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ON FUNDAMENTAL PROPERTIES OF HOM-GROUPS

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Abstract. This paper focuses on the study of Hom-groups. A Hom-group $(G, *, \alpha)$ is the non-associative generalization of the classical group G whose associativity and unitality are twisted by a compatible bijective map α. We present more properties of Hom-groups, Hom-subgroups, Hom-normal subgroups, Hom-quotient groups and Hom-group homomorphisms with examples. We prove the Zassenhaus Butterfly Lemma of Hom-groups as a major result of this paper.

Keywords: Hom-group; Hom-normal subgroup; Hom-quotient group; Hom-group homomorphism.
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1. Introduction

The concept of Hom-Lie algebras was introduced by Hartwig et al. in [6] in their study of deformations of the Witt and Virasoro algebras. Since the introduction of Hom-Lie algebras, many researchers have worked on expanding and extending the concept to include other Hom-like structures such as Hom-Lie groups, Hom-Hopf algebras, Hom-coalgebras and Hom-bialgebras. For full details, the readers should see ([2], [3], [5], [6], [11-13], [15-19], [21-24]). Hassanzadeh [8], Laurent-Gengouxa and Makhlouf [13] have shown in their works that the twisting map in the Hom-group (G, α) needs not to be invertible as it was taken in the previous works. Using the invertibiliy of α, Hassanzadeh [7] studied and obtained many interesting properties of the Hom-group including Hom-group representations and Hom-group cohomology. Following the notions introduced and used to study Hom-groups by Hassanzadeh in [7, 8], Liang et al. [14] extensively studied Hom-groups. Many basic and fundamental properties of Hom-groups were presented with interesting examples. They proved first, second and third isomorphism theorems of Hom-groups. They introduced the concept of Hom-group action and proved the first Sylow theorem for Hom-groups. In [1], Agboola et al. for the first time introduced the concept of neutrosophic Hom-group. Many examples of both the Hom-groups and neutrosophic Hom-groups were presented. They presented basic and fundamental properties of neutrosophic Hom-groups. In addition, they studied neutrosophic morphisms of neutrosophic Hom-groups along with their properties and they

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established a kind of morphism relationship that exists between a neutrosophic Hom-group $G(I)$ and a Hom-group $G \times G$. In the present paper, we follow the concepts introduced by Hassanzadeh [7, 8] and Liang et al. [14] to further study and obtain more properties of Hom-groups. Specifically, we present more properties of Hom-groups, Hom-subgroups, Hom-normal subgroups, Hom-quotient groups and Hom-group homomorphisms with examples. As a major result of this paper, we prove the Zassenhaus Butterfly Lemma of Hom-groups.

1.1. **Recalls.** In this subsection, we will recall the definition, basic notations and properties of Hom-groups as presented by Basdouri et al. in [3], Hassanzadeh in [7, 8], Liang et al. in [14] and Jiang etal. in [10] which we are going to use in the sequel.

Definition 1.1. [10] Let G be a nonempty set, $* : G \times G \to G$ a binary operation on G, $\alpha : G \to G$ a bijective set map and $1 \in G$ a distinguished element. The quadruple $(G, *, \alpha, 1)$ is called a Hom-group if the following conditions hold:

(i) The product map α satisfies the Hom-associativity property
$$\alpha(g) * (h * k) = (g * h) * \alpha(k) \ \forall g, h, k \in G.$$
(ii) The product map α is multiplicative that is
$$\alpha(g * k) = \alpha(g) * \alpha(h) \ \forall g, h \in G.$$
(iii) The element $1 \in G$ called the unit element satisfies the Hom-unitary conditions
$$g * 1 = 1 * g = \alpha(g) \ \forall g \in G.$$
(iv) For every element $g \in G$, there exists an element $g^{-1} \in G$ such that
$$g * g^{-1} = g^{-1} * g = 1.$$
If there is no confusion, we may omit the multiplication sign $*$. In that case $g * h$ will be written simply as gh.

(v) For any $g \in G$, there exists $k \in \mathbb{N}$ satisfying the Hom-invertibility condition
$$\alpha^k(gg^{-1}) = \alpha^k(g^{-1}g) = 1.$$
If only conditions (i) and (ii) are satisfied, G is called a Hom-semigroup. A Hom-semigroup with condition (iii) is called a Hom-monoid and a Hom-monoid with condition (iv) is called a Hom-group.

In (v), the smallest such k is called the invertibility index of g. If the invertibility index of $g \in G$ is k, then the invertibility index of $\alpha(g)$ is $k - 1$.

Example 1.2. *[1] Let $\mathbb{C}$ be the set of complex numbers and let $* : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be the binary operation on $\mathbb{C}$ defined by $z_1 * z_2 = \overline{z_1 + z_2} \ \forall z_1, z_2 \in \mathbb{C}$. Let $\alpha : \mathbb{C} \to \mathbb{C}$ be a mapping defined by $\alpha(z) = \overline{z} \ \forall z \in \mathbb{C}$. Then $(\mathbb{C}, *, \alpha, 0)$ is an abelian Hom-group.*

Proposition 1.3. *[14] Let G be a Hom-group.*

(a) *The unit element of G is unique.*
(b) *For any element $g, h \in G$:*
(i) $\alpha(1) = 1$*;*
(ii) g^{-1} *is unique;*
(iii) $(g^{-1})^{-1} = g$*;*
(iv) $(gh)^{-1} = h^{-1}g^{-1}$*;*
(v) $(\alpha(g))^{-1} = \alpha(g^{-1})$*.*

Proof. See [14]. □

Proposition 1.4. *[14] Let G be a Hom-group.*

(a) *If there is an element $g \in G$ such that $gg = \alpha(g) = g1$, then $g = 1$.*
(b) *If the elements $g, h, k \in G$ satisfy $gh = gk$ or $hg = kg$, then $h = k$.*
(c) *For any $g, h, k \in G$:*
(i) $\alpha^{-1}(gh) = \alpha^{-1}(g)\alpha^{-1}(h)$*;*
(ii) $(\alpha^{-1}(g)h)k = g(h\alpha^{-1}(k))$*.*

Proof. See [14]. □

Proposition 1.5. *[7] Let (G, μ) be a group and let $\alpha : G \to G$ be a group automorphism. Then $(G, \alpha \circ \mu, \alpha)$ is a Hom-group.*

Proof. See [1] □

Definition 1.6. [7] Let H be a nonempty subset of a Hom-group (G, α) that is closed under the binary operation in G. H is said to be a Hom-subgroup of G if (H, α) is itself a Hom-group under the binary operation inherited from G and we write $H \leq G$.

Proposition 1.7. *[14] Let H and K be two Hom-subgroups of a Hom-group G.*

(a) *(i)* $1_H = 1_G$*.*
(ii) *for each $h \in H$, $h_H^{-1} = h_G^{-1}$.*
(b) *(i)* $H \cap K \leq G$*.*
(ii) *$H \cup K \leq G$ if and only if $H \subset K$ or $K \subset H$.*

Proof. See [14]. □

Definition 1.8. [14] Let H be a Hom-subgroup of a Hom-group G and let $g \in G$. The sets gH an Hg are defined respectively by $\{gh : h \in H\}$ and $\{hg : h \in H\}$. gH is called a Hom-left coset of H in G while Hg is called a Hom-right coset of H in G. The set of all distinct Hom-left cosets of H in G is denoted by G/H.

Generally, $gH \neq Hg$ except if G is abelian. However, if $gH = Hg\ \forall g \in G$, then H is called a Hom-normal subgroup of G and we write $H \lhd G$.

Proposition 1.9. *[14] Let H be a Hom-subgroup of a finite Hom-group G. For all $g, h \in G$, the following statements are equivalent:*

(i) *$gH = hH$;*
(ii) *$gH \cap hH \neq \emptyset$;*
(iii) *$g^{-1}h \in H$;*
(iv) *$\alpha(h) \in gH$;*
(v) *$\alpha(g)H = \alpha(h)H$.*

Proof. See [14]. □

Proposition 1.10. *[14] Let H be a Hom-subgroup of a Hom-group G. For all $g \in G$, the following statements are equivalent:*

(i) *$gH = Hg$;*
(ii) *for $h \in H$, $(gh)\alpha(g^{-1}) \in H$;*
(iii) *$(gH)\alpha(g^{-1}) \subseteq H$;*
(iv) *$(gH)\alpha(g^{-1}) = H$;*
(v) *$\alpha(g)H = H\alpha(g)$.*

Proof. See [14]. □

Lemma 1.11. *[14] Let $(G, *, \alpha)$ be a Hom-group. Let $H \leq G$ and $N \lhd G$. Then, $NH = HN \leq G$.*

Proof. See [14]. □

Proposition 1.12. *[14] If H is a Hom-normal subgroup of a Hom-group $(G, *, \alpha)$, then $(G/H, \odot, \beta)$ is a Hom-group, where $\odot$ is defined for any $aH, bH \in G/H$ by $aH \odot bH = a * bH$ and β is defined for any $aH \in G/H$ by $\beta(aH) = \alpha(a)H$.*

Proof. See [14]. □

Definition 1.13. [14] Let (G, α) and (H, β) be two Hom-groups. The map $\phi : G \to H$ is called a Hom-group homomorphism if ϕ satisfies the following two conditions:

(i) for any $g, h \in G$, $\phi(gh) = \phi(g)\phi(h)$;
(ii) for all $g \in G$, $\beta(\phi(g)) = \phi(\alpha(g))$.

In addition, if ϕ is a bijection, then we call ϕ an isomorphism and we write $G \cong H$.

Definition 1.14. [14] The map $\phi : (G, \alpha) \to (H, \beta)$ is called a weak Hom-group homomorphism if $\phi(1_G) = 1_H$ and $\beta \circ \alpha(gk) = (\phi \circ \alpha(g))(\phi \circ \alpha(k))$ $\forall g, k \in G$.

Proposition 1.15. *[14] Let $\phi : (G, \alpha) \to (H, \beta)$ be a Hom-group homomorphism. Then,*

(i) *$\phi(1_G) = 1_H$;*
ii) *for all $g \in G$, $\phi(g^{-1}) = (\phi(g))^{-1}$;*
(iii) *$Ker\phi \lhd G$;*
(iv) *$Im\phi \leq H$.*

Proof. See [14]. □

Proposition 1.16. *Let $\phi : (G, \alpha) \to (H, \beta)$ be a Hom-group homomorphism.*

(a) *If $A \leq G$ and $B \leq H$, then,*
(i) *$\phi(A) \leq H$;*
(ii) *$\phi^{-1}(B) \leq G$.*
(b) *(i) ϕ is a monomorphism if and only if $Ker\phi = \{1_G\}$;*
(ii) *ϕ is an epimorphism if and only if $Im\phi = H$;*
(iii) *ϕ is an isomorphism if and only if $Ker\phi = \{1_G\}$ and $Im\phi = H$.*

Proof. See [14]. □

Proposition 1.17. *[14] (First Isomorphism Theorem) Let $\phi : (G, \alpha) \to (H, \beta)$ be a Hom-group homomorphism. Then, $G/Ker\phi \cong Im\phi$. If ϕ is a Hom-group epimorphism, then, $G/Ker\phi \cong H$.*

Proof. See [14]. □

Lemma 1.18. *[14] If G is a Hom-group such that $H \leq G$ and $N \lhd G$, then, $HN = NH \leq G$.*

Proof. See [14]. □

Proposition 1.19. *[14] (Second Isomorphism Theorem) Let G be a Hom-group such that $H \leq G$ and $N \lhd G$. Then,*

(i) *$H \cap N \lhd H$;*
(ii) *$N \lhd NH$;*
(iii) *$H/(H \cap N) \cong NH/N$.*

Proof. See [14]. □

Proposition 1.20. *[14] (Third Isomorphism Theorem) Let (G, α) be a Hom-group such that $M \lhd G$, $N \lhd G$ and $N \leq M$. Then,*

(i) *$(M/N, \mu) \lhd (G/N, \mu)$;*
(ii) *$(G/N)/(M/N) \cong G/M$.*

Proof. See [14]. □

Proposition 1.21. *Let (G, α) be a Hom-group and let $H, K \leq G$. If H and K satisfy the following three conditions:*

(i) *$G = HK$;*
(ii) *$H \cap K = \{1\}$;*
(iii) *$\forall h \in H$, $k \in K$, we have $hk = kh$,*

then $G \cong H \times K$.

Proof. See [14]. □

2. Main Results

In this section, we will study and give more fundamental properties of Hom-groups and provide examples.

Proposition 2.1. *Let $(G, *, \alpha)$ be a Hom-group. Then, the unital element $1 \in G$ is the only idempotent element in G.*

Proof. By definition, $1*1 = \alpha(1) = 1$ which shows that $1 \in G$ is an idempotent element. Suppose that $1 \neq u \in G$ is also another idempotent element in G. Then, $u^2 = u$ which implies that $u(u-1) = 0$. Since u cannot be 0 or 1, it follows that such a u does not exist in G and therefore, 1 is the only idempotent element in G. □

Lemma 2.2. *Let (G, α) be a Hom-group. Then for any $g \in G$:*
(i) $(\alpha^{-1}(g))^{-1} = \alpha^{-1}(g^{-1})$;
(i) $(\alpha^{-1}(g^{-1}))^{-1} = \alpha^{-1}(g)$;
(iii) $(\alpha(g)\alpha^{-1}(g^{-1}))^{-1} = \alpha^{-1}(g)\alpha(g^{-1}) = \alpha^{-1}(g)\alpha^2(g^{-1}) = g\alpha(g^{-1})$.

Proof. (i) It is sufficient to show that the inverse of $\alpha^{-1}(g)$ is $\alpha^{-1}(g^{-1})$. To this end,

$$\begin{aligned} \alpha^{-1}(g)\alpha^{-1}(g^{-1}) &= \alpha^{-1}(gg^{-1}) = 1 \text{ and also,} \\ \alpha^{-1}(g^{-1})\alpha^{-1}(g) &= \alpha^{-1}(g^{-1}g) = 1. \end{aligned}$$

The required result follows from the uniqueness of the inverse element in G.
(ii) It suffices to show that the inverse of $\alpha^{-1}(g^{-1})$ is $\alpha^{-1}(g)$. To this end,

$$\begin{aligned} \alpha^{-1}(g^{-1})\alpha^{-1}(g) &= \alpha^{-1}(g^{-1}g) = 1 \text{ and also,} \\ \alpha^{-1}(g)\alpha^{-1}(g^{-1}) &= \alpha^{-1}(gg^{-1}) = 1. \end{aligned}$$

The required result follows from the uniqueness of the inverse element in G.
(iii)

$$\begin{aligned} (\alpha(g)\alpha^{-1}(g^{-1}))^{-1} &= (\alpha^{-1}(g^{-1}))^{-1}(\alpha(g))^{-1} \\ &= \alpha^{-1}(g)\alpha(g^{-1}). \quad (1) \\ &= \alpha(\alpha^{-1}(gg^{-1})[\alpha^{-1}(g)\alpha(g^{-1})] \\ &= (\alpha^{-1}(gg^{-1})\alpha^{-1}(g))\alpha^2(g^{-1}) \\ &= \alpha^{-1}(g)\alpha^2(g^{-1}). \quad (2) \end{aligned}$$

From (1), we have

$$\begin{aligned} (\alpha(g)\alpha^{-1}(g^{-1}))^{-1} &= \alpha^{-1}(g)\alpha(g^{-1}) \\ &= (\alpha^{-1}(g)\alpha(g^{-1}))\alpha(\alpha^{-1}(gg^{-1}) \\ &= \alpha(\alpha^{-1}(g))(\alpha(g^{-1})\alpha^{-1}(gg^{-1})) \\ &= g\alpha(g^{-1}). \quad (3) \end{aligned}$$

The required results follow from (1), (2) and (3). □

Proposition 2.3. *Let (G, α) be a Hom-group and let $g, h \in G$. Then, G is an abelian Hom-group if and only if $(gh)^2 = g^2h^2$.*

Proof. Suppose that $(gh)^2 = g^2h^2$. Then, $(gh)(gh) = (gg)(hh)$ and so,

$$\begin{aligned}(gh)\alpha\alpha^{-1}(gh) &= (gg)\alpha\alpha^{-1}(hh)\\ \Rightarrow \alpha(g)(h\alpha^{-1}(gh)) &= \alpha(g)(g\alpha^{-1}(hh)).\end{aligned}$$

By premultiplying both sides by $\alpha^2(g^{-1})$, we have

$$\begin{aligned}\alpha^2(h\alpha^{-1}(gh) &= \alpha^2(g\alpha^{-1}(hh))\\ \Rightarrow (\alpha(h)\alpha(g))\alpha^2(h) &= (\alpha(g)\alpha(h))\alpha^2(h).\end{aligned}$$

By postmultiplying both sides by $\alpha^3(h^{-1})$, we have

$$\begin{aligned}\alpha^3(hg) &= \alpha^3(gh))\\ hg &= gh\end{aligned}$$

and therefore, G is abelian.

Conversely, suppose that G is abelian. Then,

$$\begin{aligned}(gh)^2 &= (gh)(gh) = (gh)\alpha\alpha^{-1}(gh) = \alpha(g)(h\alpha^{-1}(gh))\\ &= \alpha(g)(\alpha\alpha^{-1}(h)(\alpha^{-1}(g)\alpha^{-1}(h)) = \alpha(g)((\alpha^{-1}(h)(\alpha^{-1}(g))h)\\ &= \alpha(g)((\alpha^{-1}(hg)h) = \alpha(g)((\alpha^{-1}(gh)\alpha\alpha^{-1}(h))\\ &= \alpha(g)(g(\alpha^{-1}(h)\alpha^{-1}(h))) = \alpha(g)(g\alpha^{-1}(h^2))\\ &= g^2h^2.\end{aligned}$$

□

Corollary 2.4. *Let (G, α) be a Hom-group such that $g^2 = 1\ \forall g \in G$. Then G is an abelian Hom-group. More generally, if G is a Boolean Hom-group, then G is an abelian Hom-group.*

Proposition 2.5. *Let (G, α) be a Hom-group and let $g, h \in G$. G is an abelian Hom-group if and only if $(gh)^{-1} = g^{-1}h^{-1}$.*

Proof. Suppose that $(gh)^{-1} = g^{-1}h^{-1}$. Then, $h^{-1}g^{-1} = g^{-1}h^{-1}$ so that

$$\alpha^2(h^{-1})\alpha^2(g^{-1}) = \alpha^2(g^{-1})\alpha^2(h^{-1}).$$

After some manipulations, we have

$$\begin{aligned}\alpha^6(hg) &= \alpha^6(gh)\\ hg &= gh\end{aligned}$$

and therefore, G is abelian.

Conversely, suppose that G is abelian. Then, $(gh)^{-1} = (hg)^{-1} = g^{-1}h^{-1}$. □

Proposition 2.6. *Let (G, α) be a Hom-group and let g, h, k be any elements of G. If $gk = \alpha(hk)$, then $g = h$.*

Proof. Suppose that $gk = \alpha(hk)$. Then, $gk = 1(gk) = \alpha(gk) = \alpha(g)\alpha(k) = \alpha(hk) = \alpha(h)\alpha(k)$ which implies that $\alpha(g)\alpha(k) = \alpha(h)\alpha(k)$ so that $\alpha^2(g)1 = \alpha^2(h)1$ from which we obtain $\alpha^3(g) = \alpha^3(h)$ and therefore, $g = h$. □

Example 2.7. *Let $(G, *)$ be a Klein 4-group where $G = \{1, a, b, c\}$ and let $\alpha : G \to G$ be a group automorphism defined by $\alpha(1) = 1, \alpha(a) = c, \alpha(b) = a, \alpha(c) = b$. Then, by Proposition 1.5, $(G, \alpha \circ *, \alpha)$ is a Hom-group shown in the Cayley table below.*

$\alpha \circ *$	1	a	b	c
1	1	c	a	b
a	c	1	b	a
b	a	b	1	c
c	b	a	c	1

It is observed that $(G, \alpha \circ *, \alpha)$ has no nontrivial Hom-subgroup.

Example 2.8. *Let $(G, *)$ be the group of integers modulo 8 that is $G = \mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let $\alpha : G \to G$ be a group automorphism defined by $\alpha(0) = 0, \alpha(1) = 7, \alpha(2) = 6, \alpha(3) = 5, \alpha(4) = 4, \alpha(5) = 3, \alpha(6) = 2, \alpha(7) = 1$. Then, by Proposition 1.5, $(G, \alpha \circ *, \alpha)$ is a Hom-group shown in the Cayley table below.*

$\alpha \circ *$	0	1	2	3	4	5	6	7
0	0	7	6	5	4	3	2	1
1	7	6	5	4	3	2	1	0
2	6	5	4	3	2	1	0	7
3	5	4	3	2	1	0	7	6
4	4	3	2	1	0	7	6	5
5	3	2	1	0	7	6	5	4
6	2	1	0	7	6	5	4	3
7	1	0	7	6	5	4	3	2

It can easily be checked that $(G, \alpha \circ *, \alpha)$ is non-cyclic, and it has two nontrivial Hom-subgroups $H = \{0, 4\}$ and $K = \{0, 2, 4, 6\}$. Since $4 \times 0 = 0, 4 \times 1 = 4$ and $4 \times 2 = 4 + 4 = 0$, it follows that H is a cyclic Hom-subgroup of order 2 generated by 4, that is $H =< 4 >$. For K, we have $2 \times 0 = 0, 2 \times 1 = 2, 2 \times 2 = 2 + 2 = 4$; $4 \times 0 = 0, 4 \times 1 = 4, 4 \times 2 = 4 + 4 = 0$; and $6 \times 0 = 0, 6 \times 1 = 6, 6 \times 2 = 6 + 6 = 4$; it follows that K is not cyclic, that is $K \neq < 2 >, < 4 >, < 6 >$. Lastly, since $o(H) = 2$, $o(K) = 4$, and the order of the Hom-group $(G, \alpha \circ *, \alpha)$ is 8, it follows that both H and K satisfy Lagrange's theorem.

Proposition 2.9. *Let H be a subset of a Hom-group $(G, *, \alpha)$.*

(a) *H is a Hom-subgroup of G if the following conditions hold:*
 (i) *$H \neq \emptyset$;*
 (ii) *for any $g, h \in H$, then, $gh \in H$;*
 (iii) *for all $h \in H$, then, $h^{-1} \in H$.*

(b) *H is a Hom-subgroup of G if the following conditions hold:*

(i) $H \neq \emptyset$;
(ii) *for any* $g, h \in H$*, then,* $gh^{-1} \in H$.

Proof. Easy. □

Proposition 2.10. *Let* $(G, *, \alpha)$ *be an abelian Hom-group and let* H *be a subset of* G *defined by* $H = \{g \in G : g = g^{-1}\}$*. Then,* H *is a Hom-subgroup of* G *for all* $g \in G$.

Proof. Suppose that G is abelian. It is clear that $H \neq \emptyset$ since $1 = 1^{-1} \in H$. Let $g, h \in H$ be arbitrary. Then $g = g^{-1}$ and $h = h^{-1}$ so that $gh^{-1} = g^{-1}h = g^{-1}(h^{-1})^{-1} = (gh^{-1})^{-1} \in H$. According to Proposition 2.9, H is a Hom-subgroup of G. □

Example 2.11. *Let* $(G, *)$ *be a group where* $G = \{\pm 1, \pm i\}$, $i^2 = -1$ *and let* $\alpha : G \to G$ *be a group automorphism defined by* $\alpha(1) = 1, \alpha(-1) = -i, \alpha(i) = i, \alpha(-i) = -1$*. Then, by Proposition 1.5,* $(G, \alpha \circ *, \alpha)$ *is a Hom-group shown in the Cayley table below.*

$\alpha \circ *$	1	-1	i	$-i$
1	1	$-i$	i	-1
-1	$-i$	i	-1	1
i	i	-1	1	$-i$
$-i$	-1	1	$-i$	i

It is observed that $(G, \alpha\circ *, \alpha)$ *has only one nontrivial Hom-subgroup* $H = \{1, i\}$ *which is cyclic generated by* i*, that is* $H =< i >$.

Proposition 2.12. *Let* A *and* B *be Hom-normal subgroups of Hom-groups* (G, α) *and* (H, β)*, respectively. Then,* $A \times B \lhd G \times H$.

Proof. Suppose that $A \lhd G$ and $B \lhd H$. Then clearly, $A \times B \neq \emptyset$ since $(1_G, 1_H) \in A \times B$, and we have, $(ga)\alpha(g^{-1}) \in A$ and $(hb)\beta(h^{-1}) \in H$ for $(g, h) \in G \times H$ and $(a, b) \in A \times B$. Now, let $(a, b), (c, d) \in A \times B$ be arbitrary. Then, $(a, b)(c, d)^{-1} = (a, b)(c^{-1}, d^{-1}) = (ac^{-1}, bd^{-1}) \in A \times B$. This shows that $A \times B \leq G \times H$. For Hom-normality, consider $((g, h)(a, b))(\alpha(g^{-1}), \beta(h^{-1})) = (ga, hb)(\alpha(g^{-1}), \beta(h^{-1})) = ((ga)\alpha(g^{-1}), (hb)\beta(h^{-1})) \in A \times B$. Accordingly, $A \times B \lhd G \times H$. □

Definition 2.13. Let A be a subset of the Hom-group (G, α). $C(A)$, the centralizer of A in G, is defined by $C(A) = \{g \in G : ag = ga \ \forall a \in A\}$.

Proposition 2.14. (i) $C(A) \leq G$.
(ii) *If* A *is an abelian Hom-subgroup of* G*, then* $A \lhd C(A)$.

Proof. (i) $C(A) \neq \emptyset$ since $1 \in C(A)$. Next, suppose that $g, h \in C(A)$ are arbitrary. Then, $ga = ag$ and $ha = ah$ so that $h^{-1}a^{-1} = a^{-1}h^{-1}$. Now, $(gh^{-1})a = (gh^{-1})\alpha(\alpha^{-1}(a)) = \alpha(g)(h^{-1}\alpha^{-1}(a)) = \alpha(g)(\alpha^{-1}(a)h^{-1})) = (g\alpha^{-1}(a))\alpha(h^{-1}) = (\alpha^{-1}(a)g)\alpha(h^{-1}) = a(gh^{-1})$. Hence, $gh^{-1} \in C(A)$ and $C(A) \leq G$.

(ii) Suppose that A is an abelian Hom-subgroup of G. Then, $a \in A$ implies that $a \in C(A)$ $\forall a \in A$. Suppose that $g \in C(A)$. Then, $ga = ag$ and $(ga)\alpha(g^{-1}) = (ag)\alpha(g^{-1}) = \alpha(a)(gg^{-1}) = \alpha(a)1 = \alpha^2(a) \in A$. Accordingly, $A \lhd C(A)$. □

Definition 2.15. Let A be a subset of the Hom-group (G, α). $N(A)$, the normalizer of A in G, is defined by $N(A) = \{g \in G : gA = Ag\}$.

Proposition 2.16. (i) *$N(A) \leq G$.*
(ii) *If A is a Hom-subgroup of G, then $A \lhd G$ if and only if $N(A) = G$.*

Proof. (i) $N(A) \neq \emptyset$ since $1 \in N(A)$. Next, suppose that $g, h \in N(A)$ are arbitrary. Then $gA = Ag$ and $hA = Ah$ so that $h^{-1}A = Ah^{-1}$. Now, for $a \in A$, $(gh^{-1})a = (gh^{-1})\alpha(\alpha^{-1}(a)) = \alpha(g)(h^{-1}\alpha^{-1}(a)) = \alpha(g)(\alpha^{-1}(a)h^{-1}) = (g\alpha^{-1}(a))\alpha(h^{-1}) = (\alpha^{-1}(a)g)\alpha(h^{-1}) = a(gh^{-1})$. Hence, $(gh^{-1})A = A(gh^{-1})$, $gh^{-1} \in N(A)$ and $N(A) \leq G$.
(ii) The proof is the same as the case of the classical group and so omitted. □

Lemma 2.17. *Let (G, α) be a Hom-group and let $H, K \lhd G$. Then, $HK \lhd G$.*

Proof. Suppose that $H, K \lhd G$. Then, for any $g \in G$, there exist $h \in H$, $k \in K$ such that $(gh)\alpha(g^{-1}) \in H$ and $(gk)\alpha(g^{-1}) \in K$. Again, there exist $x \in H, y \in K$ such that $x = (gh)\alpha(g^{-1})$ and $y = (gk)\alpha(g^{-1})$. Now, $xy = ((gh)\alpha(g^{-1}))((gk)\alpha(g^{-1})) = (\alpha(g)(hg^{-1}))(\alpha(g)(kg^{-1})) \in HK$, and, $(g(xy))\alpha(g^{-1}) = \alpha(g)((\alpha(g)(hg^{-1}))(\alpha(g)(kg^{-1}))g^{-1}) \in HK$. Hence, $HK \lhd G$. □

Example 2.18. (i) *Let $(G, \alpha \circ *, \alpha)$ be the Hom-group of Example 2.11 and let $H = \{1, i\}$ be its Hom-subgroup. It can easily be checked that $H \lhd G$. Distinct Hom-left cosets of H in G are H and $-iH$ which form the partitions of G and $G/H = \{H, -iH\}$.*
(ii) *Let $(G, \alpha \circ *, \alpha)$ be the Hom-group of Example 2.8 and let $H = \{0, 4\}$ and $K = \{0, 2, 4, 6\}$ be its Hom-subgroups. Distinct Hom-left cosets of H in G are H, $1 + H$, $2 + H$ and $3 + H$ which are the partitions of G and $G/H = \{H, 1 + H, 2 + H, 3 + H\}$. Also, distinct Hom-left cosets of K in G are K and $1 + K$ which also form the partitions of G and therefore, $G/K = \{K, 1 + K\}$.*
(iii) *Let G/H be as defined in Example 2.18 (i) and consider the Cayley table below.*

$\odot$	H	$-iH$
H	H	$-iH$
$-iH$	$-iH$	H

It can easily be checked that $(G/H, \odot, \beta)$ is a Hom-group.

(iv) *Let G/H and G/K be as defined in Example 2.18 (ii) and consider the Cayley tables below.*

$\oplus$	H	$1+H$	$2+H$	$3+H$
H	H	$3+H$	$2+H$	$1+H$
$1+H$	$3+H$	$2+H$	$1+H$	H
$2+H$	$2+H$	$1+H$	H	$3+H$
$3+H$	$1+H$	H	$3+H$	$2+H$

,

$\oplus$	K	$1+K$
K	K	$1+K$
$1+K$	$1+K$	K

.

It can easily be checked that $(G/H, \oplus, \beta)$ and $(G/K, \oplus, \beta)$ are Hom-groups.

Proposition 2.19. *Let A and B be Hom-normal subgroups of the Hom-groups $(G, *, \alpha)$ and $(H, \star, \beta)$, respectively. Let $(G/A, \odot, \lambda)$ and $(H/B, \otimes, \mu)$ be Hom-quotient groups respectively factored by A and B. Let $(G/A) \times (H/B) = \{(gA, hB) : g \in G, h \in H\}$. Then, $(G/A) \times (H/B), \boxdot, \gamma)$ is a Hom-group.*

Proof. To prove Hom-associativity, let $(gA, hB), (pA, qB), (xA, yB) \in (G/A) \times (H/B)$ be arbitrary. Then,

$$\begin{aligned}
\gamma((gA, hB)) \boxdot ((pA, qB) \boxdot (xA, yB)) &= (\alpha(g)A, \beta(h)B) \boxdot (pxA, qyB) \\
&= (\alpha(g)(px)A, \beta(h)(qy)B) \\
&= ((gp)\alpha(x)A, (hq)\beta(y)B) \\
&= ((gA, hB) \boxdot (pA, qB)) \boxdot (\alpha(x)A, \beta(y)B) \\
&= ((gA, hB) \boxdot (pA, qB)) \boxdot \gamma(xA, yB).
\end{aligned}$$

To establish Hom-multiplication, let $(gA, hB), (pA, qB) \in (G/A) \times (H/B)$ be arbitrary. Then,

$$\begin{aligned}
\gamma(((gA, hB) \boxdot (pA, qB)) &= \gamma((gpA, hqB)) = (\alpha(gp)A, \beta(hq)B) \\
&= (\alpha(g)A, \beta(h)B) \boxdot (\alpha(p)A, \beta(q)B) \\
&= \gamma((gA, hB))\gamma((pA, qB)).
\end{aligned}$$

For Hom-neutrality, let $(gA, hB) \in (G/A) \times (H/B)$ be arbitrary. Then,

$$\begin{aligned}
(1, 1) \boxdot (gA, hB) &= (1gA, 1hB) = (g1A, h1B) = (\alpha(g)A, \beta(h)B) \\
&= \gamma((gA, hB)).
\end{aligned}$$

Lastly for Hom-invertibility, we have for all $(gA, hB) \in (G/A) \times (H/B)$,

$$\begin{aligned}
(gA, hB) \boxdot (g^{-1}A, h^{-1}B) &= (gg^{-1}A, hh^{-1}B) = (1A, 1B) = (A, B), \\
(g^{-1}A, h^{-1}B) \boxdot (gA, hB) &= (g^{-1}gA, h^{-1}hB) = (1A, 1B) = (A, B).
\end{aligned}$$

These show that for all $(gA, hB) \in (G/A) \times (H/B)$, $(g^{-1}A, h^{-1}B)$ is its inverse. Accordingly, $(G/A \times H/B, \boxdot, \gamma)$ is a Hom-group. □

Lemma 2.20. *Let $\phi : (G, \alpha) \to (H, \beta)$ be a Hom-group homomorphism and let $A \lhd G$ and $B \lhd H$. Then,*

(i) $\phi(A) \lhd H$;
(ii) $\phi^{-1}(B) \lhd G$.

Proof. Suppose that $\phi : (G, \alpha) \to (H, \beta)$ is a Hom-group homomorphism.
(i) Given that $A \lhd G$, it follows from Proposition 1.16 (a) (i) that $A \leq G$. For Hom-normality, let $\phi(a) \in \phi(A)$ and $h \in H$. Then, there exists $g \in G$ such that $h = \phi(g)$. Since $A \lhd G$, then,

$$\begin{aligned} (ga)\alpha(g^{-1}) &\in A \\ \Rightarrow (\phi(g)\phi(a))\phi(\alpha(g^{-1})) &\in \phi(A). \end{aligned}$$

Now,

$$\begin{aligned} (h\phi(a))\beta(h^{-1}) &= (\phi(g)\phi(a))\beta((\phi(g))^{-1}) = (\phi(g)\phi(a))\beta(\phi(g^{-1})) \\ &= (\phi(g)\phi(a))\phi(\alpha(g^{-1})) \in \phi(A). \end{aligned}$$

Hence, $\phi(A) \lhd H$.
(ii) Given that $B \lhd H$, it follows from Proposition 1.16 (a) (ii) that $B \leq H$. For Hom-normality, let $b \in B$ and $h \in H$. Then,

$$\begin{aligned} (hb)\beta(h^{-1}) &\in B \\ \Rightarrow (\phi^{-1}(h)\phi^{-1}(b))\phi^{-1}(\beta(h^{-1})) &\in \phi^{-1}(B). \end{aligned}$$

Now, for some $x = \phi^{-1}(h)$ and $y = \phi^{-1}(b) \in G$, we have

$$\begin{aligned} (xy)\phi^{-1}(\beta(\phi(x^{-1})) &\in \phi^{-1}(B) \\ \Rightarrow (xy)\alpha(x^{-1}) &\in \phi^{-1}(B). \end{aligned}$$

Hence, $\phi^{-1}(B) \lhd G$. □

Proposition 2.21. *Let $\phi : (G, \alpha) \to (H, \beta)$ be a Hom-group isomorphism and let N be a Hom-normal subgroup of G. Then, ϕ induces a Hom-group isomorphism between G/N and $H/\phi(N)$.*

Proof. Suppose that $\phi : (G, \alpha) \to (H, \beta)$ is a Hom-group isomorphism and $N \lhd G$. Then by Lemma 2.20 (i), $\phi(N) \lhd H$. Let $\psi : (G/N, \mu) \to (H/\phi(N), \nu)$ be a mapping defined by $\psi(xN) = \phi(x)\phi(N)$ for every $x \in G$. ψ is clearly well-defined since $N \lhd G$ and $\phi(N) \lhd H$. For Hom-homomorphism, let $xN, yN \in G/N$ be arbitrary. Then,

$$\begin{aligned} \psi(xNyN) &= \psi(xyN) = \phi(xy)\phi(N) = \phi(x)\phi(y)\phi(N) \\ &= (\phi(x)\phi(N))(\phi(y)\phi(N)) = \psi(xN)\psi(yN). \end{aligned}$$

Also,

$$\begin{aligned} \nu(\psi(xN)) &= \nu(\phi(x)\phi(N)) = \beta(\phi(x))\phi(N) = \phi(\alpha(x))\phi(N) = \mu(xN)\phi(N) \\ &= \psi(\mu(xN)). \end{aligned}$$

Accordingly, ψ is a Hom-group homomorphism. ψ is obviously onto. For 1-1,

$$\begin{aligned} \text{Ker}\psi &= \{xN \in G/N : \psi(xN) = 1_{H/\phi(N)}\} = \{xN \in G/N : \psi(xN) = \phi(N)\} \\ &= \{xN \in G/N : \phi(x) \in \phi(N)\} = \{xN \in G/N : x \in N\} \\ &= \{N\}. \end{aligned}$$

Now, ψ is 1-1 and therefore, $G/N \cong H/\phi(N)$. □

Proposition 2.22. *Let (G, α) and (H, β) be two Hom-groups. Then, $G \times H \cong H \times G$.*

Proof. Let $\phi : (G \times H, \mu) \to (H \times G, \nu)$ be a mapping defined by $\phi((g, h)) = (h, g)$ for any $(g, h) \in G \times H$. For Hom-homomorphism, let $(g, h), (u, v) \in G \times H$ be arbitrary. Then,

$$\begin{aligned} \phi((g, h)(u, v)) &= \phi((gu, hv)) = (hv, gu) = (h, g)(v, u) \\ &= \phi((g, h))\phi((u, v)). \end{aligned}$$

Also,

$$\begin{aligned} \nu(\phi((g, h))) &= \nu((h, g) = (h, g) = \phi((g, h)) \\ &= \phi(\mu((g, h))). \end{aligned}$$

The mapping ϕ is obviously onto. For 1-1,

$$\begin{aligned} \text{Ker}\phi &= \{(g, h) \in G \times H : \phi((g, h)) = 1_{H \times G}\} \\ &= \{(g, h) \in G \times H : \phi((g, h)) = (1_H, 1_G)\} \\ &= \{(g, h) \in G \times H : (h, g)) = (1_H, 1_G)\} \\ &= \{(1_G, 1_H)\}. \end{aligned}$$

Therefore, ϕ is 1-1 and, $G \times H \cong H \times G$. □

Proposition 2.23. *Let (G, α) and (H, β) be two Hom-groups and let $A \lhd G$ and $B \lhd H$. Then, $(G \times H)/(A \times B) \cong (G/A) \times (H/B)$.*

Proof. By Proposition 2.12, $A \times B \lhd G \times H$ and therefore, $(G \times H)/(A \times B)$ is a Hom-quotient group. Let $\phi : (G \times H, \mu) \to ((G/A) \times (H/B), \nu)$ be a mapping defined by $\phi((g, h)) = (gA, hB)\ \forall (g, h) \in G \times H$. Clearly, ϕ is well-defined. For Hom-homomorphism, let $(g, h), (x, y) \in G \times H$ be arbitrary. Hence,

$$\begin{aligned} \phi((g, h)(x, y)) &= \phi((gx, hy)) = (gxA, hyB) = (gA, hB)(xA, yB) \\ &= \phi((g, h))\phi((x, y)). \end{aligned}$$

Also,

$$\begin{aligned} \nu(\phi((g, h))) &= \nu((gA, hB)) = ((\alpha(g)A, \beta(h)B)) = \phi((\alpha(g), \beta(h)) \\ &= \phi((\mu(g, h)). \end{aligned}$$

Accordingly, ϕ is a Hom-group homomorphism. Clearly, ϕ is onto and therefore, ϕ is an epimorphism. Now,

$$\begin{aligned}\text{Ker}\phi &= \{(g,h) \in G \times H : \phi((g,h)) = 1_{G/A)\times(H/B)}\} \\ &= \{(g,h) \in G \times H : (gA, hB) = (A, B)\} \\ &= \{(g,h) \in G \times H : g \in A \text{ and } h \in B\} = A \times B.\end{aligned}$$

By invoking the First Isomorphism Theorem, we have $(G \times H)/(A \times B) \cong (G/A) \times (H/B)$. □

Proposition 2.24. *[20](Zassenhaus Butterfly Lemma) Let (G, α) be a Hom-group and let $H, K \leq G$. If $A \lhd H$ and $B \lhd K$, then,*

(i) $A \cap K \lhd H \cap K$;
(ii) $H \cap B \lhd H \cap K$;
(iii) $(A \cap K)B \lhd (H \cap K)B$;
(iv) $A(H \cap B) \lhd A(H \cap K)$;
(v) $A(H\cap K)/A(H\cap B) \cong (H\cap K)/(A\cap K)(H\cap B) \cong (H\cap K)B/(A\cap K)B$.

Proof. Suppose that $H, K \leq G$ and suppose that $A \lhd H$, $B \lhd K$.
(i) From Proposition 1.7 (b) (i), we have $H \cap K \leq G$ which is indeed a Hom-group. Since $A\cap K \subseteq H\cap K$, it is clear that $A\cap K \leq H\cap K$. For Hom-normality, let $x \in A \cap K$ and $y \in H \cap K$. Then, $x \in A$, $x \in K$, $y \in H$ and $y \in K$ from which we obtain $x \in A$, $y \in H$, $x \in B$ and $y \in K$. Since $A \lhd H$ and $B \lhd K$, it follows that $(yx)\alpha(y^{-1}) \in A$ and $(yx)\alpha(y^{-1}) \in K$ from which we obtain $(yx)\alpha(y^{-1}) \in A \cap K$ and therefore, $A \cap K \lhd H \cap K$.
(ii) Follows the same arguments as in (i).
(iii) Since $B \lhd K$, it follows from Lemma 1.11 that $(H \cap K)B$ is a Hom-group. To show that $(A \cap K)B$ is also a Hom-group, it suffices to show that $A \cap K$ is a Hom-subgroup of the normalizer of A which is H. To see this,

$$\begin{aligned}N(A) &= \{x \in H \cap K : xA = Ax\} = \{x \in H \cap K : A = xAx^{-1}\} \\ &= \{x \in H \cap K : x \in A \text{ since } A \lhd H\} = H.\end{aligned}$$

From the same Lemma 1.11, $(A \cap K)B$ is a Hom-group which is contained in $(H \cap K)B$ as a Hom-subgroup. For Hom-normality, this follows from Lemma 2.17 since from (i), $A \cap K \lhd H \cap K$ and we know that $B \lhd K$.
(iv) Follows the same arguments as in (iii).
(v) Let $\phi : A(H \cap K) \to (H \cap K)/(A \cap K)(H \cap B)$ be a mapping defined by

$$\phi(ah) = h(A \cap K)(H \cap B), \text{ for all } a \in A, h \in H \cap K.$$

We first show that ϕ is well-defined. Suppose that $a = b$ and $h = k$ that is $ah = bk$ where $a, b \in A$ and $h, k \in H\cap K$. We need to show that $\phi(ah) = \phi(bk)$.

To this end,

$$\begin{aligned}(a^{-1}b)^{-1}(hk^{-1}) &= (b^{-1}a)(hk^{-1}) = \alpha\alpha^{-1}((b^{-1}a))(hk^{-1})\\ &= (\alpha^{-1}((b^{-1}a))h)\alpha(k^{-1}) = [(\alpha^{-1}(b^{-1})\alpha^{-1}(a))\alpha\alpha^{-1}(h)]\alpha(k^{-1})\\ &= ((b^{-1}(\alpha^{-1}(a)\alpha^{-1}(h)))\alpha(k^{-1}) = ((b^{-1}(\alpha^{-1}(ah)))\alpha(k^{-1})\\ &= ((b^{-1}(\alpha^{-1}(bk)))\alpha(k^{-1}) = ((b^{-1}(\alpha^{-1}(b)\alpha^{-1}(k)))\alpha(k^{-1})\\ &\in A\cap K = A\cap(H\cap K)\subseteq (A\cap K)(H\cap B).\end{aligned}$$

Hence, $\phi(ah) = \phi(bk)$, and ϕ is well-defined.

For Hom-homomorphism, we first note that $(G, *, \alpha)$, $(H\cap K, *, \alpha)$, $(A(H\cap K), *, \alpha)$ and $((H\cap K)/(A\cap K)(H\cap B), \circledast, \beta)$ are Hom-groups. Let $a, b \in A$ and $h, k \in H\cap K$. We need to show that $\phi((ah)(bk)) = \phi(ah)\phi(bk)$ and $\beta(\phi(ah)) = \phi(\alpha(ah))$. Since $A \lhd H$, $(hb)\alpha(h^{-1}) \in A$ and we can take $(hb)\alpha(h^{-1}) = c = \alpha(h)(bh^{-1})$ for some $c \in A$ from which we obtain $b = (h^{-1}c)\alpha(h)$. Now,

$$\begin{aligned}(ah)(bk) &= (ah)\alpha\alpha^{-1}(bk) = \alpha(a)(h\alpha^{-1}(bk))\\ &= \alpha(a)(h(\alpha^{-1}(b)\alpha^{-1}(k))) = \alpha(a)(\alpha\alpha^{-1}(h)(\alpha^{-1}(b)\alpha^{-1}(k)))\\ &= \alpha(a)((\alpha^{-1}(h)\alpha^{-1}(b))k) = \alpha(a)((\alpha^{-1}(h)\alpha^{-1}((h^{-1}c)\alpha(h)))k)\\ &= \alpha(a)((\alpha^{-1}(hh^{-1})\alpha^{-1}(c))(hk)) = \alpha(a)(1\alpha^{-1}(c))(hk))\\ &= (\alpha(a)c)(hk).\end{aligned}$$

Therefore,

$$\begin{aligned}\phi((ah)(bk)) &= \phi((\alpha(a)c)(hk)) = hk(A\cap K)(H\cap B)\\ &= (h(A\cap K)(H\cap B))(k(A\cap K)(H\cap B)) = \phi(ah)\phi(bk).\end{aligned}$$

Next,

$$\begin{aligned}\beta(\phi(ah)) &= \beta(h(A\cap K)(H\cap B)) = \alpha(h)(A\cap K)(H\cap B))\\ &= \phi(\alpha(a)\alpha(h)) = \phi(\alpha(ah)).\end{aligned}$$

We have just shown that ϕ is a Hom-group homomorphism. We next show that ϕ is a Hom-group epimorphism. To this end, for all $h \in H$, $\phi(1h) = h(A\cap K)(H\cap B)$. This shows that every element of $(H\cap K)/(A\cap K)(H\cap B)$ under ϕ has a preimage in $H\cap K$. Hence, ϕ is a Hom-group epimorphism. Lastly,

$$\begin{aligned}\mathrm{Ker}\phi &= \{ah \in A(H\cap K) : \phi(ah) = 1_{(H\cap K)/(A\cap K)(H\cap B)} = (A\cap K)(H\cap B)\}\\ &= \{ah \in A(H\cap K) : h(A\cap K)(H\cap B) = (A\cap K)(H\cap B)\}\\ &= A(H\cap B).\end{aligned}$$

By invoking the First Isomorphism Theorem, $A(H\cap K)/A(H\cap B) \cong (H\cap K)/(A\cap K)(H\cap B)$. Using the same argument, it can be shown that $A(H\cap K)/A(H\cap B) \cong (H\cap K)B/(A\cap K)B$. Hence, $A(H\cap K)/A(H\cap B) \cong (H\cap K)/(A\cap K)(H\cap B) \cong (H\cap K)B/(A\cap K)B$. □

3. Conclusion

In this paper, we have studied and obtained more properties of Hom-groups. Specifically, we presented more properties of Hom-groups, Hom-subgroups, Hom-normal subgroups, Hom-quotient groups and Hom-group homomorphisms with examples. As a major result of this paper, we have proved the Zassenhaus Butterfly Lemma of Hom-groups.

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