarrow_S y) \cdot g = g * (x \cdot g \rightarrow_G y \cdot g)$. Therefore, $x \cdot g \rightarrow_G y \cdot g = 1$. This implies that $x \cdot g \leq_G y \cdot g$.

(iii) If G can generate a bounded semihoop, then $x \cdot 0 \leq_G 1 \cdot 0 = 0$ by $x \leq_S 1$ and (ii). Therefore, $x \cdot 0 = 0$. □

Let S be a bounded semihoop and F be a filter. We define a binary relation $\sim_F$ on S as follows: $x \sim_F y$ if and only if $x \rightarrow y \in F$ and $y \rightarrow x \in F$. In [5], it was proved that $\sim_F$ is a congruence relation on S. Then $(\frac{S}{F}, \otimes, \rightsquigarrow, \wedge_{\frac{S}{F}}, [0], [1])$ is an bounded semihoop, where $[x] \rightsquigarrow [y] = [x \rightarrow y]$, $[x] \wedge_{\frac{S}{F}} [y] = [x \wedge y]$, and $[x] \otimes [y] = [x \odot y]$, for any $x, y \in S$.

Proposition 3.9. *Let G be a module of the bounded semihoop S, where S is a chain. If F is a filter of S, then G is a module of the bounded semihoop $\frac{S}{F}$.*

Proof. Since G is a module of the bounded semihoop S, we can get that $(G, \odot_G, \rightarrow_G, \wedge_G, 1)$ is a semihoop, $(G, *, 1)$ is an Abelian group, and there exists an operation $\cdot : S \times G \rightarrow G$ defined by $(x, g) \mapsto x \cdot g$, for any $x \in S$ and $g \in G$.

We define the operation $\bullet : \frac{S}{F} \times G \rightarrow G$ by $[x] \bullet g = x \cdot g$, for any $[x] \in \frac{S}{F}$ and $g \in G$. Let $[x] = [y]$, for $x, y \in F$ and $g \in G$. We can get that $x \leq y$ or $y \leq x$, since S is a chain. If $x \leq y$, then $x \cdot g \leq_G y \cdot g$ for any $g \in G$, by Proposition 3.8. Since $[x] = [y]$, we have $x \rightarrow y = 1 \in F$ and $y \rightarrow x \in F$. Therefore, $[1] = [y \rightarrow x]$, and so $(y \rightarrow x) \cdot g = (x \rightarrow y) \cdot g = 1 \cdot g = g$. Hence, $g * (y \cdot g \rightarrow_G x \cdot g) = (y \rightarrow x) \cdot g = g$, and so $y \cdot g \rightarrow_G x \cdot g = 1$. Therefore, $y \cdot g \leq_G x \cdot g$, and so $x \cdot g = y \cdot g$. If $y \leq x$, we can use the same method to prove that $[x] = [y]$ implies $x \cdot g = y \cdot g$. Thus the operation $\cdot$ is well defined.

Subsequently, we will prove that the group $G = (G, *, 1)$ is a module of S. We have:

$(SH1)$: It is obvious that $[1] \bullet g = 1 \cdot g = g$, for any $g \in G$. Therefore, $(SH1)$ holds.

$(SH2)$: For any $[x] \in \frac{S}{F}$ and $f, g \in G$, we have $[x] \bullet (f * g) = x \cdot (f * g) = x \cdot f * x \cdot g = [x] \bullet f * [x] \bullet g$. Therefore, $(SH2)$ holds.

$(SH3)$: For any $[x], [y] \in \frac{S}{F}$ and $g \in G$, we have $([x] \otimes [y]) \bullet g = [x \odot y] \bullet g = (x \odot y) \cdot g = x \cdot (y \cdot g) = x \cdot ([y] \bullet g) = [x] \bullet ([y] \bullet g)$. Therefore, $(SH3)$ holds.

$(SH4)$: For any $[x], [y] \in \frac{S}{F}$ and $g \in G$, we have $([x] \wedge_{\frac{S}{F}} [y]) \bullet g = [x \wedge y] \bullet g = (x \wedge y) \cdot g = x \cdot g \wedge_G y \cdot g = [x] \bullet g \wedge_G [y] \bullet g$. Therefore, $(SH4)$ holds.

$(SH5)$: For any $[x], [y] \in \frac{S}{F}$ and $g \in G$, we have $([x] \rightsquigarrow [y]) \bullet g = [x \rightarrow y] \bullet g = (x \rightarrow y) \cdot g = g * (x \cdot g \rightarrow_G y \cdot g) = g * ([x] \bullet g \rightarrow_G [y] \bullet g)$. Therefore, $(SH5)$ holds.

Therefore, the group $G = (G, *, 1)$ is a module of $\frac{S}{F}$, according to the definition of modules of semihoops.

□

Definition 3.10. Let G be a module of the semihoop S and H be a subalgebra of G. If $x \cdot h \in H$, for any $x \in S$ and $h \in H$, then it is called a submodule of G.

Note: If H is a submodule of G and $H \neq G$, then we call H is a proper submodule of G.

Example 3.11. *$H = \{\alpha, 1\}$ can generate a submodule of G in Example 3.2. But $H = \{a, 1\}$ can not generate a submodule of G in Example 3.5, since $a * a = 0 \notin H$ for $a \in H$.*

Let G be a module of the bounded semihoop S and H be a submodule of G. Since $(G, *)$ is an Abelian group and H is a subgroup of G. We define a binary relation $\sim_H$ on G as follows: $x \sim_H y$ if and only if $x * y^{-1} \in H$. In [4], it was proved that $\sim_H$ is a congruence relation on G. So it is clear that $(\frac{G}{H}, *_{\frac{G}{H}})$ is an Abelian group, where $(f * H) *_{\frac{G}{H}} (g * H) = (f * g) * H$, for every $f, g \in G$.

Definition 3.12. Let G be a module of the semihoop S. If H is a submodule of G and is also a filter of $(G, \odot, \to, \wedge, 1)$, then H is called a filter of module G.

Example 3.13. $H = \{\alpha, 1\}$ *can generate a filter of the module* G *in Example* 3.2. *But* $H = \{a, 1\}$ *can not generate a filter of the module* G *in Example* 3.5.

Theorem 3.14. *Let* G *be a module of the bounded semihoop* S. *If* H *is a submodule of module* G, *then* $\frac{G}{H}$ *is a module of* S.

Proof. Since G is a module of the bounded semihoop S, there exists an operation $\cdot : S \times G \to G$ by $(x, g) \mapsto x \cdot g$ for any $x \in S$ and $g \in G$. We define the operation $\bullet : S \times \frac{G}{H} \to \frac{G}{H}$ by $x \bullet (g * H) = x \cdot g * H$, for every $x \in S$ and $g * H \in \frac{G}{H}$. For any $x \in S$ and $g_1, g_2 \in G$, Let $x = y$ and $g_1 * H = g_2 * H$. Then $g_1 * g_2^{-1} \in H$. Since H is a filter of module G and $x \cdot (g_1 * g_2^{-1}) \leq x \cdot g_1 * x \cdot g_2^{-1}$, we can get that $x \cdot g_1 * x \cdot g_2^{-1} \in H$ and so $x \bullet g_1 * H = x \bullet g_1 * H$. It means that $\bullet$ is well-defined.

For any $f * H, g * H \in \frac{G}{H}$ and $x, y \in S$, we have:

$(SH1)$: It is obvious that $1 \bullet (g * H) = 1 \cdot g * H = g * H$. Therefore, $(SH1)$ holds.

$(SH2)$: $x \bullet ((f * H) *_{\frac{G}{H}} (g * H)) = x \bullet ((f * g) * H) = x \cdot (f * g) * H = (x \cdot f * x \cdot g) * H = (x \cdot f * H) *_{\frac{G}{H}} (x \cdot g * H) = x \bullet (f * H) *_{\frac{G}{H}} x \bullet (g * H)$. Therefore, $(SH2)$ holds.

$(SH3)$: $(x \odot_S y) \bullet (g * H) = (x \odot_S y) \cdot g * H = x \cdot (y \cdot g) * H = x \bullet (y \cdot g * H) = x \bullet (y \bullet (g * H))$. Therefore, $(SH3)$ holds.

$(SH4)$: $(x \wedge_S y) \bullet (g * H) = (x \wedge_S y) \cdot g * H = (x \cdot g \wedge_G y \cdot g) * H = ((x \cdot g * H) \wedge_{\frac{G}{H}} (y \cdot g * H) = x \bullet (g * H) \wedge_{\frac{G}{H}} y \bullet (g * H)$. Therefore, $(SH4)$ holds.

$(SH5)$: $(x \to_S y) \bullet (g * H) = (x \to_S y) \cdot g * H = (g * (x \cdot g \to_G y \cdot g)) * H = (g * H) *_{\frac{G}{H}} ((x \cdot g \to_G y \cdot g) * H) = (g * H) *_{\frac{G}{H}} ((x \cdot g * H) \to_{\frac{G}{H}} (y \cdot g * H)) = (g * H) *_{\frac{G}{H}} (x \bullet (g * H) \to_{\frac{G}{H}} y \bullet (g * H))$. Therefore, $(SH5)$ holds.

Therefore, $\frac{G}{H}$ is a module of the semihoop S. □

Definition 3.15. Let G be a module of the semihoop S and H be a proper submodule of G. Then H is called a prime submodule of G, if $x \cdot g \in H$, then $g \in H$ or $x \in (H : G) = \{x \in S | x \cdot G \subseteq H\}$, for $x \in S$ and $g \in G$.

Example 3.16. $H = \{\alpha, 1\}$ *can generate a submodule of* G *in Example* 3.2. *It is clear that* $H = \{\alpha, 1\}$ *can generate a prime submodule of* G.

Note: Let G be a module of the semihoop S, $A \subseteq S$, and $B \subseteq G$. We define $A \cdot B = \{a \cdot b | a \in A, b \in B\}$.

Proposition 3.17. *Let* G *be a module of the semihoop* S *and* H *be a proper submodule of* G. *If* H *is a prime submodule of* G, *then* $A \cdot E \subseteq H$ *implies* $E \subseteq H$ *or* $A \subseteq (H : G)$, *for any filter* A *of* S *and submodule* E *of* G.

Proof. If H is a prime submodule of G and $A \cdot E \subseteq H$ for any submodule E of G. We want to prove that $E \subseteq H$ or $A \subseteq (H : G)$. Now, we assume that $E \nsubseteq H$ and $A \nsubseteq (H : G)$. Then there exist $e \in E \nsubseteq H$ and $x \in A \nsubseteq (H : G)$. Since $A \cdot E \subseteq H$, we have $x \cdot e \in H$. Since H is a prime submodule of G and $e \notin H$, we can conclude that $x \in (H : G)$, which is a contradiction. Therefore, $A \cdot E \subseteq H$ implies $E \subseteq H$ or $A \subseteq (H : G)$, for any filter A of S and submodule E of G.

□

Lemma 3.18. *Let G be a module of the bounded semihoop S and H be a proper submodule of G. If H is a filter of G, then $C = (H : G) \cup \{1\}$ is a filter of S.*

Proof. Let G be a module of the bounded semihoop S and H be a proper submodule of G. We want to prove that $C = (H : G) \cup \{1\}$ is a filter of S. For any $g \in G$, we have:

(1) It is clear that the set $(H : G) \cup \{1\}$ is a nonempty subset of S.

(2) If $x, y \in C = (H : G) \cup \{1\}$. We want to prove that $x \odot y \in C = (H : G) \cup \{1\}$. We will encounter the following four situations:

When $x = 1$ and $y = 1$, we have $x \odot y = 1 \in C = (H : G) \cup \{1\}$.

When $x = 1$ and $y \neq 1$, we have $y \cdot g \in H$ for any $g \in G$. We can get that $(x \odot y) \cdot g = (1 \odot y) \cdot g = y \cdot g \in H$. Therefore, $x \odot y \in (H : G)$ and $x \odot y \in C$.

When $x \neq 1$ and $y = 1$, we can use the same method to prove $x \odot y \in C$.

When $x \neq 1$ and $y \neq 1$, We have $x \cdot g, y \cdot g \in H$ for any $g \in G$. We can conclude that $(x \odot y) \cdot g = x \cdot (y \cdot g) \in H$. Since H is a proper submodule of G, we have $x \cdot (y \cdot g) \in H$. Therefore, $x \odot y \in C$.

(3) If $x \in C = (H : G) \cup \{1\}$ and $x \leq y$. We will encounter the following two situations:

When $x = 1$, we have $y = 1 \in C = (H : G) \cup \{1\}$, since $x \leq y$.

When $x \neq 1$, we have $x \cdot g \in H$ for any $g \in G$. Since $x \leq_S y$, we can get that $x \cdot g \leq_G y \cdot g$, according to Proposition 3.8. Since H is a filter of G, we can conclude $y \cdot g \in H$. Therefore, $y \in C = (H : G) \cup \{1\}$.

Therefore, $C = (H : G) \cup \{1\}$ is a filter of S. □

Proposition 3.19. *Let G be a module of the linear bounded semihoop S and H be a proper submodule of G. If H is a filter of the module G and F is a filter of S, then $\frac{G}{H}$ is a module of $\frac{S}{F}$.*

Proof. Since G is a module of the linear bounded semihoop S, there exists an operation $\cdot : S \times G \to G$ by $(x, g) \mapsto x \cdot g$ for any $x \in S$ and $g \in G$. We define an operation $\bullet : \frac{S}{F} \times \frac{G}{H} \to \frac{G}{H}$ by $[x] \bullet (g * H) = x \cdot g * H$ for any $[x] \in \frac{S}{F}$ and $g * H \in \frac{G}{H}$. It is obvious that $\bullet$ is well defined, according to Proposition 3.9 and Theorem 3.14.

For any $f * H, g * H \in \frac{G}{H}$ and $[x], [y] \in \frac{S}{F}$, we have:

$(SH1)$: It is obvious that $[1] \bullet (g * H) = 1 \cdot g * H = g * H$. Therefore, $(SH1)$ holds.

$(SH2)$: $[x] \bullet ((f * H) *_{\frac{G}{H}} (g * H)) = [x] \bullet ((f * g) * H) = x \cdot (f * g) * H = (x \cdot f * x \cdot g) * H = (x \cdot f * H) *_{\frac{G}{H}} (x \cdot g * H) = [x] \bullet (f * H) *_{\frac{G}{H}} [x] \bullet (g * H)$. Therefore, $(SH2)$ holds.

$(SH3)$: $([x] \odot_{\frac{S}{F}} [y]) \bullet (g * H) = [x \odot_S y] \bullet (g * H) = (x \odot_S y) \cdot g * H = x \cdot (y \cdot g) * H = [x] \bullet (y \cdot g * H) = [x] \bullet ([y] \bullet (g * H))$. Therefore, $(SH3)$ holds.

$(SH4)$: $([x] \wedge_{\frac{S}{F}} [y]) \bullet (g * H) = [x \wedge_S y] \bullet (g * H) = (x \wedge_S y) \cdot g * H = (x \cdot g \wedge_S y \cdot g) * H = (x \cdot g * H) \wedge_{\frac{G}{H}} (y \cdot g * H) = [x] \bullet (g * H) \wedge_{\frac{G}{H}} [y] \bullet (g * H)$. Therefore, $(SH4)$ holds.

$(SH5)$: $([x] \rightarrow_{\frac{S}{F}} [y]) \bullet (g * H) = [x \rightarrow_S y] \bullet (g * H) = (x \rightarrow_S y) \cdot g * H = (g * (x \cdot g \rightarrow_S y \cdot g)) * H = (g * H) *_{\frac{G}{H}} ((x \cdot g \rightarrow_G y \cdot g) * H) = (g * H) *_{\frac{G}{H}} ((x \cdot g * H) \rightarrow_{\frac{G}{H}} (y \cdot g * H)) = (g * H) *_{\frac{G}{H}} ([x] \bullet (g * H) \rightarrow_{\frac{G}{H}} [y] \bullet (g * H))$. Therefore, $(SH5)$ holds. Therefore, $\frac{G}{H}$ is a module of the semihoop $\frac{S}{F}$. □

Definition 3.20. Let G be a module of the semihoop S. Then G is called a torsion free module, if $x \cdot g = 1$ implies $x = 1$ or $g = 1$, for any $x \in S$ and $g \in G$.

Example 3.21. *$H = \{1\}$ can generate a torsion free submodule of G in Example* 2.2.

Theorem 3.22. *Let G be a module of the linear bounded semihoop S and H be a filter of the module G. If H is a prime submodule of G if and only if $C = (H : G) \cup \{1\}$ is a filter of S and $\frac{G}{H}$ is a torsion free module of $\frac{S}{C}$.*

Proof. Since G is a module of the linear bounded semihoop S, there exists an operation $\cdot : S \times G \to G$ defined by $(x, g) \mapsto x \cdot g$.

If H is a prime submodule of G and is a filter of G, then $C = (H : G) \cup \{1\}$ is a filter of S, according to Lemma 3.18, then we define an operation $\bullet : \frac{S}{C} \times \frac{G}{H} \to \frac{G}{H}$ by $[x] \bullet (g * H) = x \cdot g * H$ for any $[x] \in \frac{S}{C}$ and $g * H \in \frac{G}{H}$. We have proved that $\frac{G}{H}$ is a module of $\frac{S}{C}$. We want to prove that $\frac{G}{H}$ is a torsion free module of $\frac{S}{C}$. Assume $[x] \bullet (g * H) = 1 * H = H$, we can get that $[x] \bullet (g * H) = x \cdot g * H = H$ and so $x \cdot g \in H$. Since H is a prime submodule of G, we have $x \in (H : G)$ or $g * H = H$. Since $x \to 1, 1 \to x \in C = (H : G) \cup \{1\}$, we can conclude that $[x] = [1]$. We can get that if $[x] \bullet (g * H) = 1 * H = H$, then $[x] = [1]$ or $g * H = H = 1 * H$. Therefore, $\frac{G}{H}$ is a torsion free module of $\frac{S}{C}$.

If $C = (S : G) \cup \{1\}$ is a prime ideal of S and $\frac{G}{H}$ is a torsion free module of $\frac{S}{C}$, then we define an operation $\bullet : \frac{S}{C} \times \frac{G}{H} \to \frac{G}{H}$ by $[x] \bullet (g * H) = x \cdot g * H$ for any $[x] \in \frac{S}{C}$ and $g * H \in \frac{G}{H}$. We want to prove that H is a prime submodule of G. If $x \cdot g \in H$, for $x \in S$ and $g \in G$, then we have $[x] \bullet (g * H) = x \cdot g * H = H$. Since $\frac{G}{H}$ is a torsion free module of $\frac{S}{C}$, we have $[x] = [1]$ or $g * H = H$. We have $x \to 1, 1 \to x \in C = (H : G) \cup \{1\}$. Therefore, $x \in (H : G)$ or $g \in H$. □

Proposition 3.23. *Let $S_1 = (S_1, \odot_{S_1}, \rightarrow_{S_1}, \wedge_{S_1}, 1)$ and $S_2 = (S_2, \odot_{S_2}, \rightarrow_{S_2}, \wedge_{S_2}, 1)$ be two semihoops and $f : S_1 \to S_2$ be a homomorphism. If G is a module of S_2, then G is a module of S_1.*

Proof. Let $S_1 = (S_1, \odot_{S_1}, \to_{S_1}, \wedge_{S_1}, 1)$ and $S_2 = (S_2, \odot_{S_2}, \to_{S_2}, \wedge_{S_2}, 1)$ be two semihoops and $f : S_1 \to S_2$ be a homomorphism. Since G is a module of S_2, we have the operation of modules $\cdot : S_2 \times G \to G$ defined by $(a, g) \mapsto a \cdot g$ for any $a \in S_2$ and $g \in G$. Now, we define the operation $\cdot_1 : S_1 \times G \to G$ by $x \cdot_1 g = f(x) \cdot g$, for any $x \in S_1$ and $g \in G$. For $x, y \in S_1$, if $x = y$, then $f(x) = f(y)$ and so $x \cdot_1 g = f(x) \cdot g = f(y) \cdot g = y \cdot_1 g$. It means that $\cdot_1$ is well-defined. We prove that G is a module of S_1. For any $x, y \in S_1$ and $g \in G$, we have:

$(SH1)$ $1_{S_1} \cdot_1 g = f(1_{S_1}) \cdot g = 1_{S_2} \cdot g = g$;

$(SH2)$ $x \cdot_1 (g * f) = f(x) \cdot (g * f) = f(x) \cdot g * f(x) \cdot f = x \cdot_1 g * x \cdot_1 f$;

$(SH3)$ $(x \odot_{S_1} y) \cdot_1 g = f(x \odot_{S_1} y) \cdot g = (f(x) \odot_{S_2} f(y)) \cdot g = f(x) \cdot (f(y) \cdot g) = f(x) \cdot (y \cdot_1 g) = x \cdot_1 (y \cdot_1 g)$;

$(SH4)$ $(x \wedge_{S_1} y) \cdot_1 g = f(x \wedge_{S_1} y) \cdot g = (f(x) \wedge_{S_2} f(y)) \cdot g = f(x) \cdot g \wedge_G f(y) \cdot g = x \cdot_1 g \wedge_G y \cdot_1 g$;

$(SH5)$ $(x \to_{S_1} y) \cdot_1 g = f(x \to_{S_1} y) \cdot g = (f(x) \to_{S_2} f(y)) \cdot g = g * (f(x) \cdot g \to_G f(y) \cdot g) = g * (x \cdot_1 g \to_G y \cdot_1 g)$.

Therefore, G is a module of the semihoop S_1. □

Proposition 3.24. *Let S_1 and S_2 be two bounded semihoops and $f : S_1 \to S_2$ a homomorphism. Then $kerf = \{x \in S_1 | f(S_1) = 1_{S_2}\}$ and $Imf = \{f(x) | x \in S_1\}$ are semihoops.*

Proof. $kerf = \{x \in S_1 | f(S_1) = 1_{S_2}\}$ is a nonempty subset of S_1, since $f(1_{S_1}) = 1_{S_2}$, we have $1_{S_1} \in kerf$. We only need to prove that $kerf$ is closed under operations $\odot$, $\to$, and $\wedge$. For any $x, y \in kerf$, we have:

(1) $f(x \odot_{S_1} y) = f(x) \odot_{S_2} f(y) = 1_{S_2} \odot_{S_2} 1_{S_2} = 1_{S_2}$.

(2) $f(x \to_{S_1} y) = f(x) \to_{S_2} f(y) = 1_{S_2} \to_{S_2} 1_{S_2} = 1_{S_2}$.

(3) $f(x \wedge_{S_1} y) = f(x) \wedge_{S_2} f(y) = 1_{S_2} \wedge_{S_2} 1_{S_2} = 1_{S_2}$.

Therefore, $kerf = \{x \in S_1 | f(S_1) = 1_{S_2}\}$ is a semihoop.

It is clear that $Imf = \{f(x) | x \in S_1\} \subseteq S_2$. We only need to prove that $0, 1 \in Imf$ and Imf are closed under operations $\odot$, $\to$, and $\wedge$. Since $f(1) = 1_{S_2}$, we have $1_{S_2} \in Imf$.

Let $f(x), f(y) \in Imf$ for any $x, y \in S_1$. Since S_1 is a bounded semihoop, we have $x \odot_{S_1} y \in S_1$, $x \to_{S_1} y \in S_1$, and $x \wedge_{S_1} y \in S_1$. Then,

(1) $f(x) \odot_{S_2} f(y) = f(x \odot_{S_1} y) \in Imf = \{f(x) | x \in S_1\}$.

(2) $f(x) \to_{S_2} f(y) = f(x \to_{S_1} y) \in Imf = \{f(x) | x \in S_1\}$.

(3) $f(x) \wedge_{S_2} f(y) = f(x \wedge_{S_1} y) \in Imf = \{f(x) | x \in S_1\}$.

(4) $0_{S_2} = f(0) \in Imf$.

Hence Imf is closed under operations $\odot$, $\to$, and $\wedge$. We can conclude that Imf is a subalgebra of S_2. Therefore, $Imf = \{f(x) | x \in S_1\}$ is a bounded semihoop. □

Definition 3.25. Let G_1 and G_2 be two modules of the semihoop S. Then the map $f : G_1 \to G_2$ is called a module homomorphism if for any $g_1, g_2 \in G_1$ and $x \in S$, it satisfies with

(*i*) $f(g_1 *_{G_1} g_2) = f(g_1) *_{G_2} f(g_2)$;
(*ii*) $f(x \cdot g_1) = x \cdot f(g_1)$.

Example 3.26. *Let $G_1 = \{0, a, 1\}$ and $G_2 = \{0, \alpha, 1\}$ be modules of the semihoop S. We can get that $(G_1, \odot_{G_1}, \rightarrow_{G_1}, \wedge_{G_1}, 1)$ and $(G_2, \odot_{G_2}, \rightarrow_{G_2}, \wedge_{G_2}, 1)$ are semihoops and we define $G_1 = (\{0, a, 1\}, *_{G_1}, 1)$ and $G_2 = (\{0, \alpha, 1\}, *_{G_2}, 1)$ are Abelian groups and group operations on G_2 and G_2 as shown in Table* 17 *and Table* 18*, respectively.*

$*_{G_1}$	0	a	1
0	a	1	0
a	1	0	a
1	0	a	1

Table 17

$*_{G_2}$	0	α	1
0	α	1	0
α	1	0	α
1	0	α	1

Table 18

We define the operation $\cdot_{G_1} : S \times G_1 \rightarrow G_1$ by $x \cdot g_1 = g_1$ and the operation $\cdot_{G_2} : S \times G_2 \rightarrow G_2$ by $x \cdot y_2 = y_2$, for every $x \in S$, $g_1 \in G_1$, and $y_2 \in G_2$.

There exists a map $f : G_1 \rightarrow G_2$, in which $f(1) = 1$, $f(0) = 0$, and $f(a) = \alpha$. We can easily examine that $f(g_1 * g_2) = f(g_1) * f(g_2)$ and $f(x \cdot g_1) = f(g_1) = x \cdot f(g_1)$, for any $g_1, g_2 \in G_1$ and $x \in S$. Therefore, the map f is a module homomorphism, according to Definition 3.25.

Note: If $f : G_1 \rightarrow G_2$ is a surjective mapping, then we call the module homomorphism $f : G_1 \rightarrow G_2$ is a module surjective homomorphism.

Proposition 3.27. *Let G_1 and G_2 be two modules of the semihoop S and $f : G_1 \rightarrow G_2$ be a homomorphism of semihoops. If $f : G_1 \rightarrow G_2$ is also a module surjective homomorphism, then*

(*i*) *$kerf = \{g \in G_1 | f(g) = 1\}$ is a submodule of G_1. If H is a submodule of G_2, then $f^{-1}(H)$ is a submodule of G_1.*

(*ii*) *$Imf = \{f(g) | g \in G_1\}$ is a submodule of G_2. If H is a submodule of G_1, then $f(H)$ is a submodule of G_2.*

Proof. (*i*) For any $x \in S$ and $g \in kerf$, we have $f(x \cdot g) = x \cdot f(g) = x \cdot 1 = 1$. We can get that $x \cdot g \in kerf$. Therefore, $kerf = \{g \in G_1 | f(g) = 1\}$ is a submodule of G_1.

Let H be a submodule of G_2. It is clear that $f^{-1}(H) \subseteq G_1$. For any $x \in S$ and $g \in f^{-1}(H)$, we can get that $x \cdot h \in H$ and $g = f^{-1}(h)$ for $h \in H$. We have $x \cdot g = x \cdot f^{-1}(h) = f^{-1}(x \cdot h) \in f^{-1}(H)$. Then $f^{-1}(H)$ is a submodule of G_1.

(*ii*) For any $x \in S$ and $t \in Imf = \{f(g) | g \in G_1\}$, there is $t = f(g)$ for $g \in G_1$. So, we have $x \cdot t = x \cdot f(g) = f(x \cdot g)$. Since G_1 is a module of the semihoop S, we have $x \cdot g \in G_1$. Therefore, $x \cdot t \in Imf$. Therefore, $Imf = \{f(g) | g \in G_1\}$ is a submodule of G_2.

If H is a submodule of G_1. It is clear that $f(H) \subseteq G_2$. For any $x \in S$ and $g \in f(H)$, we can get that $g = f(h)$ and $x \cdot h \in H$ for $h \in H$. We have $x \cdot g = x \cdot f(h) = f(x \cdot h) \in f(H)$. Therefore, $f(H)$ is a submodule of G_2. □

Theorem 3.28. *Let G_1 and G_2 be two modules of the semihoop S and $f : G_1 \to G_2$ be a homomorphism of semihoops. If $f : G_1 \to G_2$ is also a module surjective homomorphism, then H is a prime submodule of G_2 if and only if $f^{-1}(H)$ is a prime submodule of G_1.*

Proof. Let G_1 and G_2 be two modules of the semihoop S and $f : G_1 \to G_2$ be a module surjective homomorphism.

Let H be a prime submodule of G_2. $f^{-1}(H)$ is a submodule of G_1, according to Proposition 3.27. If $x \cdot g \in f^{-1}(H)$, for $x \in S$ and $g \in G_1$. We have $x \cdot g = f^{-1}(h)$ for $h \in H$. We can conclude that $x \cdot f(g) = f(x \cdot g) = ff^{-1}(h) = h \in H$. Since H is a prime submodule of G_2, we have $f(g) \in H$ or $x \in (H : G_2) = \{x \in S | x \cdot G_2 \in H\}$. If $f(g) \in H$, then we have $f(g) = h$ for $h \in H$. We can get that $f^{-1}(f(g)) = f^{-1}(h) \in f^{-1}(H)$. If $x \in (H : G_2) = \{x \in S | x \cdot G_2 \in H\}$, then we have $x \cdot g_2 = h$ for $h \in H$ and any $g_2 \in G_2$. We can conclude that $x \cdot f^{-1}(g_2) = f^{-1}(x \cdot g_2) = f^{-1}(h) \in f^{-1}(H)$ for any $f^{-1}(g_2) \in G_1$. Therefore, $g \in f^{-1}(H)$ or $x \in (f^{-1}(H) : G_1) = \{x \in S | x \cdot G_1 \in f^{-1}(H)\}$.

Let $f^{-1}(H)$ be a prime submodule of G_1. We have $ff^{-1}(H) = H$ is a submodule of G_2, according to Proposition 3.27. If $x \cdot g \in H$, for $x \in S$ and $g \in G_2$. We can get that $x \cdot f^{-1}(g) = f^{-1}(x \cdot g) \in f^{-1}(H)$. Since $f^{-1}(H)$ is a prime submodule of G_1, we have $f^{-1}(g) \in f^{-1}(H)$ or $x \in (f^{-1}(H) : G_1) = \{x \in S | x \cdot G_1 \in f^{-1}(H)\}$, according to Definition 3.15. If $f^{-1}(g) \in f^{-1}(H)$, then we have $f^{-1}(g) = f^{-1}(h)$ for $h \in H$. We can get that $g = ff^{-1}(g) = ff^{-1}(h) = h \in H$. If $x \in (f^{-1}(H) : G_1) = \{x \in S | x \cdot G_1 \in f^{-1}(H)\}$, then we have $x \cdot g_1 = f^{-1}(h)$ for $h \in H$ and any $g_1 \in G_1$. We can conclude that $x \cdot f(g_1) = f(x \cdot g_1) = ff^{-1}(h) = h \in H$. Therefore, $g \in H$ or $x \in (H : G_2) = \{x \in S | x \cdot G_2 \in H\}$. □

Definition 3.29. Let G be a module of the semihoop S. If for any submodule H of G and there exists a filter F of S such that $H = F \cdot G$, then we call G is a multiplication module.

Example 3.30. *In example 3.3, if we define $\cdot : S \times G \to G$ by $(x, g) \mapsto x \cdot g$, in which, $1 \cdot g = g$, $a \cdot g = g$, $0 \cdot 0 = 0$, $0 \cdot 1 = 0$, and $0 \cdot \alpha = 0$, for any $x \in S$ and $g \in G$, then we can examine the group $G = (\{0, \alpha, 1\}, *, 1)$ is a module of S. Since $0 \cdot 1 = 0 \in \{1\}$, $H = \{1\}$ is not a submodule of G. We can find the submodule of G only has itself and we have $G = F \cdot G$ for filter F of S. Therefore, the module G is a multiplication module.*

Proposition 3.31. *Let G be a module of the semihoop S. If G is a multiplication module, then $H = (H : G) \cdot G$, for any submodule H of G.*

Proof. Let G be a module of the semihoop S and H be a submodule of G. Since G is a multiplication module, there exists a filter F of S such that $F \cdot G = H$. Hence, we have $F \subseteq (H : G)$. We can get that $H = F \cdot G \subseteq (H : G) \cdot G$.

If $x \in (H : G) \cdot G$, then $x = y \cdot g$ for some $y \in (H : G)$ and $g \in G$. Since $y \in (H : G)$, we have $x = y \cdot g \in H$. Therefore, $(H : G) \cdot G \subseteq H$. □

Definition 3.32. Let G be a module of the semihoop S. G is called a comultiplication module, if there exists a filter F of S such that $H = \{g \in G | F \cdot g = 1\}$ is a submodule of G.

Example 3.33. *It is obvious that $F = \{b, 1\}$ is a filter of S in Example* 3.2. *Since we define $\cdot : S \times G \to G$ by $x \cdot g = g$, for any $x \in S$ and $g \in G$, we have $x \cdot 1 = 1$ and $x \cdot g = g \neq 1$ for any $x \in F$ and $g \in G \setminus \{1\}$. Therefore, $H = \{1\}$ and it is clear that $H = \{1\}$ is a submodule of G. Then G is a comultiplication module.*

However, in Example 3.30*, we can find that $F_1 = \{1\}$ and $F_2 = \{0, a, 1\}$ are all filters of S and we can get that $H_1 = \{g \in G | F_1 \cdot g = 1\} = \{1\}$ is not a submodule of G, since $0 \cdot 1 = 0 \notin \{1\}$ and $H_2 = \{g \in G | F_2 \cdot g = 1\} = \emptyset$ is also not a submodule of G. Therefore, G is not a comultiplication module.*

Note: Let G be a module of the semihoop S and H be a submodule of G. We define $A_H = \{x \in S | x \cdot H = 1\}$.

Proposition 3.34. *Let G be a comultiplication module of the semihoop S. Then there exists a submodule H of G such that $H = \{g \in G | A_H \cdot g = 1\}$.*

Proof. If G is a comultiplication module of the semihoop S, then there exists a filter F of S such that $H = \{g \in G | F \cdot g = 1\}$ is a submodule of G. Hence, we can get that $F \cdot H = \{1\}$.

For any $b \in F$ and $h \in H$, we have $b \cdot h = 1$. This implies that $b \in A_H = \{x \in S | x \cdot H = 1\}$. Hence, we can conclude that $F \subseteq A_H$. For any $y \in \{g \in G | A_H \cdot g = 1\}$, we have $A_H \cdot y = 1$ and $F \cdot y = 1$, since $F \subseteq A_H$. Therefore, $y \in \{g \in G | F \cdot g = 1\}$, and so $\{g \in G | A_H \cdot g = 1\} \subseteq \{g \in G | F \cdot g = 1\} = H$. For any $h \in H \subseteq G$, we have $A_H \cdot h = 1$, and so $h \in \{g \in G | A_H \cdot g = 1\}$. Hence, $H \subseteq \{g \in G | A_H \cdot g = 1\}$.

Therefore, $H = \{g \in G | A_H \cdot g = 1\}$. □

Definition 3.35. A module G of the semihoop S satisfies the ascending chain condition on submodules if for every chain $G_1 \subseteq G_2 \subseteq G_3 \cdots$ of submodules of G, there exists $n \in \mathbb{N}$ such that $G_k = G_n$ for any $k \geq n$. A module G of the semihoop S satisfies the descending chain condition on submodules if for every chain $G_1 \supseteq G_2 \supseteq G_3 \cdots$ of submodules of G, there exists $n \in \mathbb{N}$ such that $G_k = G_n$ for any $k \geq n$.

Example 3.36. *We can find that $G_1 = \{1\}$ and $G_2 = \{0, \alpha, 1\}$ are all submodules of the module G in Example* 3.3. *Since $S_1 \subseteq S_2$, the module G satisfies the ascending chain condition.*

Theorem 3.37. *Let G be a comultiplication module of the semihoop S with the ascending chain condition. If for any submodule H of G such that $H = \{g \in G | A_H \cdot g = 1\}$, then G satisfies the descending chain condition.*

Proof. Assume $G_1, G_2, G_3, \ldots$are submodules of G such that $G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots \supseteq G_n \supseteq \cdots$ is a descending chain of the module G. If $G_i \supseteq G_j$, where $i \leq j$ and for any $x \in \{s \in S | s \cdot G_i = 1\}$, then we have $x \cdot G_i = 1$. We can conclude that $x \cdot G_j = 1$ and $x \in \{s \in S | s \cdot G_j = 1\}$. Therefore, $\{s \in S | s \cdot G_i = 1\} \subseteq \{s \in S | s \cdot G_j = 1\}$. We can get that $\{s \in S | s \cdot G_1 = 1\} \subseteq \{s \in S | s \cdot G_2 = 1\} \subseteq \cdots \subseteq \{s \in S | s \cdot G_n = 1\} \subseteq \cdots$. We can conclude that $A_{G_1} \subseteq A_{G_2} \subseteq A_{G_3} \subseteq \cdots \subseteq A_{G_n} \subseteq \cdots$, which is an ascending chain of S. Since S satisfies the ascending chain condition, there exists $n \in N$ such that $A_{G_k} = A_{G_n}$, for any $k \geq n$.

We have $G_n = \{g \in G | A_{G_n} \cdot g = 1\}$, since $H = \{g \in G | A_H \cdot g = 1\}$ for any submodule H of G. For any $g \in G_n$. Then $A_{G_n} \cdot g = A_{G_k} \cdot g = 1$, that is, $g \in G_k = \{g \in G | A_{G_k} \cdot g = 1\}$. Hence $G_n \subseteq G_k$, for any $k \geq n$. We can get that $G_n = G_k$, for any $k \geq n$. Therefore, G satisfies the descending chain condition. □

4. Conclusion

In this paper, we have introduced the definitions of modules on semihoops and provided some examples to illustrate it. Moreover, we find that if the semihoop G can generate an Abelian group, then G is a module of any subalgebra S of the semihoop G. Subsequently, we have introduced the concepts of prime submodules, torsion free modules, multiplication modules, and comultiplication modules and explored the relationships among them through employing the concepts homomorphism, filter, and quotient structure. Finally, we find that if G is a comultiplication module of S with the ascending chain condition, then G satisfies the descending chain condition. We use modules of semihoops, to investigate their impact on the substructure of semihoops and further study their representation. We also hope to extend the structure of modules of semihoops to fuzzy logic and uncertain information. In future work, we will study the fuzzy submodules and topological submodules of semihoops, give characterizations regarding them, and investigate the relationship between these special submodules and the descending chain condition.

5. Author Contributions

All authors contributed to this article.

6. Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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8. Ethical considerations

We declare that we have complied with the ethical standards for publishing articles in this journal.

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10. Conflict of interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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