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THE COALESCENCE OF MULTI-WHEEL AND STARLIKE GRAPHS IS DLS

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Abstract. The Laplacian spectrum of a graph is obtained by taking the difference of the adjacency spectrum from the diagonal matrix of degrees. If a graph has a unique Laplacian spectrum, it means that it can be identified by this spectrum, it is called *DLS*. In this article, we first introduce the graph resulting from the integration of a starlike tree and a multi-wheel graph at the vertices with the maximum degree of these two graphs. Then, we check whether it is *DLS*.

Keywords: Laplacian spectrum, *DLS* graph, Coalescence of graphs.
2020 MSC: Primary 05C50.

1. Introduction

Suppose that G is a simple graph with n vertex, m edge, and the set of vertex $\{v_1, v_2, \ldots, v_n\}$. The square matrix $A = [a_{ij}]$ whose a_{ij} equals the number of edges between two vertices v_i and v_j is called the adjacency matrix of graph G. The polynomial $\chi(G, \lambda) = |\lambda I - A|$, where I is the identity matrix of order n, is known as the characteristic polynomial of the graph. The roots of this polynomial, denoted as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, are referred to as the adjacency spectrum of the graph.

Consider a graph G with a degree sequence given by $d_1, d_2, \ldots, d_n$. The diagonal matrix of the degrees of the graph G is $D = Diag[d_1, d_2, \ldots, d_n]$. The Laplacian matrix L of graph G is obtained by subtracting the adjacency matrix from the degree matrix. The roots of the polynomial $\psi(L, \mu) = |\mu I - L|$ represent the spectrum of the Laplacian of a graph, which is denoted as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. If two graphs, G_1 and G_2, have the same Laplacian spectrum, they are called L-cospectral. A graph is called *DLS* if its unique Laplacian spectrum can recognize it.

This article defines the disjoint union of graphs and their join as $G_1 \cup G_2$ and $G_1 \nabla G_2$, respectively. Complete graphs, cycles, and paths with n vertices are denoted by K_n, C_n, and P_n, respectively.

If the set of vertices and edges of the graph H is a subset of the vertices and edges of graph G, then H is called a subgraph of G. Suppose G and H are two

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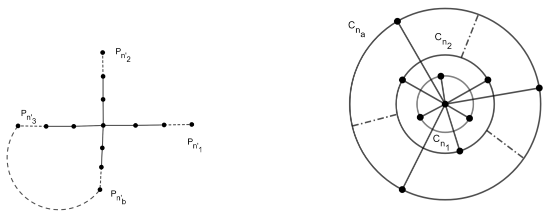
graphs, the notation $N_G(H)$ represents the number of subgraphs of G that are isomorphic to H. $N_G(C_3)$ denoted the number of triangles in graph G, where C_3 is a cycle of length three.

Suppose that G_1 and G_2 are disjoint graphs, with sets of vertices V_1 and V_2, and sets of edges E_1 and E_2, respectively. If $v_1 \in V_1$ and $v_2 \in V_2$, by identifying v_1 and v_2 with the vertex v^*, in $G_1 \cup G_2$, then the vertex $v*$ is adjacent to all the neighbors of v_1 and v_2, coalescence graph is obtained and denoted by $G_1 \circ G_2$.

A tree with a single vertex of a degree greater than two is called a starlike tree. A wheel graph is created by joining a cycle and K_1. If more than one cycle is joined to K_1, the resulting graph is called a multi-wheel, as shown in Figure 1.



(a) starlike tree (b) multi-wheel graph

FIGURE 1. starlike tree and multi-wheel graph

The study of graphs determined by the Laplacian spectrum is motivated by the desire to understand the structural and dynamic properties of networks across various disciplines, including mathematics, physics, and computer science. The Laplacian spectrum provides deep insights into the connectivity and robustness of a graph, revealing critical information about its topology. This spectral analysis finds applications in numerous fields, such as assessing the stability of chemical compounds, optimizing communication networks, and enhancing machine learning algorithms through improved data clustering.

Additionally, Laplacian spectra are instrumental in detecting community structures within social networks and facilitating effective routing in transportation and logistics. By knowing these spectral properties, researchers can develop algorithms that efficiently analyze and manipulate complex systems, thereby paving the way for innovations in technology and science.

Analyzing graphs with specific characteristics has always been of interest to researchers. Van Dam and Haemers [9] conjectured that most graphs are uniquely determined by their spectrum, meaning they are the only graphs with that particular spectrum up to isomorphism. The paper [1] has established that no two non-isomorphic starlike trees have identical Laplacian spectra. Furthermore, the work [11] by Yuanping Zhang et al, utilizing the results from [8],

shows that multi-wheel graphs can be identified using their Laplacian spectrum.
If $G_1 = (C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_a}) \bigtriangledown K_1$, where for $1 \leq i \leq a$, C_{n_i} is the cycle of length n_i. The graph G_2 is a starlike graph obtained by coalescing paths $P_{n'_1}, P_{n'_2}, \ldots, P_{n'_b}$. When these two graphs are merged, the resulting graph is denoted as $G = G_1 \circ G_2 = C(n, s, r, a, b)$. Here, $s = \sum_{i=1}^{a} n_i$, $r = \sum_{i=1}^{b} n'_i$, see Figure 2. We prove that this graph is DLS.

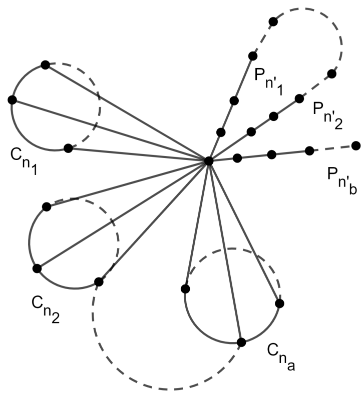


Figure 2. $G = C(n, s, r, a, b)$

2. Preliminaries

In this section, we would like to present the initial definitions and theorems used to establish the main results of the article.

Theorem 2.1. *[7] [9] [10] The Laplacian spectrum of a graph can reveal its number of vertices, edges, components, spanning trees, and the sum of squares of degrees of vertices.*

Theorem 2.2. *[2] If G is a graph with at least one vertex, then $d_1(G) \leq \mu_1(G) - 1$. Furthermore, if $d_1(G) = n - 1$, then equality is achieved.*

Theorem 2.3. *[5] Let G be a graph with more than one edge, then $\mu_1(G) \leq d_1(G) + d_2(G)$. If G is connected, then $\mu_2(G) \geq d_2(G)$.*

Theorem 2.4. *[6] Let v is a vertex of graph G and graph G_v is the result of removing vertex v, then*

$$\mu_i(G) \geq \mu_i(G_v) \geq \mu_{i+1}(G) - 1 \quad 1 \leq i \leq n - 1. \tag{1}$$

Lemma 2.5. *[5] If G is a connected graph with $d_2(G) = 2$, then $\mu_2(G) < 4$.*

Lemma 2.6. *[5] If graph G has m edges,n vertices, and τ number of triangles, then*
(2)
$$\sum_{i=1}^{n}\mu_i=\sum_{i=1}^{n}d_i=2m,\ \sum_{i=1}^{n}\mu_i^2=2m+\sum_{i=1}^{n}d_i^2,\ \sum_{i=1}^{n}\mu_i^3=\sum_{i=1}^{n}d_i^3+3\sum_{i=1}^{n}d_i^2-6\tau.$$

Lemma 2.7. *[4] Let G and H be two L-cospecteal graphs and $deg(G)=(d_1,d_2,\ldots,d_n)$, $deg(H)=(d'_1,d'_2,\ldots,d'_n)$, then*
$$N_G(C_3)-\frac{1}{6}\sum_{i=1}^{n}(d_i-2)^3=N_H(C_3)-\frac{1}{6}\sum_{i=1}^{n}(d'_i-2)^3.$$

Suppose the average degree of vertices adjacent to v is m_v, then the following inequality holds.

Lemma 2.8. *[5] [7] Let G be a connected graph, then $\mu_1(G)\le max_v[deg(v)+m_v]$.*

Lemma 2.9. *[1] No two non-isomorphic starlike trees have the same L-spectrum, meaning they are not L-cospectral.*

Lemma 2.10. *[3] Suppose that G is a connected graph, $v\in V_G$ and $v_1,v_2,\ldots,v_r$ are pendant vertices that adjacent v. If G^* is the graph resulting from the addition of t $(1\le t\le\binom{r}{2})$ edges between $v_1,v_2,\ldots,v_r$ vertices, then $\mu_1(G)=\mu_1(G^*)$.*

3. Checking whether a graph is *DLS*

In this section, we first prove that the degree sequence of the L-cospectral graph is the same as the graph resulting from integrating the starlike tree and the multi-wheel graph. By definition, if $G=C(n,s,r,a,b)$, then the degree sequence of this graph is as follows
$$deg(G)=(s+b,\overbrace{3,\ldots,3}^{s},\underbrace{2,\ldots,2}_{r-2b},\overbrace{1,\ldots,1}^{b}).$$

Thus, the graph's total number of vertices and edges are $s+r-b+1$ and $2s+r-b$, respectively. A connected graph G with n vertices and m edges is called k-cyclic if $m=n+k-1$. In this context, $G=C(n,s,r,a,b)$ is a s-cyclic graph.

Lemma 3.1. *If H and $G=C(n,s,r,a,b)$ are L-cospectral, then*

- $s+b+1\le\mu_1(H)<s+b+3$,
- $3\le\mu_2(H)\le 5$.

Proof. • By using (2.2), (2.8), and $b\ge 1$,
$$s+b+1\le\mu_1(G)\le s+b+\frac{3s+2b}{s+b}=s+b+\frac{2s+2b}{s+b}+\frac{s}{s+b}<s+b+3.$$
.

- Let v be the vertex with the maximum degree in G, then the graph G_v contains paths and cycles. By Theorems (2.4), (2.3), and Lemma (2.5)

$$2 \leq \mu_2(G) - 1 = \mu_2(H) - 1 \leq \mu_1(G_v) \leq 4$$
$$3 \leq \mu_2(H) \leq 5.$$

□

Example 3.2. *Let* $G = C(18, 10, 8, 3, 3)$*, then* $\mu_2(G) = 5$ *and* $\mu_1(G) \approx 14.019$*, Figure* (3).

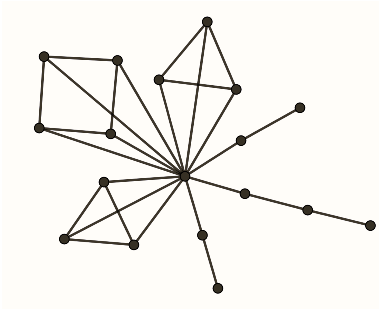


Figure 3. $G = C(18, 10, 8, 3, 3)$

Lemma 3.3. *Let* $G = C(n, s, r, a, b)$ *and* H *are two L-cospectral graphs and* $b \geq 3$*, then the highest degree of the vertices of the graph* H *is not* $s + b - 1$*.*

Proof. Suppose that the degree sequence of the vertices in graph H is $d_1 \geq d_2 \geq \cdots \geq d_n$. Let n_i represent the number of vertices with degree i in graph H and $N_H(C_3) = \kappa$, according to (2) and Lemma (2.7), then

$$\sum_{i=1}^{d_1(H)} n_i = s + r - b + 1, \tag{3}$$

$$\sum_{i=1}^{d_1(H)} in_i = 2m = 4s + 2r - 2b, \tag{4}$$

$$\sum_{i=1}^{d_1(H)} i^2 n_i = (s + b)^2 - 7b + 4r + 9s, \tag{5}$$

$$\sum_{i=1}^{d_1(H)} i^3 n_i = (s + b)^3 - 15b + 8r + 21s + 6\kappa, \tag{6}$$

$$\kappa = \frac{1}{6}[5s + b - (s + b - 2)^3 + \sum_{i=1}^{n} (d_i - 2)^3]. \tag{7}$$

Proof by contradiction: let $d_1 = s + b - 1$ and $n_{s+b-1} \geq 2$, then $s = b = 3$. By (3),(4), (5) and (6),

$$\begin{cases} n_1 = n_5 - \kappa + 2, \\ n_2 = -4n_5 + r + 3\kappa, \\ n_3 = 6n_5 - 3\kappa - 8, \\ n_4 = -4n_5 + \kappa + 7. \end{cases}$$

The degree sequence of the graph's vertices is non-negative, therefore $\kappa = 0$. According to the sequence of degrees above $\frac{4}{3} \leq n_5 \leq \frac{7}{4}$, which is impossible. Let $d_1 = s + b - 1$ and $n_{s+b-1} = 1$, then $s + b \geq 7$. By (3),(4), (5) and (6),

$$(8) \qquad \begin{cases} 2n_1 = 2n_5 - (s+b)^2 + 9(s+b) - 2\kappa - 6, \\ 2n_2 = -8n_5 + 3(s+b)^2 - 25s - 23b + r + 6\kappa + 28, \\ 2n_3 = 12n_5 - 3(s+b)^2 + 25s + 17b - 6\kappa - 11, \\ 2n_4 = -8n_5 + (s+b)^2 - 7s - 5b + 2\kappa + 6. \end{cases}$$

According to (8), the expression $-(s+b)^2 + 29s + 19b - 26 \geq 6\kappa$ indicates that the left side is negative for all values of $s + b \geq 7$, except at the point $(s, b) = (4, 3)$. The sequence of degrees for the point $(s, b) = (4, 3)$ is as follows.

$$\begin{cases} n_1 = n_5 + 1, \\ n_2 = -4n_5 + r + 3, \\ n_3 = 6n_5 - 9, \\ n_4 = -4n_5 + 6. \end{cases}$$

Therefore $n_5 = 1.5$, so for all values of $s + b \geq 7$, It comes to a contradiction.

□

Lemma 3.4. *If graph H is L-cospectral with graph $G = C(n, s, r, a, b)$, and $b \geq 3$, then the maximum vertex degree of H cannot be $s + b - 2$.*

Proof. Suppose it is not so, then $d_1(H) = s + b - 2$ and $n_{s+b-2} \geq 2$. In this case, (s, b) is equal to one of the ordered pairs of $(3, 3)$, $(3, 4)$, and $(4, 3)$.

- If $s = b = 3$, the following relations will be obtained, which are impossible.

$$\begin{cases} n_1 = -n_4 + 9, \\ n_2 = 3n_4 + r - 21, \\ n_3 = -3n_4 + 13, \\ \kappa = n_4 - 7 \Rightarrow n_3 < 0. \end{cases}$$

- Let $(s, b) = (3, 4)$, then the graph vertices' maximum degree equals 5. Therefore $n_3 = 6n_5 - 3\kappa - 33$, $n_4 = -4n_5 + \kappa + 17$ and $\kappa \leq \frac{-17}{3}$, this is a contradiction.
- If $(s, b) = (4, 3)$, then similar to the previous case, the number of triangles of the graph H will be negative, which results in contradiction.

Let $d_1(H) = s + b - 2$ and $n_{s+b-2} = 1$, then $s + b > 7$. By (3),(4), (5) and (6),

$$\begin{cases} n_1 = n_5 - (s+b)^2 + 9(s+b) - \kappa - 16, \\ n_2 = -4n_5 + 3(s+b)^2 - 25s - 24b + r + 3\kappa + 39, \\ n_3 = 6n_5 - 3(s+b)^2 + 24s + 20b - 3\kappa - 32, \\ n_4 = -4n_5 + (s+b)^2 - 7s - 6b + \kappa + 9. \end{cases} \tag{9}$$

By (9), $-3(s+b)^2 + 30(s+b) - s - 55 \geq 3\kappa$. For $s + b > 7$, the left side of this inequality is always negative,, this is a contradiction. The assumption $d_1(H) = s + b - 2$ is contradicted in every possible state, thus completing the proof. □

Lemma 3.5. *Suppose H and $G = C(n, s, r, a, b)$ be two L-cospectral graphs, where $b \geq 3$, then $d_1(H) \neq s + b - 3$.*

Proof. Let $d_1(H) = s + b - 3$, where $n_{s+b-3} \geq 2$, then $s + b \leq 8$ and $(s, b) \in \{(3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (5, 3)\}$.

- If $s = b = 3$, then $d_1(H) = d_2(H) = 3$, by (3),(4) and (5), $deg(H) = (\underbrace{3, 3, \ldots, 3}_{9}, \overbrace{2, 2, \ldots, 2}^{r-21}, \underbrace{1, 1, \ldots, 1}_{13})$, so $\kappa < 0$, this is a contradiction.
- Let $(s, b) = (3, 4)$, then, 4 is the maximum degree of the vertices in the graph H. The sequence of degrees of the graph and the number of triangles H is as follows:

$$\begin{cases} n_1 = -n_4 + 14, \\ n_2 = 3n_4 + r - 32, \\ n_3 = -3n_4 + 18, \\ \kappa = n_4 - 17. \end{cases}$$

  The vertices with a degree of one will be negative, which is a contradiction.
- If $s = 3$ and $b = 5$, then $d_1(H) = d_2(H) = 5$, so $n_1 = n_5 - \kappa - 12$, $n_4 = -4n_5 + \kappa + 32$, therefore,

$$n_5 \geq \kappa + 12,$$
$$4n_5 \leq \kappa + 32,$$
$$\kappa \leq \frac{-16}{3}.$$

  This is a contradiction.
- Let $(s, b) = (4, 3)$, then $d_1(H) = d_2(H) = 4$, by using (3),(4) and (5),

$$\begin{cases} n_1 = -n_4 + 13, \\ n_2 = 3n_4 + r - 30, \\ n_3 = -3n_4 + 19, \\ \kappa = n_4 - 17. \end{cases}$$

So $n_1 < 0$ and it is impossible.

- If $s = b = 4$, then $d_1(H) = d_2(H) = 5$. By (3),(4), (5) and (6), have a contradiction.

$$\begin{aligned} n_1 = n_5 - \kappa - 12 \Rightarrow 4n_5 \geq 4\kappa + 48, \\ n_4 = -4n_5 + \kappa + 31 \Rightarrow 4n_5 \leq \kappa + 31, \\ \kappa \leq \frac{-17}{3}. \end{aligned}$$

- Let $(s, b) = (5, 3)$, then seam the last case $\kappa \leq \dfrac{-18}{3}$, which is impossible.

Let $d_1(H) = s + b - 3$ and $n_{s+b-3} = 1$, then $s + b \geq 9$. By (3),(4), (5) and (6),

$$(10) \qquad \begin{cases} 2n_1 = 2n_5 - 3(s+b)^2 + 29(s+b) - 2\kappa - 62, \\ 2n_2 = -8n_5 + 9(s+b)^2 - 81s - 79b + 2r + 2\kappa + 156, \\ 2n_3 = 12n_5 - 9(s+b)^2 + 77s + 69b - 6\kappa - 132, \\ n_4 = -8n_5 + 3(s+b)^2 - 23s - 21b + 2\kappa + 38. \end{cases}$$

By using (10),

$$\begin{aligned} 24n_5 + 154s + 138b \geq 264 + 18(s+b)^2 + 12\kappa, \\ 24n_5 + 69s + 63b \leq 114 + 9(s+b)^2 + 6\kappa. \end{aligned}$$

So $\kappa \leq \frac{1}{6}[-9(s+b)^2 + 85s + 75b - 150] < 0$. This case also leads to a contradiction, so the proof is complete. □

Lemma 3.6. *There is no L-cospectral graph with $G = C(n, s, r, a, b)$, where $b \geq 3$, that has a vertex with a maximum degree of $s + b - 4$.*

Proof. Suppose that H is an L-cospectral graph with $G = C(n, s, r, a, b)$ and $deg(H) = (d_1, d_2, \ldots, d_n)$, where $d_1 = s + b - 4$. If $n_{s+b-4} \geq 2$, then $s + b \leq 9$, so

$$(11) \quad (s, b) \in \{(3,3), (3,4), (3,5), (3,6), (4,3), (4,4), (4,5), (5,3), (5,4), (6,3)\}.$$

- If $(s, b) = (3, 3)$, then $d_1(H) = d_2(H) = 2$. Therefore, G and H are paths or cycles, which is impossible.
- According to equation (11), suppose that $(s, b) = (3, 4)$ and $d_1(H) = d_2(H) = 3$. By (3),(4), and (5) the degree sequence of H is

$$(\underbrace{3, 3, \ldots, 3}_{18}, \overbrace{2, 2, \ldots, 2}^{r-32}, \underbrace{1, 1, \ldots, 1}_{14}).$$

  Now using (7), $\kappa = -17$, it is impossible.

- If (s, b) equals $(3, 5)$, then $d_1(H) = d_2(H) = 4$. Therefore, by (3),(4) and (5) we have

$$\begin{cases} n_1 = 20 - n_4, \\ n_2 = r - 45 + 3n_4, \\ n_3 = 24 - 3n_4. \end{cases}$$

by (7), $\kappa = -32 + n_4$, so $n_1 < 0$, this is impossible.
- Let $(s, b) = (3, 6)$, then, the maximum degree of the vertices of the graph H is 5. The degree sequence of the graph H is given by the equations (3),(4),(5), and (6)

$$\begin{cases} n_1 = n_5 - \kappa - 26, \\ n_2 = -4n_5 + 3\kappa + r + 99, \\ n_3 = 6n_5 - 3\kappa - 128, \\ n_4 = -4n_5 + \kappa + 53. \end{cases} \tag{12}$$

By (12) and the fact that the degree sequence of a graph is non-negative,

$$\begin{aligned} n_5 \geq& \kappa + 26, \\ 4n_5 \leq& \kappa + 53. \end{aligned}$$

So $\kappa \leq -17$, it is impossible.
- According to equation (11), let $(s, b) = (4, 3)$, then the maximum degree of the vertices of the graph H is equal to 3. From equations(3),(4) and (5), the degree sequence is given by $(\underbrace{3, 3, \ldots, 3}_{19}, \overbrace{2, 2, \ldots, 2}^{r-30}, \underbrace{1, 1, \ldots, 1}_{13})$.
So $\kappa = -16$, this is impossible.
- If $(s, b) = (4, 4)$, then $d_1(H) = d_2(H) = 4$, by (3),(4), (5), and (7),

$$\begin{cases} n_1 = -n_4 + 19, \\ n_2 = 3n_4 + r - 43, \\ n_3 = -3n_4 + 25, \\ \kappa = n_4 - 17. \end{cases}$$

So $n_1 < 0$, this is a contradiction.
- Let $s = 4$ and $b = 5$, then $d_1(H) = d_2(H) = 5$, therefore

$$\begin{aligned} n_1 = n_5 - \kappa - 26 \Rightarrow n_5 \geq \kappa + 26, \\ n_4 = -4n_5 + \kappa + 52 \Rightarrow 4n_5 \leq \kappa + 52, \\ \frac{-52}{3} \geq \kappa. \end{aligned}$$

This is impossible.

- By (11), if $(s, b) = (5, 3)$, then $d_1(H) = d_2(H) = 4$, by (3),(4),(5), and (7),
$$\begin{cases} n_1 = -n_4 + 18, \\ n_2 = 3n_4 + r - 41, \\ n_3 = -3n_4 + 26, \\ \kappa = n_4 - 30. \end{cases}$$
Therefore $n_1 < 0$, it is impossible.
- If $(s, b) = (5, 4)$, then the maximum degree of vertices is 5. Using Eqs (3),(4), (5) and (6), we have $n_1 = n_5 - \kappa - 26$, and $n_5 \geq \kappa + 26$. Additionally, $n_4 = -4n_5 + \kappa + 51$, and $4n_5 \leq \kappa + 51$. Therefore $\kappa \leq \frac{-53}{3}$, this is a contradiction.
- Let $(s, b) = (6, 3)$, then $d_1(H) = d_2(H) = 5$. By equations (3),(4), (5), and (6), $n_1 = n_5 - \kappa - 26$, also $n_4 = -4n_5 + \kappa + 50$. Therefore $\kappa \leq \frac{-54}{3}$, which is impossible.

If $d_1(G) = s + b - 4$ and $n_{s+b-4} = 1$, then $s + b \geq 10$. By (3),(4), (5) and (6),
$$\begin{cases} n_1 = n_5 - 2(s+b)^2 + 21(s+b) - \kappa - 52, \\ n_2 = -4n_5 + 6(s+b)^2 - 59s - 58b + r + \kappa + 134, \\ n_3 = 6n_5 - 6(s+b)^2 + 56s + 52b - 3\kappa - 116, \\ n_4 = -4n_5 + 2(s+b)^2 - 17s - 16b + \kappa + 34. \end{cases}$$
Due to the $n_3, n_4 \geq 0$,
$$12n_5 + 112s + 104b \geq 232 + 12(s+b)^2 + 6\kappa, \tag{13}$$
$$12n_5 + 51s + 48b \leq 102 + 6(s+b)^2 + 3\kappa. \tag{14}$$
By (13) and (14), $\kappa \leq \frac{1}{3}[-2(s+b)^2 + 61s + 56b - 130] < 0$, it is impossible. Therefore, all possible scenarios result in a contradiction, thus negating the existence of the H graph. □

Theorem 3.7. *If $G = C(n, s, r, a, b)$ and H are two L-cospectral graphs and $b \geq 3$, they have the same degree sequence.*

Proof. According to Lemma (3.1), we have $d_1 \leq s + b + 2$. Using equations (2.3), (3.3), (3.4), (3.5), (3.6), and the fact that $d_2(H) \leq 5$, we can conclude that $s + b \leq d_1(H) \leq s + b + 1$.

Let $d_1(H) = s + b + 1$, then $n_{s+b+1} = 1$. Using equations (3),(4), (5) and (6), the degree sequence of the graph is as follows.
$$2n_4 = -8n_5 - (s+b)^2 + s + 3b + 2\kappa - 2 \geq 0,$$
$$\Rightarrow -8n_5 - (s+b)^2 + 3(s+b) - 2 \geq 0.$$
The left side of the above inequality is always negative for $s, b \geq 3$, this is a contradiction.

Let $d_1 = s+b$, then, $n_{s+b} = 1$. By (3),(4), (5) and (6),

$$\begin{cases} n_1 = n_5 + s + b - \kappa, \\ n_2 = -4n_5 - 3s - 2b + r + 3\kappa, \\ n_3 = 6n_5 + 4s + 20b - 3\kappa, \\ n_4 = -4n_5 - s - 6b + \kappa. \end{cases}$$

The number of vertices of a graph is non-negative, therefore $\kappa \geq s + 4n_5$. Since that the graph H is s-cyclic, $\kappa = s$ and $4n_5 = 0$. The degree sequence of the graph H is as follows.

$$\begin{cases} n_1 = b, \\ n_2 = -2b + r, \\ n_3 = s, \\ n_{s+b} = 1. \end{cases} \tag{15}$$

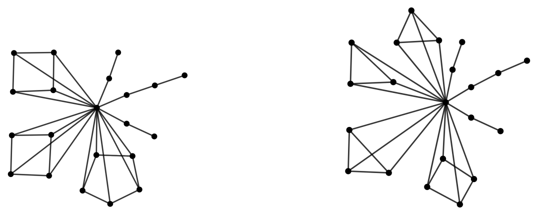
□

Lemma 3.8. *Let $G = C(n,s,r,a,b)$ and H are L-cospectral, $b \geq 3$, then $H = C(n,s,r,a',b)$.*

Proof. Using Theorem (3.7), we have $deg(H) = (s+b, \overbrace{3,\ldots,3}^{s}, \underbrace{2,\ldots,2}_{r-2b}, \overbrace{1,\ldots,1}^{b})$, and $N_H(C_3) = s$. Let $deg_v = s+b$. Then, the components of graph $H - v$ are paths or cycles. Since cycle components must contain vertices of degree two, $H - v$ comprises b paths and a' cycles. Hence, all the vertices of degree 3 of the graph in s are triangles, and this means $H = C(n,s,r,a',b)$. □

The converse of Theorem (3.7) and Lemma (3.8) is not always valid, as illustrated by the following example.

Example 3.9. *Graphs $G_1 = (21, 13, 10, 3, 3)$ and $G_2 = (21, 13, 10, 4, 3)$, Figure (4), with the same degree sequence, have different Laplacian spectra.*



(a) G_1 (b) G_2

FIGURE 4. G_1 and G_2

If $G = C(n, s, r, a, b)$, the graph resulting from removing edges in a triangle not adjacent to the vertex of maximum degree is called $St(G)$. By Lemma (2.10), $\mu_1(G) = \mu_1(St(G))$.

Lemma 3.10. *If $G = C(n, s, r, a, b)$ and $H = C(n, s, r, a', b)$ are L-cospectral, then $St(G) = St(H)$.*

Proof. Let $St(G) \neq St(H)$, by (2.9) $\mu_1(St(G)) \neq \mu_1(St(H))$, then $\mu_1(G) \neq \mu_1(H)$, which is a contradiction. □

Theorem 3.11. *Let $G = C(n, s, r, a, b)$ and $H = C(n, s, r, a', b)$ are L-cospectral, and $b \geq 3$, then $G = H$.*

Proof. In Lemma (3.10), it was proven that the starlike parts of the graphs must be the same. Let's prove this by induction on s. If $s = 3$ according to $N_G(C_3) = N_H(C_3)$, $G = H$. Suppose the rule holds for $s = k - 1$, and we prove that it also holds for $s = k$. If $s = k$, and we reduce the length of one of the arbitrary cycles by one, it becomes $s = k - 1$. According to the induction assumption, the length of the resulting cycles and the rest of the cycles should be equal to each other. So, by adding the deleted vertex, the graphs will be equal. This confirms the rule. □

4. Conclusions and open problems

This article showed that its Laplacian spectrum can uniquely identify the graph resulting from integrating a multi-wheel graph and a Starlike tree. It can also be checked whether this graph can be identified by other spectra such as Adjacency, signless Laplacian, etc.

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6. Conflict of interest

The authors declare no conflict of interest.

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