Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

EUROPEAN LOOKBACK OPTION PRICING WITH FLOATING STRIKE PRICE UNDER FRACTIONAL BLACK-SCHOLES MODELS

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Article type: Research Article
(Received: 19 June 2024, Received in revised form 06 October 2024)
(Accepted: 17 January 2025, Published Online: 18 January 2025)

Abstract. One of the most famous path-dependent options is the Lookback option. This option is a useful financial instrument to hedge against the risks associated with high volatility in the market. Since empirical studies on the statistical properties of logarithmic returns show the dependence of returns and stock price volatilities on different days; we need a suitable model for pricing the Lookback option to illustrate this phenomenon. Partial differential equations with fractional order derivatives can be useful tools to describe the long memory effect in the financial markets. Hence, we want to price the European floating strike Lookback option (FSLO) under fractional Black-Scholes (FBS) models using a numerical method: implicit difference scheme (IDS). Also, the stability and convergence analysis of the proposed method are investigated using Fourier series expansion. Numerical results are provided to show the efficiency of the method.

Keywords: Fractional Black-Scholes equation, Lookback option, Path-dependent option, Implicit difference scheme.
2020 MSC: 91G20, 65M06, 35R11, 65M12.

1. Introduction

Lookback options are one of the most popular exotic options for risk management or trading in the financial market. These options are traded in the over-the-counter (OTC) market. Lookback options are called path-dependent options because their payoffs depend on the path followed by the underlying asset price (S) during the option's lifetimes. Since holders of Lookback options can buy or sell at the lowest or highest underlying asset price over the options lifetimes, these options are never out-of-the-money, and their holders can always exercise them [11]. So, Lookback options act as insurance against the risks associated with high volatility of asset price during the life of the contract. This ability of Lookback options makes these options more expensive

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https://doi.org/10.22103/jmmr.2025.23656.1672
Publisher: Shahid Bahonar University of Kerman
How to cite: M. Rezaei, A.R. Yazdanian, *European Lookback option pricing with floating strike price under fractional Black-Scholes models*, J. Mahani Math. Res. 2025; 14(2): 113-135.

than vanilla options, but due to their high premiums, the volume of transactions in the OTC market is reduced [6, 11, 18, 35]. These options have a more complex structure than standard options.

There are two types of Lookback options: FSLOs and fixed strike Lookback options. The payoff functions of these options are according to Table 1 where T and K are the expiration date and strike price, respectively; and

$$\widetilde{M} = \max_{0\leq t\leq T} S(t), \quad \widetilde{m} = \min_{0\leq t\leq T} S(t).$$

Table 1. Payoff functions of Lookback options.

Option name	Floating strike price	Fixed strike price
Put option	$V(T, S(T), \widetilde{M}) = \max(\widetilde{M} - S(T), 0)$	$V(T, S(T), \widetilde{m}) = \max(K - \widetilde{m}, 0)$
Call option	$V(T, S(T), \widetilde{m}) = \max(S(T) - \widetilde{m}, 0)$	$V(T, S(T), \widetilde{M}) = \max(\widetilde{M} - K, 0)$

Holders of floating strike Lookback put (call) options (FSLPOs) have the right to sell (buy) at the highest (lowest) realized asset price during a certain time. Since $\widetilde{M} \geq S(T)$ and $S(T) \geq \widetilde{m}$, payoff functions of Lookback put and call options are $\widetilde{M} - S(T)$ and $S(T) - \widetilde{m}$, respectively. At first, Goldman et al. [13] have priced Lookback options under the Black-Scholes model. Then, Conze [7] has obtained closed-form formulas for various European Lookback options by using probabilistic approaches under the continuous-time asset model with Black and Scholes assumptions. Wong and Chan [30] have analytically evaluated floating and fixed strike Lookback options, in-the-money Lookback spread options, and dynamic fund protection under a multiscale stochastic volatility model. They have demonstrated that this model captures the volatility surface with high quality. Also, they have investigated the effects of stochastic volatility on Lookback options. Kim et al. [17] have introduced a binomial tree method for pricing European and American Lookback options with jump-diffusion models and have demonstrated its equivalence to a certain explicit difference scheme. They have proved the existence and convergence of the optimal exercise boundary in the binomial tree method for American Lookback options. Park and Taksar [21] have obtained a semi-analytic pricing formula for Lookback options under a general stochastic volatility model using probabilistic tools together with the homotopy analysis method.

The above studies do not show the memory effect and other properties in the financial market. Various empirical studies on the financial time series show that the logarithmic returns of financial assets are non-independent, non-normal, nonlinear, heavy tails, self-similar, long-range dependence, and volatility clustering [2, 5, 8–10, 15, 26]. Long-range dependence or long memory is one of the phenomena observed between returns and stock price volatilities on different days. There are several reasons for the appearance of this phenomenon in the financial markets, such as long-range dependence appears in high frequency financial data [1,20]. In addition to fundamentals, other market characteristics such as speculative behaviors (e.g., technical analysis) and speculative bubbles in stock markets play an important role in determining prices [32]. Since traders are bounded rational, they either underreact or continually overreact to the news [29]. Because the financial system is a complex system with a lot of flexibility, investors do not make a decision immediately after receiving financial information, but wait for the information to reach its threshold limit value [33]. This behavior may lead to the feature of "long-range dependence" in the financial market. Mandelbrot [19] has first found the existence of long-range dependence in asset returns. To capture this phenomenon, effective instruments such as fractional Brownian motion and fractional calculus are used. Chen et al. [4] have obtained the bullish and bearish Lookback option pricing formula for the mixed jump-diffusion fractional Brownian motion with a floating strike price. Yang [34] has compounded the Brownian motion, fractional Brownian motion, and Poisson process to suggest the default probability of the American Lookback option under a mixed jump-diffusion model. He has considered two-sided predetermined strikes to activate the American Lookback option. In addition to fractional Brownian motion, partial differential equations with fractional derivatives also provide a powerful tool for describing memory and hereditary characteristics of stock price variations. So far, various FBS models have been presented. First FBS model is proposed in the following form to price the European call options [31]:

$$\frac{\partial^\alpha V}{\partial t^\alpha}(t,S) = rV(t,S) - \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}(t,S) - rS\frac{\partial V}{\partial S}(t,S), \tag{1}$$

where σ and r are the volatility of the asset returns and interest rate, respectively. The second FBS model is derived by using Itô Lemma and fractional Taylor's series of $V(S,t)$ as follows [16]:

$$\frac{\partial^\alpha V}{\partial t^\alpha}(t,S) = \left(rV(t,S) - rS\frac{\partial V}{\partial S}(t,S)\right)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}(t,S). \tag{2}$$

The third FBS model is presented to describe the trend memory effects in the following form [12]:

$$\frac{\partial^\alpha V}{\partial t^\alpha}(t,S) = \left(V(t,S) - \frac{\sigma^2 S^2}{2\Gamma^2(1+\alpha)}\frac{\partial^2 V}{\partial S^2}(t,S) - rS\frac{\partial V}{\partial S}(t,S)\right)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{3}$$

The fractional derivative in the above equations is the following modified Riemann-Liouville derivative [16]

$$\frac{\partial^\alpha V}{\partial t^\alpha}(t,S) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{V(\kappa,S)-V(T,S)}{(\kappa-t)^\alpha}d\kappa, & 0<\alpha<1, \\ \frac{\partial V(t,S)}{\partial t}, & \alpha = 1. \end{cases} \tag{4}$$

Note that by taking $\alpha = 1$ in the above FBS models, they will be the classical Black-Scholes model.

At the time of trading, traders look at past asset price information and make decisions based on it. Therefore, information about past asset prices is required. The classic Black-Scholes equation with non-fractional derivatives is not able to show this feature. But the Black-Scholes equation with fractional order derivative uses past information of underlying asset price. This point lies in the integral of relation (4). Therefore, the Black-Scholes equation with time-fractional derivative can be efficient for traders determining option price, implied volatility, and Greek letters. The importance of this issue led to extensive studies on these models: Chen et al. [3] have proposed a new operator splitting method for pricing American options under FBS models (1), (2), and (3). In [27], American options are evaluated based on the FBS equation (1) under the CEV model using a compact difference scheme when the dividend yield is paid on assets during that period. Samareh et al. [14] have priced the European option under the first FBS model by using the finite difference formula for time discretization and the spectral method based on Chelyshkov wavelets for space discretization. Rezaei and Izadi [23] have obtained an analytical solution for the time-space FBS model arising in the financial market. We will use models (1), (2), and (3) to price the European FSLO. The purpose of this study is to price the European FSLO under FBS models (1), (2), and (3). Consider first FBS model

$$\frac{\partial^\alpha V}{\partial t^\alpha}(t,S,\varrho) = rV(t,S,\varrho) - \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}(t,S,\varrho) - rS\frac{\partial V}{\partial S}(t,S,\varrho), \tag{5}$$

with the following terminal payoff and boundary conditions

$$\begin{cases} \dfrac{\partial V}{\partial \varrho}(t,S,\varrho) = 0, \quad S=\varrho, \quad 0\le t<T, \quad \varrho>0, \\ V(t,0,\varrho) = \varrho e^{-r(T-t)}, \quad 0\le t<T, \quad \varrho>0, \\ V(T,S,\varrho) = \varrho - S(T), \quad 0\le S<\varrho, \end{cases}$$

where $V(t,S,\varrho)$ is the FSLPO price, and $\varrho := \max_{0\le u\le t} S(u)$. By converting $\tau = T-t$ and $V(t,S,\varrho) = V(T-\tau,S,\varrho) = G(\tau,S,\varrho)$, the boundary problem with the terminal condition becomes a boundary problem with the initial condition. For $0<\alpha<1$, we have

$$\begin{aligned} \frac{\partial^\alpha V}{\partial t^\alpha}(t,S,\varrho) &= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{V(\kappa,S,\varrho)-V(T,S,\varrho)}{(\kappa-t)^\alpha}d\kappa \\ &\overset{\kappa=T-\iota}{=\!=\!=} \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial \tau}\int_\tau^0 \frac{V(T-\iota,S,\varrho)-V(T,S,\varrho)}{(\tau-\iota)^\alpha}d\iota \\ &= -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial \tau}\int_0^\tau \frac{G(\iota,S,\varrho)-G(0,S,\varrho)}{(\tau-\iota)^\alpha}d\iota \end{aligned}$$

$$= -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\tau}\int_0^{\tau}\left[G(\iota,S,\varrho)-G(0,S,\varrho)\right]d\frac{(\tau-\iota)^{1-\alpha}}{-(1-\alpha)}$$

$$= -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial\tau}\int_0^{\tau}\frac{(\tau-\iota)^{1-\alpha}}{1-\alpha}\frac{\partial G(\iota,S,\varrho)}{\partial\iota}d\iota$$

$$= -\frac{1}{\Gamma(1-\alpha)}\int_0^{\tau}\frac{1}{(\tau-\iota)^{\alpha}}\frac{\partial G(\iota,S,\varrho)}{\partial\iota}d\iota = -{}_0^C D_\tau^\alpha G(\tau,S,\varrho), \tag{6}$$

where ${}_0^C D_\tau^\alpha G(\tau,S,\varrho)$ is the Liouville-Caputo fractional derivative [22]. By (6), $\frac{\partial^2 V}{\partial S^2}(t,S,\varrho)=\frac{\partial^2 G}{\partial S^2}(\tau,S,\varrho)$, and $\frac{\partial V}{\partial S}(t,S,\varrho)=\frac{\partial G}{\partial S}(\tau,S,\varrho)$, equation (5) becomes

$${}_0^C D_\tau^\alpha G(\tau,S,\varrho) = \frac{\sigma^2}{2}S^2\frac{\partial^2 G}{\partial S^2}(\tau,S,\varrho)+rS\frac{\partial G}{\partial S}(\tau,S,\varrho)-rG(\tau,S,\varrho), \tag{7}$$

with initial and boundary conditions

$$\begin{cases} \frac{\partial G}{\partial\varrho}(\tau,S,\varrho)=0, \quad S=\varrho, \quad 0<\tau\le T, \quad \varrho>0,\\ G(\tau,0,\varrho)=\varrho e^{-r\tau}, \quad 0<\tau\le T, \quad \varrho>0,\\ G(0,S,\varrho)=\varrho-S(0), \quad 0\le S<\varrho. \end{cases} \tag{8}$$

Equation (7) with initial and boundary conditions (8) is 3-dimensional and its numerical solution will be complex. Hence, we transform the above problem into a 2-dimensional problem by using the linear scaling property of the FSLO price as follows [28]

$$G(\tau,\lambda S,\lambda\varrho)=\lambda G(\tau,S,\varrho), \quad \text{for all } \lambda>0.$$

Let

$$U(\tau,z)=G(\tau,z,1), \quad 0\le\tau\le T, \quad 0\le z\le 1,$$

then

$$G(\tau,S,\varrho)=\varrho G(\tau,\frac{S}{\varrho},1)=\varrho U(\tau,\frac{S}{\varrho}), \quad 0\le\tau\le T, \quad 0\le S\le\varrho, \quad \varrho>0. \tag{9}$$

Now, we compute the partial derivatives of equation (7) from (9)

$${}_0^C D_\tau^\alpha G(\tau,S,\varrho) = \frac{1}{\Gamma(1-\alpha)}\int_0^{\tau}\frac{1}{(\tau-\iota)^{\alpha}}\frac{\partial G(\iota,S,\varrho)}{\partial\iota}d\iota$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^{\tau}\frac{1}{(\tau-\iota)^{\alpha}}\frac{\partial}{\partial\iota}\left[\varrho U(\iota,\frac{S}{\varrho})\right]d\iota$$

$$= \varrho\cdot\frac{1}{\Gamma(1-\alpha)}\int_0^{\tau}\frac{1}{(\tau-\iota)^{\alpha}}\frac{\partial}{\partial\iota}\left[U(\iota,\frac{S}{\varrho})\right]d\iota = \varrho\,{}_0^C D_\tau^\alpha U(\tau,\frac{S}{\varrho}), \tag{10}$$

$$\frac{\partial G}{\partial S} = \frac{\partial}{\partial S}\left[\varrho U(\tau, \frac{S}{\varrho})\right] = \varrho \frac{\partial U}{\partial z}(\tau, \frac{S}{\varrho}) \cdot \frac{\partial}{\partial S}\left(\frac{S}{\varrho}\right) = \frac{\partial U}{\partial z}(\tau, \frac{S}{\varrho}), \tag{11}$$

$$\frac{\partial^2 G}{\partial S^2} = \frac{\partial}{\partial S}\left[\frac{\partial U}{\partial z}(\tau, \frac{S}{\varrho})\right] = \frac{\partial^2 U}{\partial z^2}(\tau, \frac{S}{\varrho}) \cdot \frac{\partial}{\partial S}\left(\frac{S}{\varrho}\right) = \frac{1}{\varrho}\frac{\partial^2 U}{\partial z^2}(\tau, \frac{S}{\varrho}), \tag{12}$$

$$\frac{\partial G}{\partial \varrho} = \frac{\partial}{\partial \varrho}\left[\varrho U(\tau, \frac{S}{\varrho})\right] = U(\tau, \frac{S}{\varrho}) - \varrho \frac{\partial U}{\partial z}(\tau, \frac{S}{\varrho}) \cdot \frac{\partial}{\partial \varrho}\left(\frac{S}{\varrho}\right)$$

$$= U(\tau, \frac{S}{\varrho}) - \frac{S}{\varrho}\frac{\partial U}{\partial z}(\tau, \frac{S}{\varrho}). \tag{13}$$

By substituting (9), (10), (11), and (12) into (7), we have

$$\varrho\, {}_0^C D_\tau^\alpha U(\tau, \frac{S}{\varrho}) = \frac{\sigma^2}{2} S^2 \cdot \frac{1}{\varrho}\frac{\partial^2 U}{\partial z^2}(\tau, \frac{S}{\varrho}) + rS \cdot \frac{\partial U}{\partial z}(\tau, \frac{S}{\varrho}) - r\varrho U(\tau, \frac{S}{\varrho}),$$

or

$$\varrho\left[{}_0^C D_\tau^\alpha U(\tau, \frac{S}{\varrho}) - \frac{\sigma^2}{2}\frac{S^2}{\varrho^2}\frac{\partial^2 U}{\partial z^2}(\tau, \frac{S}{\varrho}) - r\frac{S}{\varrho}\frac{\partial U}{\partial z}(\tau, \frac{S}{\varrho}) + rU(\tau, \frac{S}{\varrho})\right] = 0. \tag{14}$$

By omitting ϱ ($\varrho > 0$ and $\varrho \neq 0$) and changing the variable $z = \frac{S}{\varrho}$, we get

$${}_0^C D_\tau^\alpha U(\tau, z) = \frac{\sigma^2}{2} z^2 \frac{\partial^2 U}{\partial z^2}(\tau, z) + rz\frac{\partial U}{\partial z}(\tau, z) - rU(\tau, z). \tag{15}$$

Also, by substituting (9) and (13) into (8), we obtain the following initial and boundary conditions for (15):

$$\begin{cases} \dfrac{\partial U}{\partial z}(\tau, 1) = U(\tau, 1), & 0 < \tau \leq T, \\ U(\tau, 0) = e^{-r\tau}, & 0 < \tau \leq T, \\ U(0, z) = 1 - z(0), & 0 \leq z < 1. \end{cases} \tag{16}$$

Finding an analytical solution to problems (15)-(16) is difficult, so we use a numerical method to price the FSLO. In Section 2, we present an IDS for problems (15)-(16). In Section 3, we investigate the stability analysis of the suggested method. In Section 4, we demonstrate the convergence analysis of the suggested method. In Section 5, we report some numerical examples. Finally, some remarks are presented in Section 6.

2. Numerical method for FBS models

We describe a numerical method for equation (15) to price the FSLO. At first, we let ($N, M \in \mathbb{N}$)

$$\xi = \frac{T}{M}, \quad \tau_\varsigma = \varsigma\xi, \quad \varsigma = 0, 1, \ldots, M,$$
$$\rho = \frac{1}{N}, \quad z_\upsilon = \upsilon\rho, \quad \upsilon = 0, 1, \ldots, N.$$

We use the notation $U_\upsilon^\varsigma = U(\tau_\varsigma, z_\upsilon)$ to show the numerical solution of equation (15) in the uniform grid of mesh points $(\tau_\varsigma, z_\upsilon)$ and approximate time and space derivatives at this point. The time-fractional derivative ${}_0^C D_\tau^\alpha U(\tau, z)$ is approximated at the point $(\tau_\varsigma, z_\upsilon)$ as follows [22, 26]

$$(17)\quad {}_0^C D_\tau^\alpha U(\tau, z)\Big|_{\tau=\tau_\varsigma,\, z=z_\upsilon} = \varphi_{\alpha,\xi} \sum_{\omega=1}^{\varsigma} \chi_\omega^{(\alpha)} \left(U_\upsilon^{\varsigma-\omega+1} - U_\upsilon^{\varsigma-\omega}\right) + O\left(\xi^{2-\alpha}\right),$$

with

$$\varphi_{\alpha,\xi} = \frac{1}{\xi^\alpha \Gamma(2-\alpha)},$$
$$\chi_\omega^{(\alpha)} = \omega^{1-\alpha} - (\omega-1)^{1-\alpha}, \quad \omega = 1, \ldots, \varsigma.$$

We approximate derivatives $\frac{\partial^2 U}{\partial z^2}$ and $\frac{\partial U}{\partial z}$ as follows:

$$(18a)\quad \frac{\partial^2 U}{\partial z^2}\Big|_{\tau=\tau_\varsigma,\, z=z_\upsilon} = \frac{U_{\upsilon-1}^\varsigma - 2U_\upsilon^\varsigma + U_{\upsilon+1}^\varsigma}{\rho^2} + O\left(\rho^2\right),$$

$$(18b)\quad \frac{\partial U}{\partial z}\Big|_{\tau=\tau_\varsigma,\, z=z_\upsilon} = \frac{U_{\upsilon+1}^\varsigma - U_{\upsilon-1}^\varsigma}{2\rho} + O\left(\rho^2\right).$$

By substituting (17), (18a), and (18b) into equation (15), we get

$$\begin{aligned}\varphi_{\alpha,\xi} \sum_{\omega=1}^{\varsigma} \chi_\omega^{(\alpha)} \left(U_\upsilon^{\varsigma-\omega+1} - U_\upsilon^{\varsigma-\omega}\right) &= \frac{1}{2}\sigma^2 z_\upsilon^2 \frac{U_{\upsilon-1}^\varsigma - 2U_\upsilon^\varsigma + U_{\upsilon+1}^\varsigma}{\rho^2} \\ &+ r z_\upsilon \frac{U_{\upsilon+1}^\varsigma - U_{\upsilon-1}^\varsigma}{2\rho} - rU_\upsilon^\varsigma + O\left(\xi^{2-\alpha} + \rho^2\right). \qquad (19)\end{aligned}$$

We can rearrange the IDS (19) as follows

$$(20)\quad \begin{cases} \left[\zeta_\upsilon - \beta_\upsilon r\right] U_{\upsilon-1}^1 - \left[2\zeta_\upsilon + r + \varphi_{\alpha,\xi}\right] U_\upsilon^1 + \left[\zeta_\upsilon + \beta_\upsilon r\right] U_{\upsilon+1}^1 \\ \quad = -\varphi_{\alpha,\xi} U_\upsilon^0 + O\left(\xi^{2-\alpha} + \rho^2\right), \quad \varsigma = 1, \\ \left[\zeta_\upsilon - \beta_\upsilon r\right] U_{\upsilon-1}^\varsigma - \left[2\zeta_\upsilon + r + \varphi_{\alpha,\xi}\right] U_\upsilon^\varsigma + \left[\zeta_\upsilon + \beta_\upsilon r\right] U_{\upsilon+1}^\varsigma \\ \quad = \varphi_{\alpha,\xi} \sum_{\omega=1}^{\varsigma-1} \left(\chi_{\omega+1}^{(\alpha)} - \chi_\omega^{(\alpha)}\right) U_\upsilon^{\varsigma-\omega} \\ \qquad -\varphi_{\alpha,\xi} \chi_\varsigma^{(\alpha)} U_\upsilon^0 + O\left(\xi^{2-\alpha} + \rho^2\right), \quad \varsigma \geq 2, \end{cases}$$

where

$$\zeta_\upsilon := \frac{\sigma^2}{2\rho^2} z_\upsilon^2, \quad \beta_\upsilon := \frac{1}{2\rho} z_\upsilon.$$

The discretization of initial and boundary conditions are:

$$(21)\quad \begin{cases} \dfrac{U_{N+1}^\varsigma - U_{N-1}^\varsigma}{2\rho} + O\left(\rho^2\right) = U_N^\varsigma, \\ U_0^\varsigma = e^{-r\tau_\varsigma}, \\ U_\upsilon^0 = 1 - z_\upsilon. \end{cases}$$

Then, the following IDS is obtained for equation (15) by omitting the truncation errors from (20)

$$(22)\qquad \begin{cases} a_{\upsilon-1}\widetilde{U}^{1}_{\upsilon-1} + b_{\upsilon}\widetilde{U}^{1}_{\upsilon} + c_{\upsilon+1}\widetilde{U}^{1}_{\upsilon+1} = -\varphi_{\alpha,\xi}\widetilde{U}^{0}_{\upsilon}, \quad \varsigma = 1, \\ a_{\upsilon-1}\widetilde{U}^{\varsigma}_{\upsilon-1} + b_{\upsilon}\widetilde{U}^{\varsigma}_{\upsilon} + c_{\upsilon+1}\widetilde{U}^{\varsigma}_{\upsilon+1} \\ \quad = \varphi_{\alpha,\xi}\sum_{\omega=1}^{\varsigma-1}\left(\chi^{(\alpha)}_{\omega+1} - \chi^{(\alpha)}_{\omega}\right)\widetilde{U}^{\varsigma-\omega}_{\upsilon} - \varphi_{\alpha,\xi}\chi^{(\alpha)}_{\varsigma}\widetilde{U}^{0}_{\upsilon}, \quad \varsigma \geq 2, \end{cases}$$

where

$$a_{\upsilon-1} := \zeta_{\upsilon} - \beta_{\upsilon} r, \quad b_{\upsilon} := -\left[2\zeta_{\upsilon} + r + \varphi_{\alpha,\xi}\right], \quad c_{\upsilon+1} := \zeta_{\upsilon} + \beta_{\upsilon} r,$$

$$\upsilon = 1, 2, \ldots, N-1,$$

$$a_{N-1} := 2\zeta_N, \quad b_N := -\left[2\zeta_N + r + \varphi_{\alpha,\xi}\right] + 2\rho\left[\zeta_N + \beta_N r\right].$$

U^{ς}_{υ} and $\widetilde{U}^{\varsigma}_{\upsilon}$ are exact solutions of equations (20) and (22), respectively. Also, initial and boundary conditions are:

$$(23)\qquad \begin{cases} \widetilde{U}^{\varsigma}_{N+1} = \widetilde{U}^{\varsigma}_{N-1} + 2\rho\widetilde{U}^{\varsigma}_{N}, \\ \widetilde{U}^{\varsigma}_{0} = e^{-r\tau_{\varsigma}}, \\ \widetilde{U}^{0}_{\upsilon} = 1 - z_{\upsilon}. \end{cases}$$

We can rewrite the IDS (22) in the following matrix form

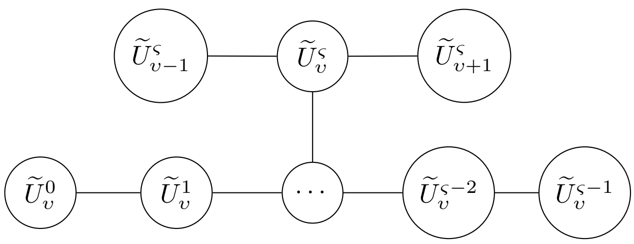
$$\begin{cases} A\widetilde{U}^{1} = -\varphi_{\alpha,\xi}\widetilde{U}^{0} - F^{1}, \quad \varsigma = 1, \\ A\widetilde{U}^{\varsigma} = \varphi_{\alpha,\xi}\sum_{\omega=1}^{\varsigma-1}\left(\chi^{(\alpha)}_{\omega+1} - \chi^{(\alpha)}_{\omega}\right)\widetilde{U}^{\varsigma-\omega} - \varphi_{\alpha,\xi}\chi^{(\alpha)}_{\varsigma}\widetilde{U}^{0} - F^{\varsigma}, \quad \varsigma \geq 2, \end{cases}$$

where

$$A = \begin{bmatrix} b_1 & c_2 & & & 0 \\ a_1 & b_2 & c_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & & c_N \\ 0 & & & a_{N-1} & b_N \end{bmatrix},$$

$$\widetilde{U}^{\varsigma} = \left(\widetilde{U}^{\varsigma}_1, \widetilde{U}^{\varsigma}_2, \ldots, \widetilde{U}^{\varsigma}_N\right)^t, \qquad F^{\varsigma} = \left(a_0\widetilde{U}^{\varsigma}_0, 0, \ldots, 0\right)^t.$$

The graph of the IDS (22) is as follows:



3. Stability analysis of IDS

We discuss the unconditional stability of the suggested IDS (22)-(23) using Fourier analysis [24,25]. We suppose that $\widehat{U}_{\upsilon}^{\varsigma}$ is the approximate solution of the proposed IDS. Let $\varepsilon = \widetilde{U} - \widehat{U}$, $\varepsilon_{\upsilon}^{\varsigma} = \widetilde{U}_{\upsilon}^{\varsigma} - \widehat{U}_{\upsilon}^{\varsigma}$, $\upsilon = 0, 1, \dots, N$; $\varsigma = 0, 1, \dots, M$, then $\varepsilon_{\upsilon}^{\varsigma}$ satisfies (22)-(23)

$$
\left\{
\begin{array}{l}
a_{\upsilon-1}\varepsilon_{\upsilon-1}^{1} + b_{\upsilon}\varepsilon_{\upsilon}^{1} + c_{\upsilon+1}\varepsilon_{\upsilon+1}^{1} = -\varphi_{\alpha,\xi}\varepsilon_{\upsilon}^{0}, \quad \varsigma = 1, \\
a_{\upsilon-1}\varepsilon_{\upsilon-1}^{\varsigma} + b_{\upsilon}\varepsilon_{\upsilon}^{\varsigma} + c_{\upsilon+1}\varepsilon_{\upsilon+1}^{\varsigma} \\
\quad = \varphi_{\alpha,\xi}\sum_{\omega=1}^{\varsigma-1}\left(\chi_{\omega+1}^{(\alpha)} - \chi_{\omega}^{(\alpha)}\right)\varepsilon_{\upsilon}^{\varsigma-\omega} - \varphi_{\alpha,\xi}\chi_{\varsigma}^{(\alpha)}\varepsilon_{\upsilon}^{0}, \quad \varsigma \geq 2,
\end{array}
\right. \tag{24}
$$

and $\varepsilon_{0}^{\varsigma} = 0$. We can expand $\varepsilon^{\varsigma}(z)$ in a Fourier series for grid function

$$
\varepsilon^{\varsigma}(z) = \left\{
\begin{array}{ll}
\varepsilon_{\upsilon}^{\varsigma}, & z \in \left(z_{\upsilon} - \frac{\rho}{2}, z_{\upsilon} + \frac{\rho}{2}\right] \bigcup \left(1 - \frac{\rho}{2}, 1\right], \\
0, & z \in \left[0, \frac{\rho}{2}\right],
\end{array}
\right.
$$

with period $L = 1$ as follows

$$
\varepsilon^{\varsigma}(z) = \sum_{\upsilon=-\infty}^{+\infty} \theta_{\upsilon}^{\varsigma} e^{i\frac{2\pi\upsilon z}{L}} \quad (i^2 = -1), \quad \varsigma = 0, 1, \dots, M,
$$

$$
\theta_{\upsilon}^{\varsigma} = \frac{1}{L}\int_{0}^{L} \varepsilon^{\varsigma}(z) e^{i\frac{2\pi\upsilon z}{L}} dz, \quad \upsilon \in \mathbb{Z}.
$$

We let $\varepsilon^{\varsigma} = (\varepsilon_{1}^{\varsigma}, \varepsilon_{2}^{\varsigma}, \dots, \varepsilon_{N}^{\varsigma})^{t}$, and introduce the following norm

$$
\|\varepsilon^{\varsigma}\|_{2}^{2} = \sum_{\upsilon=1}^{N} \rho \left|\varepsilon_{\upsilon}^{\varsigma}\right|^{2} = \int_{0}^{L} \left|\varepsilon^{\varsigma}(z)\right|^{2} dz = \|\varepsilon^{\varsigma}(z)\|_{L^2}^{2}, \quad \varsigma = 0, 1, \dots, M.
$$

According to the Parseval identity

$$
\|\varepsilon^{\varsigma}(z)\|_{L^2}^{2} = L \sum_{\upsilon=-\infty}^{+\infty} \left|\theta_{\upsilon}^{\varsigma}\right|^{2}, \quad \varsigma = 0, 1, \dots, M,
$$

one gets

$$
\|\varepsilon^{\varsigma}\|_{2}^{2} = \sum_{\upsilon=1}^{N} \rho \left|\varepsilon_{\upsilon}^{\varsigma}\right|^{2} = L \sum_{\upsilon=-\infty}^{+\infty} \left|\theta_{\upsilon}^{\varsigma}\right|^{2}. \tag{25}
$$

We suppose that $\varepsilon_{\upsilon}^{\varsigma}$ in equation (24) has the form $\varepsilon_{\upsilon}^{\varsigma} = \theta^{\varsigma} e^{iq\upsilon\rho}$ ($q = \frac{2\pi l}{L}$, $l \in \mathbb{Z}$). Substituting this relation into (24), we derive

$$
\left\{
\begin{array}{l}
\left[a_{\upsilon-1}e^{-iq\rho} + b_{\upsilon} + c_{\upsilon+1}e^{iq\rho}\right]\theta^{1} = -\varphi_{\alpha,\xi}\theta^{0}, \quad \varsigma = 1, \\
\left[a_{\upsilon-1}e^{-iq\rho} + b_{\upsilon} + c_{\upsilon+1}e^{iq\rho}\right]\theta^{\varsigma} \\
\quad = \varphi_{\alpha,\xi}\sum_{\omega=1}^{\varsigma-1}\left(\chi_{\omega+1}^{(\alpha)} - \chi_{\omega}^{(\alpha)}\right)\theta^{\varsigma-\omega} - \varphi_{\alpha,\xi}\chi_{\varsigma}^{(\alpha)}\theta^{0}, \quad \varsigma \geq 2.
\end{array}
\right. \tag{26}
$$

Lemma 3.1. *The coefficients $\chi_{\omega}^{(\alpha)}$ of relation (17) satisfy*

(1) $\chi_\omega^{(\alpha)} > 0, \quad \omega = 1, 2, \ldots,$
(2) $1 = \chi_1^{(\alpha)} > \chi_2^{(\alpha)} > \ldots > \chi_\omega^{(\alpha)}, \quad \chi_\omega^{(\alpha)} \to 0, \quad as\ \omega \to +\infty.$
(3) $\sum_{\omega=1}^{\varsigma-1}(\chi_\omega^{(\alpha)} - \chi_{\omega+1}^{(\alpha)}) + \chi_\varsigma^{(\alpha)} = 1.$

Proof. See [25]. □

Lemma 3.2. *Suppose θ^ς $(\varsigma = 1, 2, \ldots, M)$ is the solution* (26), *we have*

$$|\theta^\varsigma| \le \left|\theta^0\right|, \quad \varsigma = 1, 2, \ldots, M.$$

Proof. We can write equation (26) by using $\sin^2(\frac{q\rho}{2}) = -\frac{1}{4}(e^{iq\rho} - 2 + e^{-iq\rho})$ as follows

$$\left[-4\zeta_\upsilon \sin^2(\frac{q\rho}{2}) + 2i\beta_\upsilon r \sin(q\rho) - r - \varphi_{\alpha,\xi}\right]\theta^1 = -\varphi_{\alpha,\xi}\theta^0,$$

where $\zeta_\upsilon, \beta_\upsilon, r, \varphi_{\alpha,\xi} > 0$, $(\upsilon = 1, 2, \ldots, N)$ imply

$$\left|-4\zeta_\upsilon \sin^2(\frac{q\rho}{2}) + 2i\beta_\upsilon r \sin(q\rho) - r - \varphi_{\alpha,\xi}\right| \ge \varphi_{\alpha,\xi},$$

that results $|\theta^1| \le |\theta^0|$. We suppose that $|\theta^n| \le |\theta^0|$, $n = 2, 3, \ldots, \varsigma - 1$ and apply the mathematical induction to prove $|\theta^\varsigma| \le |\theta^0|$. With equation (26) for $j \ge 2$ and Lemma 3.1, we have

$$\begin{aligned}
&\left|\left[-4\zeta_\upsilon \sin^2(\frac{q\rho}{2}) + 2i\beta_\upsilon r \sin(q\rho) - r - \varphi_{\alpha,\xi}\right]\theta^\varsigma\right| \\
&= \left|\varphi_{\alpha,\xi}\sum_{\omega=1}^{\varsigma-1}\left(\chi_{\omega+1}^{(\alpha)} - \chi_\omega^{(\alpha)}\right)\theta^{\varsigma-\omega} - \varphi_{\alpha,\xi}\chi_\varsigma^{(\alpha)}\theta^0\right| \\
&\le \varphi_{\alpha,\xi}\left|\sum_{\omega=1}^{\varsigma-1}\left(\chi_{\omega+1}^{(\alpha)} - \chi_\omega^{(\alpha)}\right)\theta^{\varsigma-\omega}\right| + \varphi_{\alpha,\xi}\chi_\varsigma^{(\alpha)}\left|\theta^0\right| \\
&\le \varphi_{\alpha,\xi}\sum_{\omega=1}^{\varsigma-1}\left(\chi_\omega^{(\alpha)} - \chi_{\omega+1}^{(\alpha)}\right)\left|\theta^{\varsigma-\omega}\right| + \varphi_{\alpha,\xi}\chi_\varsigma^{(\alpha)}\left|\theta^0\right| \\
&\le \varphi_{\alpha,\xi}\left[\sum_{\omega=1}^{\varsigma-1}\left(\chi_\omega^{(\alpha)} - \chi_{\omega+1}^{(\alpha)}\right) + \chi_\varsigma^{(\alpha)}\right]\left|\theta^0\right| = \varphi_{\alpha,\xi}\left|\theta^0\right|,
\end{aligned}$$

then $|\theta^\varsigma| \le |\theta^0|$. □

Theorem 3.3. *The IDS* (22)-(23) *is unconditionally stable.*

Proof. By applying Lemma 3.1 and 3.2 in (25), we obtain

$$\|\varepsilon^\varsigma\|_2^2 = L\sum_{\upsilon=-\infty}^{+\infty}|\theta_\upsilon^\varsigma|^2 \le L\sum_{\upsilon=-\infty}^{+\infty}\left|\theta_\upsilon^0\right|^2 = \left\|\varepsilon^0\right\|_2^2.$$

That is, $\|\varepsilon^\varsigma\|_2 \le \|\varepsilon^0\|_2$ for $\varsigma = 1, 2, \ldots, M$. Hence, the proof is finished. □

4. **Convergence analysis of IDS**

We discuss the convergence of the IDS (22)-(23). We assume that U_υ^ς and $\widetilde{U}_\upsilon^\varsigma$ are solutions of (20)-(21) and (22)-(23), respectively. Subtracting (20) from (22), we get the following equation

$$(27)\quad \begin{cases} a_{\upsilon-1}E_{\upsilon-1}^1 + b_\upsilon E_\upsilon^1 + c_{\upsilon+1}E_{\upsilon+1}^1 = -\varphi_{\alpha,\xi}E_\upsilon^0 + R_\upsilon^1, \quad \varsigma = 1, \\ a_{\upsilon-1}E_{\upsilon-1}^\varsigma + b_\upsilon E_\upsilon^\varsigma + c_{\upsilon+1}E_{\upsilon+1}^\varsigma \\ \quad = \varphi_{\alpha,\xi}\sum_{\omega=1}^{\varsigma-1}\left(\chi_{\omega+1}^{(\alpha)} - \chi_\omega^{(\alpha)}\right)E_\upsilon^{\varsigma-\omega} - \varphi_{\alpha,\xi}\chi_\varsigma^{(\alpha)}E_\upsilon^0 + R_\upsilon^\varsigma, \quad \varsigma \geq 2, \end{cases}$$

where $E_\upsilon^\varsigma = U_\upsilon^\varsigma - \widetilde{U}_\upsilon^\varsigma$ and $R_\upsilon^\varsigma = O(\xi^{2-\alpha} + \rho^2)$, $\upsilon = 0, 1, \ldots, N$; $\varsigma = 0, 1, \ldots, M$. Applying equation (21) and (23), we have

$$\begin{cases} E_0^\varsigma = 0, \quad \varsigma = 0, 1, \ldots, M, \\ E_\upsilon^0 = 0, \quad \upsilon = 0, 1, \ldots, N. \end{cases}$$

Now, we multiply $\varpi := \varphi_{\alpha,\xi}^{-1}$ on both sides (27) and rewrite as follows

$$(28)\quad \begin{cases} \varpi a_{\upsilon-1}E_{\upsilon-1}^1 + \varpi b_\upsilon E_\upsilon^1 + \varpi c_{\upsilon+1}E_{\upsilon+1}^1 = -E_\upsilon^0 + \xi^\alpha R_\upsilon^1, \quad \varsigma = 1, \\ \varpi a_{\upsilon-1}E_{\upsilon-1}^\varsigma + \varpi b_\upsilon E_\upsilon^\varsigma + \varpi c_{\upsilon+1}E_{\upsilon+1}^\varsigma \\ \quad = \sum_{\omega=1}^{\varsigma-1}\left(\chi_{\omega+1}^{(\alpha)} - \chi_\omega^{(\alpha)}\right)E_\upsilon^{\varsigma-\omega} - \chi_\varsigma^{(\alpha)}E_\upsilon^0 + \xi^\alpha R_\upsilon^\varsigma, \quad \varsigma \geq 2. \end{cases}$$

Similar to stability analysis, we first define the following two grid functions

$$E^\varsigma(z) = \begin{cases} E_\upsilon^\varsigma, \quad z \in \left(z_\upsilon - \frac{\rho}{2}, z_\upsilon + \frac{\rho}{2}\right] \bigcup \left(1 - \frac{\rho}{2}, 1\right], \\ 0, \quad z \in \left[0, \frac{\rho}{2}\right], \end{cases}$$

and

$$R^\varsigma(z) = \begin{cases} R_\upsilon^\varsigma, \quad z \in \left(z_\upsilon - \frac{\rho}{2}, z_\upsilon + \frac{\rho}{2}\right] \bigcup \left(1 - \frac{\rho}{2}, 1\right], \\ 0, \quad z \in \left[0, \frac{\rho}{2}\right], \end{cases}$$

for $\varsigma = 0, 1, \ldots, M$. Then, we can write the Fourier series expansion of $E^\varsigma(z)$ and $R^\varsigma(z)$:

$$E^\varsigma(z) = \sum_{\upsilon=-\infty}^{+\infty} \vartheta_\upsilon^\varsigma e^{i\frac{2\pi\upsilon z}{L}} \quad (i^2 = -1), \quad \varsigma = 0, 1, \ldots, M,$$

and

$$R^\varsigma(z) = \sum_{\upsilon=-\infty}^{+\infty} \nu_\upsilon^\varsigma e^{i\frac{2\pi\upsilon z}{L}} \quad (i^2 = -1), \quad \varsigma = 0, 1, \ldots, M,$$

where $L = 1$, $\vartheta_\upsilon^\varsigma = \frac{1}{L}\int_0^L E^\varsigma(z)e^{i\frac{2\pi\upsilon z}{L}}dz$, and $\nu_\upsilon^\varsigma = \frac{1}{L}\int_0^L R^\varsigma(z)e^{i\frac{2\pi\upsilon z}{L}}dz$, $\upsilon \in \mathbb{Z}$. Let $E^\varsigma = (E_1^\varsigma, E_2^\varsigma, \ldots, E_N^\varsigma)^t$ and $R^\varsigma = (R_1^\varsigma, R_2^\varsigma, \ldots, R_N^\varsigma)^t$, then define the following norms

$$\|E^{\varsigma}\|_2^2 = \sum_{\upsilon=1}^{N} \rho \left|E_{\upsilon}^{\varsigma}\right|^2 = \int_0^L \left|E^{\varsigma}(z)\right|^2 dz = \|E^{\varsigma}(z)\|_{L^2}^2, \quad \varsigma = 0, 1, \ldots, M,$$

and

$$\|R^{\varsigma}\|_2^2 = \sum_{\upsilon=1}^{N} \rho \left|R_{\upsilon}^{\varsigma}\right|^2 = \int_0^L \left|R^{\varsigma}(z)\right|^2 dz = \|R^{\varsigma}(z)\|_{L^2}^2, \quad \varsigma = 0, 1, \ldots, M.$$

Applying the Parseval identity, we have

$$\|E^{\varsigma}(z)\|_{L^2}^2 = L \sum_{\upsilon=-\infty}^{+\infty} \left|\vartheta_{\upsilon}^{\varsigma}\right|^2, \quad \varsigma = 0, 1, \ldots, M,$$

$$\|R^{\varsigma}(z)\|_{L^2}^2 = L \sum_{\upsilon=-\infty}^{+\infty} \left|\nu_{\upsilon}^{\varsigma}\right|^2, \quad \varsigma = 0, 1, \ldots, M,$$

which leads to

$$\|E^{\varsigma}\|_2^2 = \sum_{\upsilon=1}^{N} \rho \left|E_{\upsilon}^{\varsigma}\right|^2 = L \sum_{\upsilon=-\infty}^{+\infty} \left|\vartheta_{\upsilon}^{\varsigma}\right|^2, \quad \varsigma = 0, 1, \ldots, M, \tag{29a}$$

$$\|R^{\varsigma}\|_2^2 = \sum_{\upsilon=1}^{N} \rho \left|R_{\upsilon}^{\varsigma}\right|^2 = L \sum_{\upsilon=-\infty}^{+\infty} \left|\nu_{\upsilon}^{\varsigma}\right|^2, \quad \varsigma = 0, 1, \ldots, M. \tag{29b}$$

Suppose E_{υ}^{ς} and R_{υ}^{ς} in equation (28) have the forms $E_{\upsilon}^{\varsigma} = \vartheta^{\varsigma} e^{iq\upsilon\rho}$ and $R_{\upsilon}^{\varsigma} = \nu^{\varsigma} e^{iq\upsilon\rho}$ ($q = \frac{2\pi l}{L}$, $l \in \mathbb{Z}$), respectively. By inserting these relations into (28) results

$$\begin{cases} \left[\varpi a_{\upsilon-1} e^{-iq\rho} + \varpi b_{\upsilon} + \varpi c_{\upsilon+1} e^{iq\rho}\right] \vartheta^1 = \xi^{\alpha} \nu^1, \quad \varsigma = 1, \\ \left[\varpi a_{\upsilon-1} e^{-iq\rho} + \varpi b_{\upsilon} + \varpi c_{\upsilon+1} e^{iq\rho}\right] \vartheta^{\varsigma} \\ \qquad = \sum_{\omega=1}^{\varsigma-1} \left(\chi_{\omega+1}^{(\alpha)} - \chi_{\omega}^{(\alpha)}\right) \vartheta^{\varsigma-\omega} + \xi^{\alpha} \nu^{\varsigma}, \quad \varsigma \geq 2. \end{cases} \tag{30}$$

Using $\sin^2(\frac{q\rho}{2}) = -\frac{1}{4}(e^{iq\rho} - 2 + e^{-iq\rho})$, we can rewrite the above formula as follows

$$\begin{cases} \left[-4\varpi\zeta_{\upsilon} \sin^2(\frac{q\rho}{2}) + 2i\varpi\beta_{\upsilon} r \sin(q\rho) - \varpi r - 1\right] \vartheta^1 = \xi^{\alpha} \nu^1, \quad \varsigma = 1, \\ \left[-4\varpi\zeta_{\upsilon} \sin^2(\frac{q\rho}{2}) + 2i\varpi\beta_{\upsilon} r \sin(q\rho) - \varpi r - 1\right] \vartheta^{\varsigma} \\ \qquad = \sum_{\omega=1}^{\varsigma-1} \left(\chi_{\omega+1}^{(\alpha)} - \chi_{\omega}^{(\alpha)}\right) \vartheta^{\varsigma-\omega} + \xi^{\alpha} \nu^{\varsigma}, \quad \varsigma \geq 2. \end{cases}$$

Lemma 4.1. *Suppose that ϑ^{ς} is a solution of* (30)*, then there is a positive constant C_1 such that*

$$|\vartheta^{\varsigma}| \leq C_1 \frac{\xi^{\alpha} \left|\nu^1\right|}{\chi_{\varsigma}^{(\alpha)}}, \quad \varsigma = 1, 2, \ldots, M.$$

Proof. Because of $R_\upsilon^\varsigma = O(\xi^{2-\alpha} + \rho^2)$, there is a positive constant C_2 such that

$$|R_\upsilon^\varsigma| \le C_2 \left(\xi^{2-\alpha} + \rho^2\right), \quad \upsilon = 0, 1, \ldots, N; \quad \varsigma = 0, 1, \ldots, M.$$

According to (29b), we obtain

$$\|R^\varsigma\|_2 \le C_2 \sqrt{L} \left(\xi^{2-\alpha} + \rho^2\right), \quad \varsigma = 1, 2, \ldots, M. \tag{31}$$

With regards to the convergence of the series on the right-hand side of (29b), it yields

$$|\nu^\varsigma| \equiv |\nu_\upsilon^\varsigma| \le C_3 \left|\nu_\upsilon^1\right| \equiv C_3 \left|\nu^1\right|, \quad C_3 \in \mathbb{R}^+, \quad \varsigma = 1, 2, \ldots, M.$$

Due to the

$$\left|-4\varpi\zeta_\upsilon \sin^2\left(\frac{q\rho}{2}\right) + 2i\varpi\beta_\upsilon r \sin(q\rho) - \varpi r - 1\right| \ge 1, \quad (\upsilon = 1, 2, \ldots, N),$$

we get

$$\left|\vartheta^1\right| = \frac{\xi^\alpha \left|\nu^1\right|}{\left|-4\varpi\zeta_\upsilon \sin^2(\frac{q\rho}{2}) + 2i\varpi\beta_\upsilon r \sin(q\rho) - \varpi r - 1\right|} \le \xi^\alpha \left|\nu^1\right| = \frac{\xi^\alpha \left|\nu^1\right|}{\chi_1^{(\alpha)}}.$$

Applying the mathematical induction, we can suppose that $|\vartheta^\omega| \le C_1 \frac{\xi^\alpha |\nu^1|}{\chi_\omega^{(\alpha)}}$, $\omega = 2, 3, \ldots, \varsigma - 1$ and prove $|\vartheta^\varsigma| \le C_1 \frac{\xi^\alpha |\nu^1|}{\chi_\varsigma^{(\alpha)}}$, where $C_1 = \max\{1, C_3\}$,

$$\begin{aligned}
|\vartheta^\varsigma| &= \frac{\left|\sum_{\omega=1}^{\varsigma-1} \left(\chi_{\omega+1}^{(\alpha)} - \chi_\omega^{(\alpha)}\right) \vartheta^{\varsigma-\omega} + \xi^\alpha \nu^\varsigma\right|}{\left|-4\varpi\zeta_\upsilon \sin^2(\frac{q\rho}{2}) + 2i\varpi\beta_\upsilon r \sin(q\rho) - \varpi r - 1\right|} \\
&\le \left|\sum_{\omega=1}^{\varsigma-1} \left(\chi_{\omega+1}^{(\alpha)} - \chi_\omega^{(\alpha)}\right) \vartheta^{\varsigma-\omega} + \xi^\alpha \nu^\varsigma\right| \le \left|\sum_{\omega=1}^{\varsigma-1} \left(\chi_{\omega+1}^{(\alpha)} - \chi_\omega^{(\alpha)}\right) \vartheta^{\varsigma-\omega}\right| + \xi^\alpha |\nu^\varsigma| \\
&\le \sum_{\omega=1}^{\varsigma-1} \left(\chi_\omega^{(\alpha)} - \chi_{\omega+1}^{(\alpha)}\right) \left|\vartheta^{\varsigma-\omega}\right| + C_3 \xi^\alpha \left|\nu^1\right| \\
&\le \sum_{\omega=1}^{\varsigma-1} \left(\chi_\omega^{(\alpha)} - \chi_{\omega+1}^{(\alpha)}\right) \frac{C_1 \xi^\alpha \left|\nu^1\right|}{\chi_{\varsigma-\omega}^{(\alpha)}} + C_3 \xi^\alpha \left|\nu^1\right| \\
&\le \frac{C_1 \xi^\alpha \left|\nu^1\right|}{\chi_\varsigma^{(\alpha)}} \sum_{\omega=1}^{\varsigma-1} \left(\chi_\omega^{(\alpha)} - \chi_{\omega+1}^{(\alpha)}\right) + C_3 \xi^\alpha \left|\nu^1\right| \\
&\le \frac{C_1 \xi^\alpha \left|\nu^1\right|}{\chi_\varsigma^{(\alpha)}} \left(\sum_{\omega=1}^{\varsigma-1} \left(\chi_\omega^{(\alpha)} - \chi_{\omega+1}^{(\alpha)}\right) + \chi_\varsigma^{(\alpha)}\right) = C_1 \frac{\xi^\alpha \left|\nu^1\right|}{\chi_\varsigma^{(\alpha)}}.
\end{aligned}$$

The proof is finished. □

Theorem 4.2. *Suppose that $U(\tau, z)$ and $\widetilde{U}(\tau, z)$ are solutions of* (15) *and* (22)*, respectively. Then the proposed IDS* (22) *is convergent, and the convergence order is $O(\xi^{2-\alpha} + \rho^2)$.*

Proof. Since $\frac{1}{\chi_\varsigma^{(\alpha)}\varsigma^\alpha} \le \frac{1}{1-\alpha}$, $\varsigma = 1, 2, \dots, M$, then Lemma 4.1 gives

$$|\vartheta^\varsigma| \le C_1 \frac{\xi^\alpha \left|\nu^1\right|}{\chi_\varsigma^{(\alpha)}} = \frac{C_1}{\chi_\varsigma^{(\alpha)}\varsigma^\alpha}\varsigma^\alpha\xi^\alpha \left|\nu^1\right| \le \frac{C_1}{1-\alpha}(\varsigma\xi)^\alpha \left|\nu^1\right| \le \frac{C_1}{1-\alpha}T^\alpha \left|\nu^1\right|.$$

Also, using (29a), (29b) and (31), we derive

$$\|E^\varsigma\|_2 \le \frac{C_1}{1-\alpha}T^\alpha \left\|R^1\right\|_2 \le \frac{C_1}{1-\alpha}T^\alpha C_2\sqrt{L}\left(\xi^{2-\alpha} + \rho^2\right) = C\left(\xi^{2-\alpha} + \rho^2\right),$$

where $C = \frac{C_1 C_2}{1-\alpha}T^\alpha\sqrt{L}$. The proof is complete. □

5. **Numerical results**

Options are derivative financial instruments that allow traders the right to buy or sell an asset in the future at a specified price. This market can be very volatile due to its sensitivity to various factors such as volatility of the underlying asset price, time to maturity, interest rates, etc. In this section, we see the effect of changes in each of these parameters on the FSLPO price. We present different numerical results to show the performance of the proposed IDS and compare the European and Lookback option prices governed by different FBS models. We focus on the first FBS model to show the effect of different parameters on the European Lookback put option with a floating strike price. All results in this study are performed using MATLAB R2015a and no specific packages are used for the fractional calculus.

Figure 1 shows the European option price under different FBS models at $\alpha = 0.6$, $K = 105$, $T = 3$ months, $r = 0.023$, $\sigma = 0.45$, and $S_{\max} = 3K$. In this figure, the first model has the highest price, and the price of the second model is higher than the price of the third model. The put option holder earns a profit by reducing the price of the underlying asset. Because the first model has the highest price, at the same price, the option price under the first model will be more profitable for the investor. Note that the three fractional models are equal to the classical Black-Scholes model at $\alpha = 1$. Figure 2 displays the price of European FSLPOs under the first, second, and third FBS models, and compares these prices with the European put option price under the first FBS model at $\alpha = 0.7$, $T = 5$ months, $r = 0.016$, and $\sigma = 0.5$. Here, the strike price of the European put option is 70. This figure shows that the Lookback option price is higher than the European option price because the holders of the FSLPOs can exercise the contract at the highest realized price of the asset during a certain time. The first model also has the highest Lookback option price, and the Lookback option price of the second model is higher than the third model. Figure 3 indicates the FSLPO price under the first FBS model for $\alpha = \{0.1, 0.3, 0.5, 0.7, 0.9\}$ with an expiration date of six months and one year, respectively. We see that as the expiration date increases, the value of the FSLPO increases with increasing α. Also, when the expiration date decreases, the value of the FSLPO decreases with increasing α. Hence, at longer maturities, Lookback options will be priced higher under a fractional model with an α close to one. Figure 4 shows the FSLPOs with different α values under the first, second, and third FBS models. We used the parameter set $\sigma = 0.47$, $r = 0.019$, $T = 13$ months, and $N = M = 110$. In this figure, the option price increases as the fractional order derivative parameter α increases.

We investigate the impact of volatility, interest rate, and expiration date changes on the FSLPO price using Figures 5-7. Figure 5 illustrates that the Lookback put option price increases with increasing σ because the probability of making large gains in the future increases with increasing volatility. Volatility is defined as a measure of risk in financial management and investment. In other words, stocks with high volatility impose more risk than stocks with low volatility. And more risk can mean more profit. The high volatility of the base asset means that in a certain time, the stock can fluctuate in a larger range and go up and down. So there is a greater possibility that the derivative contract will change from out-the-money to in-the-money in a shorter period or vice versa. Also, high volatility provides more opportunities to fluctuate from contracts and move between contracts or exit positions to increase the efficiency of strategies. On the other hand, long-term investors prefer less volatility.

Figure 6 demonstrates that the Lookback put option price decreases with decreasing r because interest rate's growth will reduce the current value of cash flows in the future. Figure 7 shows that with increasing the expiration date, the FSLPO price increases. The main reason for this is that the value of the option increases over time.

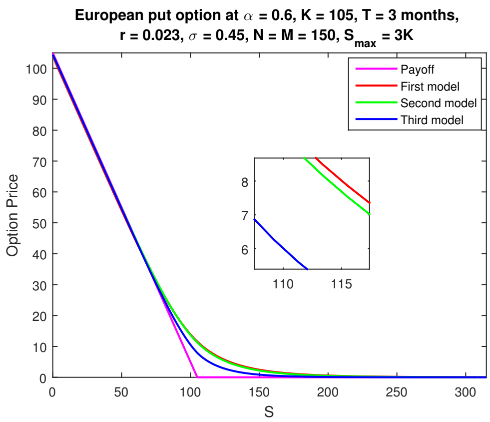


FIGURE 1. European put option prices under three FBS models.

Delta is one of the most important and widely used Greeks, which represents a linear representation of how option prices react to changes in the price of the underlying asset. For example, the contractual Delta is 75 percent. That is, if

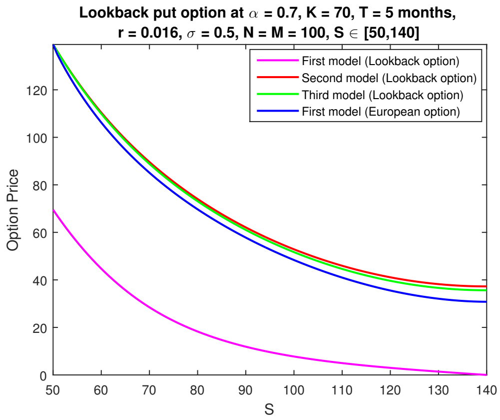


FIGURE 2. European put option price under the first FBS model and European Lookback put option prices under first, second, and third FBS models at $T - t = 5$ months.

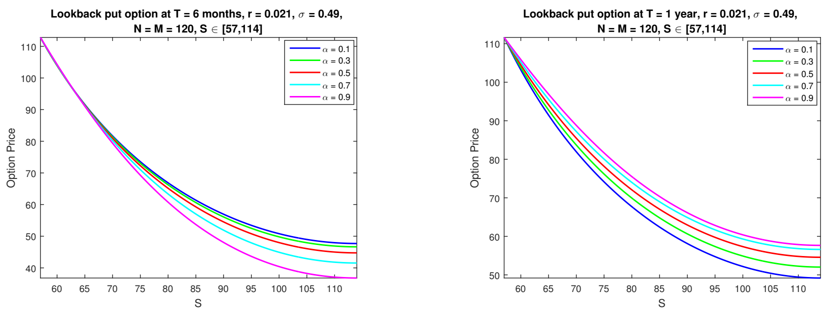


FIGURE 3. European Lookback put option prices under the first FBS model for different α with a maturity time of six months (at $T - t = 6$ months) and one year (at $T - t = 1$ year).

the share price changes by 50 dollars and, for example, from 200 dollars to 250 dollars, the option price must change by 75 percent of this change. 75 percent of 50 dollars becomes 37 dollars. Another use of Delta is in the calculation of leverage. Delta is also used to hedge a position. When the implied volatility decreases, the value of the contract also decreases, and the Delta decrease also occurs. But this Delta decrease is more in at-the-money and out-the-money contracts. Delta is not constant over time, and it is affected by price changes in the base and time decay. In buy positions, as the stock price moves, the Delta changes faster and favors the option holder. This causes the profit or loss to

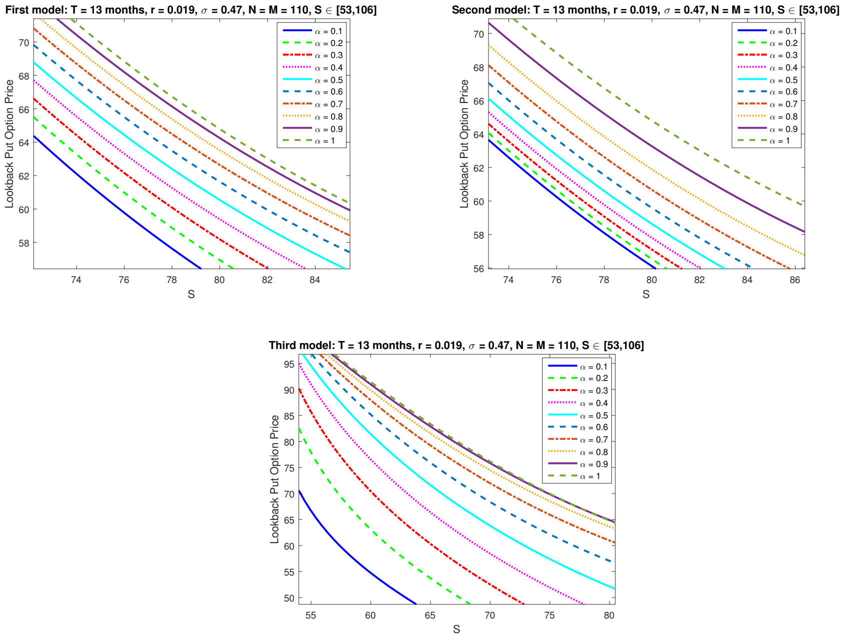


FIGURE 4. European Lookback put option prices under the first, second, and third FBS models for different α with 13 months maturity time (at $T - t = 13$ months).

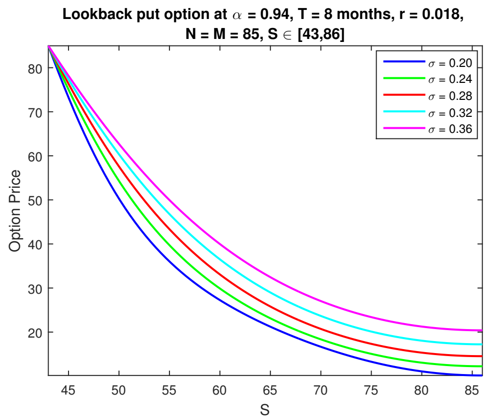


FIGURE 5. European Lookback put option prices under the first FBS model for different σ with 8 months maturity time (at $T - t = 8$ months).

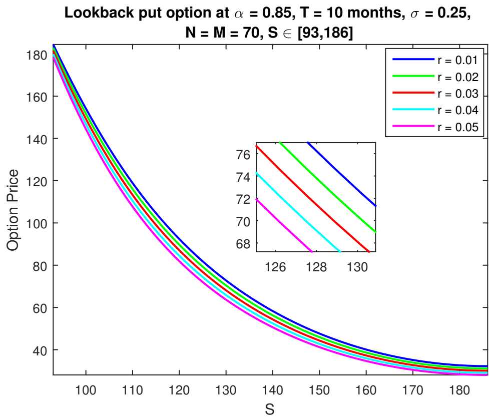


FIGURE 6. European Lookback put option prices under the first FBS model for different r at $T - t = 10$ months.

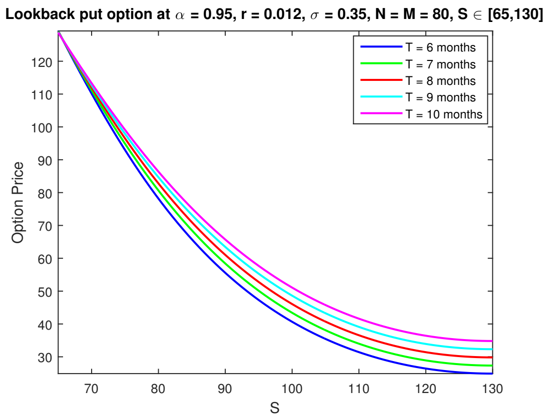


FIGURE 7. European Lookback put option prices under the first FBS model for different maturity time.

increase more rapidly. In sell positions, rapid Delta changes are detrimental to the option seller. This can increase the risk for the seller.

Gamma evaluates the change in the Delta of the option according to the change in the price of the underlying asset. In simpler terms, it shows the acceleration of the option and the slope of the Delta graph. A large value of Gamma indicates that the Delta will change significantly even for small changes in the price of the underlying asset. Gamma is a dynamic concept that changes as the stock price changes and the expiration date approaches. One of the most important uses of Gamma for option sellers is Delta hedging. So the bigger the

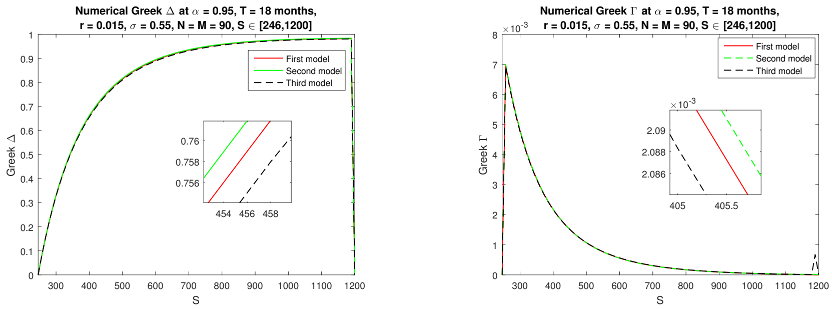


FIGURE 8. Numerical Delta and Gamma Greek of European Lookback call option at $T - t = 18$ months.

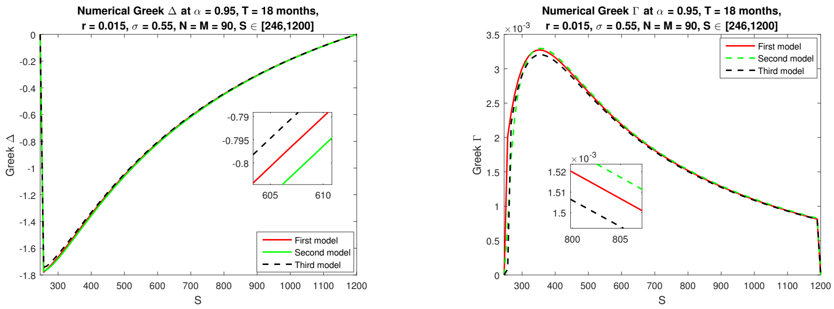


FIGURE 9. Numerical Delta and Gamma Greek of European Lookback put option at $T - t = 18$ months.

Gamma you will lose from the price movement in both directions. Remember that in multiple sell positions, having less Gamma means less momentum during price changes. Risk hedging by taking a buy position in stocks that have high price fluctuations will be profitable regardless of the direction in which the prices move.

Delta is sensitive to stock price changes. If the option Gamma is small (close to zero), that means Delta does not change much due to stock price changes. But if the Gamma is large, then small changes in the stock price will cause minor changes in the trading option. So Delta hedged positions are at risk due to price changes and must be rebalanced frequently in order to maintain a neutral position in relation to Delta.

Figures 8 and 9 show Delta and Gamma for FSLO. Greek indices are tools that are used to measure and compare prices against various changes in the financial market. These indicators help investors to better understand and manage factors related to their portfolio.

We check the spatial convergence rate for the FSLPOs under the first, second, and third FBS models. The convergence rate is computed by

$$\text{Rate} = \log_2 \left(\frac{e^{N,M}}{e^{2N,M}} \right),$$

where $e^{N,M}$ is the maximum norm. We define the following maximum norm

$$e^{N,M} = \max_{\substack{0 \le \upsilon \le N \\ 0 \le \varsigma \le M}} \left\| \bar{U}(\tau_\varsigma, z_\upsilon) - \widehat{U}_\upsilon^\varsigma \right\|_\infty = \max_{0 \le \varsigma \le M} \left(\max_{0 \le \upsilon \le N} \left| \bar{U}(\tau_\varsigma, z_\upsilon) - \widehat{U}_\upsilon^\varsigma \right| \right),$$

where $\widehat{U}$ is the numerical solution of equation (22). Since, equations (1)-(3) have no exact solution, we consider $\bar{U}$ as the approximate solution when M is fixed and sufficiently large. We list the maximum norm error and spatial convergence rate of the proposed scheme for three models in Table 2.

Table 2. Convergence rates of the proposed IDS at $\alpha = 0.9$, $r = 0.01$, $\sigma = 0.5$, $T = 1$, and $M = 100$.

N	First FBS model		Second FBS model		Third FBS model	
	Error	Rate	Error	Rate	Error	Rate
32	0.0468	-	0.0477	-	0.0438	-
64	0.0119	1.9704	0.0122	1.9692	0.0112	1.9740
128	0.0030	1.9925	0.0031	1.9922	0.0028	1.9934
256	7.5120e-04	1.9981	7.6711e-04	1.9980	7.0094e-04	1.9984
512	1.8786e-04	1.9995	1.9184e-04	1.9995	1.7529e-04	1.9996

6. Conclusion

FSLOs are never out-of-the-money because their holders can exercise at the lowest (for floating Lookback call options) or highest (for floating Lookback put options) realized price during the life of the contract. Due to the long memory property in the financial market, we evaluated European Lookback put options by the floating strike price under different FBS models. Because of the complexity of the Lookback option pricing, we introduced an IDS to solve these problems. Also, we investigated the stability and convergence of the IDS using Fourier analysis. Numerical results demonstrate the relationship between the Lookback option price and the parameters.

We know that traders make decisions under the influence of the trend memory effect phenomenon in financial markets. Therefore, they can use the theory presented in this article to price other options under the time-fractional Black-Scholes model. They can also use the theory of this article to estimate implied volatility and calculate Greek letters.

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