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DISCRETE DUALITIES FOR MONTEIRO'S TETRAVALENT MODAL ALGEBRAS

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Abstract. Discrete duality refers to a type of duality in which a class of abstract relational systems serves as the dual counterpart to a class of algebras. These relational systems are called *frames*, following the terminology of non-classical logic. No topology is required in the construction of these frames; hence, they can be considered to have a discrete topology. In 1978, A. Monteiro introduced a class of algebras known as tetravalent modal algebras, which represent a generalization of the three-valued Łukasiewicz algebras defined by Moisil. The theory of these tetravalent modal algebras was initially developed by I. Loureiro, with significant contributions from A. V. Figallo, and was later expanded by the work of J. Font and M. Rius, and more recently by the work of M. Coniglio and M. Figallo. In this paper, we present two discrete dualities for Monteiro's tetravalent modal algebras, each corresponding to a different class of frames and a different complex algebra.

Keywords: De Morgan algebras, frames, discrete duality
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1. Introduction and Preliminaries

A discrete duality represents a relationship between classes of algebras and classes of relational systems (frames). If **Alg** is a class of algebras and **Frm** is a class of frames, establishing a discrete duality between these two classes involves the following steps:

- For each algebra L in **Alg**, associate a canonical frame $\mathcal{X}(\mathcal{C}(L))$ to the algebra and show that it belongs to **Frm**.
- For each frame X in **Frm**, associate a complex algebra $\mathcal{C}(\mathcal{X}(X))$ and show that it belongs to **Alg**.
- Prove two representation theorems:
  - For each $L \in$ **Alg**, there exists an embedding $h : L \hookrightarrow \mathcal{C}(\mathcal{X}(L))$.
  - For each $X \in$ **Frm**, there exists an embedding $k : X \hookrightarrow \mathcal{X}(\mathcal{C}(X))$.

Canonical frames correspond to the dual spaces of algebras in the Priestley-style duality [24]; however, they are not equipped with a topology and can thus be considered as having a discrete topology. In the context of duality theory

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in algebra and logic, a complex algebra is an algebraic structure constructed from a given relational frame [16].

A discrete duality leads to what is referred to as *duality via truth* in [21]. Duality via truth asserts that the notion of truth associated with the algebraic semantics of a formal language determined by the class **Alg** of algebras, and the notion of truth associated with its relational (Kripke-style) semantics determined by the class **Frm** of relational systems, are equivalent. That is, the same formulas hold true in both classes of semantic structures. General principles and applications of discrete duality are outlined in [22].

Tetravalent modal algebras were first introduced by A. Monteiro and later studied in depth by I. Loureiro, A. V. Figallo, P. Landini, A. Ziliani, S. Celani, A. Figallo–Orellano, and I. Pascual (see [2, 3, 7–13, 17–19]). In 2000, J. M. Font and M. Rius [14] showed an interest in the logics stemming from the lattice structures of these algebras. They introduced a sequent calculus (for one of these logics) whose associated sentential logic coincides with the matrix logic defined by the two matrices formed by the four-element tetravalent modal algebra with each of its prime filters. Additionally, the groundbreaking contributions by M. Coniglio and M. Figallo (see [4, 5]) have, in our view, significantly revitalized and advanced the study and development of tetravalent modal algebras.

The main goal of this paper is to establish two discrete dualities for tetravalent modal algebras. To achieve this, we extend the discrete duality given in [6] for De Morgan algebras. This paper is organized as follows: in the remainder of this section, we review a discrete duality for De Morgan algebras. In Section 2, we introduce tetravalent modal algebras along with their basic properties, which will be essential in the following sections. In Sections 3 and 4, we develop discrete dualities for tetravalent modal algebras. Each of these dualities involves a distinct class of frames and a unique definition of a complex algebra.

1.1. **Discrete Duality for De Morgan Algebras.**

In this section, we recall the discrete duality described in [6] for De Morgan algebras.

Recall that an algebra $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra if the reduct $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, and $\sim$ is a unary operation on L satisfying the following identities: $\sim (x \vee y) = \sim x \wedge \sim y$, $\sim\sim x = x$, $\sim 0 = 1$.

A proper lattice filter F of a De Morgan algebra $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ is said to be a *prime filter* if it satisfies the additional condition: If $a \vee b \in F$, then $a \in F$ or $b \in F$.

Given a relational structure $\langle X, \leq\rangle$, where $X \neq \emptyset$ and $\leq$ is a reflexive, antisymmetric, and transitive binary relation on X (i.e., a poset), we denote by $[\leq]U$ the set $\{x \in X : \forall y, x \leq y \Rightarrow y \in U\}$, where U is a subset of X. Additionally, we denote by $[Y)$ $((Y])$ the set $\{x \in X : \exists y \in Y\, y \leq x\}$

$(\{x \in X : \exists y \in Y\, x \leq y\})$, for any $Y \subseteq X$. In particular, if Y is the singleton set $\{x\}$, we will write $[x)$ instead of $[\{x\})$.

A De Morgan frame is a structure $\langle X, \leq, g\rangle$, where $\langle X, \leq\rangle$ is a poset and $g : X \longrightarrow X$ is a function satisfying:

- $g(g(x)) = x$,
- if $x \leq y$, then $g(y) \leq g(x)$.

Let $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ be a De Morgan algebra, and let $\mathcal{X}(L)$ be the set of all prime filters of L (see [1, Definition 3.7]). It is known that $\langle \mathcal{X}(L), \leq^c, g^c\rangle$ is a De Morgan frame, where $\leq^c$ is defined as $\subseteq$, and $g^c : \mathcal{X}(L) \longrightarrow \mathcal{X}(L)$ is the involution defined by

$$g^c(S) = \{x \in L : \sim x \notin S\}, \text{ for all } S \in \mathcal{X}(L). \tag{1}$$

Let $S \in \mathcal{X}(L)$. It is clear that $g^c(S)$ is non-empty set since $\sim 1 = 0 \notin S$. Furthermore, if $\langle X, \leq, g\rangle$ is a De Morgan frame, then

$$\langle \mathcal{C}(X), \cap, \cup, \sim^c, \emptyset, X\rangle$$

is a De Morgan algebra, where $\mathcal{C}(X) = \{U \subseteq X : [\leq]U = U\}$ and $\sim^c : \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$ is defined by

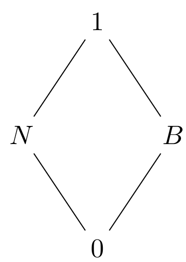
$$\sim^c U = X \setminus g(U), \text{ for every } U \in \mathcal{C}(X). \tag{2}$$

These results enable us to obtain a discrete duality for De Morgan algebras by defining the embeddings as follows:

- $h : L \longrightarrow \mathcal{C}(\mathcal{X}(L))$, defined by $h(a) = \{S \in \mathcal{X}(L) : a \in S\}$,
- $k : X \longrightarrow X(\mathcal{C}(X))$, defined by $k(x) = \{U \in \mathcal{C}(X) : x \in U\}$.

1.2. **Tetravalent Modal Algebras.**

In 1966, L. Monteiro (see [20]) proved that the axiomatization proposed by A. Monteiro for the variety of three-valued Łukasiewicz algebras was independent. To demonstrate the independence of one of these axioms, he considered the four-valued tetravalent modal algebra $T_4 = \langle T, \wedge, \vee, \sim, \Box, 0\rangle$, where the lattice $T = \{1, N, B, 0\}$ is defined as follows:



with the following conditions:

$$\sim N = N, \quad \sim B = B, \quad \sim 0 = 1, \quad \sim 1 = 0,$$

and a unary operator defined by:

$$\Box a = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{if } a \neq 1. \end{cases}$$

Motivated by the previous example, A. Monteiro considered the class of algebras generated by T_4, which he called *tetravalent modal algebras*.

A tetravalent modal algebra is an algebra $\langle L, \wedge, \vee, \sim, \Box, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ such that its non-modal reduct $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra, and the unary operation $\Box$ satisfies the following for all $a \in L$:

(t1) $\Box a \wedge \sim a = 0$,
(t2) $\sim \Box a \wedge a = \sim a \wedge a$.

Equivalently, a tetravalent modal algebra can be defined as an algebra $\langle L, \wedge, \vee, \sim, \Diamond, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ such that its non-modal reduct $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra, and the unary operation $\Diamond$ satisfies the following for all $x \in L$:

(t1)′ $\sim x \vee \Diamond x = 1$,
(t2)′ $\sim x \wedge x = \sim x \wedge \Diamond x$.

On the other hand, let us recall that a 3-valued Łukasiewicz–Moisil algebra (see [1]) is an algebra $\langle L, \vee, \wedge, \sim, \Box, 0, 1\rangle$ such that $\langle L, \vee, \wedge, \sim, 0, 1\rangle$ is a Kleene algebra, and $\Box$ is a unary operation on L satisfying the following conditions

(Ł1) $\Box a \wedge \sim a = 0$,
(Ł2) $\sim a \wedge a = a \wedge \sim \Box a$,
(Ł3) $\Box a \vee \Box b \leq \Box(a \vee b)$.

Next, we will present some results on tetravalent modal algebras that will be necessary for what follows.

Lemma 1.1. ([14]) *In every tetravalent modal algebra $\langle L, \wedge, \vee, \sim, \Box, 0, 1\rangle$ and for all $a, b \in L$, the following hold:*

(t3) $\sim \Box a \vee a = 1$,
(t4) $\Box a \vee \sim a = a \vee \sim a$,
(t5) $\Box a \vee \sim \Box a = 1$,
(t6) $\Box a \wedge \sim \Box a = 0$,
(t7) $\Box a \leq a$,
(t8) $\Box 1 = 1$,
(t9) $\Box 0 = 0$,
(t10) $a \leq b \Rightarrow \Box a \leq \Box b$,
(t11) $\Box\Box a = \Box a$,
(t12) $\Box(\Box a \wedge \Box b) = \Box a \wedge \Box b$,
(t13) $\Box(a \wedge b) = \Box a \wedge \Box b$,
(t14) $\Box \sim \Box a = \sim \Box a$,
(t15) $\Box(a \vee \Box b) = \Box a \vee \Box b$,
(t16) $\sim a \wedge \Box a = 0$.

Proposition 1.2. ([18, Lemma 2.2]) *Let $\langle L, \vee, \wedge, \sim, \Diamond, 0, 1\rangle$ be a tetravalent modal algebra. Then for all $S \in \mathcal{X}(L)$, the following conditions are equivalent:*

(a) $\Diamond a \in S$,

(b) $a \in S$ *or* $a \in g^c(S)$.

Proposition 1.3. ([18, Lemma 2.4]) *On a De Morgan algebra $\langle L, \vee, \wedge, \sim, 0, 1\rangle$, it is possible to define a tetravalent modal algebra structure if and only if the De Morgan frame $\langle \mathcal{X}(L), \leq^c, g^c\rangle$ satisfies the following condition:*

(R) $S \subseteq T \iff S = T$ *or* $g^c(S) = T$.

2. A Discrete Duality for Tetravalent Modal Algebras I

In this section, we establish a discrete duality for tetravalent modal algebras, taking into account the results presented in [18].

Definition 2.1. A Loureiro frame is a De Morgan frame $\langle X, \leq, g\rangle$ satisfying the following additional condition:

(K) If $x \leq y$, then $x = y$ or $g(x) = y$.

Definition 2.2. The complex algebra of a Loureiro frame $\langle X, \leq, g\rangle$ is a structure

$$\langle \mathcal{C}(X), \cup, \cap, \sim^c, \Diamond^c, \emptyset, X\rangle,$$

where $\langle \mathcal{C}(X), \cup, \cap, \sim^c, \emptyset, X\rangle$ is the complex algebra of the De Morgan frame $\langle X, \leq, g\rangle$ and for any $U \in \mathcal{C}(X)$, $\Diamond^c(U) = U \cup g(U)$.

Definition 2.3. The canonical Loureiro frame of a tetravalent modal algebra $\langle L, \wedge, \vee, \sim, \Diamond, 0, 1\rangle$ is

$$\langle \mathcal{X}(L), \leq^c, g^c\rangle,$$

where $\langle \mathcal{X}(L), \leq^c, g^c\rangle$ is the canonical frame of the De Morgan reduct of L.

Lemma 2.4. *The canonical Loureiro frame of a tetravalent modal algebra is a Loureiro frame.*

Proof. This is a direct consequence of the results established in Subsection 1.1 and Proposition 1.3. □

The following result is necessary for the proof of Lemma 2.4.

Lemma 2.5. *Let $\langle X, \leq, g\rangle$ be a Loureiro frame. Then, the set $\mathcal{C}(X)$ is closed under the operation $\Diamond^c$.*

Proof. We will prove that $\mathcal{C}(X)$ is closed under the operation $\Diamond^c$, i.e., for any $U \in \mathcal{C}(X)$, $[\leq](U \cup g(U)) = (U \cup g(U))$. The inclusion $\subseteq$ follows from the reflexivity of $\leq$. Now, we will show that $U \cup g(U) \subseteq [\leq](U \cup g(U))$. Let $x, y \in X$ such that $x \in U \cup g(U)$ and $x \leq y$. Then, $x \in U$ or $x \in g(U)$. In the first case, since $U = [\leq]U$, we have that $y \in U$. In the other case, applying (K), we have $x = y$ or $g(x) = y$, from which it follows in both cases that $y \in U \cup g(U)$. □

3. A Discrete Duality for Tetravalent Modal Algebras II

In this section, we describe a discrete duality for tetravalent modal algebras, based on the duality indicated in Subsection 1.1 for De Morgan algebras. To this end, we introduce the following definitions:

Definition 3.1. A structure $\langle X, \leq, g, R\rangle$ is a tetravalent modal frame if $\langle X, \leq, g\rangle$ is a De Morgan frame and R is a binary relation on X such that

(K1) R is reflexive,
(K2) $(\leq \circ R \circ \leq) \subseteq R$,
(K3) If $(x, y) \in R$, then $x \leq y$ or $g(x) \leq y$.
(K4) $g(x) \in R(x)$, for any $x \in X$.

Remark 3.2. If, in the previous definition, the De Morgan frame $\langle X, \leq, g\rangle$ is a *Kleene frame*, that is, a De Morgan frame satisfying the condition $x \leq g(x)$ or $g(x) \leq x$ for all $x \in X$, we obtain the notion of a *3-valued Łukasiewicz–Moisil frame* as defined in [23].

Definition 3.3. The complex algebra of a tetravalent modal frame $\langle X, \leq, g, R\rangle$ is a structure

$$\langle \mathcal{C}(X), \cup, \cap, \sim^c, \Box^c, \emptyset, X\rangle,$$

where $\langle \mathcal{C}(X), \cup, \cap, \sim^c, \emptyset, X\rangle$ is the complex algebra of the De Morgan frame $\langle X, \leq, g\rangle$, and $\Box^c(U) = \{x \in X : R(x) \subseteq U\}$ for all $U \in \mathcal{C}(X)$.

Definition 3.4. A canonical tetravalent modal frame of a tetravalent modal algebra $\langle L, \vee, \wedge, \sim, \Box, 0, 1\rangle$ is a structure

$$\langle \mathcal{X}(L), \leq^c, g^c, R^c\rangle,$$

where:

(a) $\langle \mathcal{X}(L), \leq^c, g^c\rangle$ is the canonical frame associated with $\langle L, \vee, \wedge, \sim, 0, 1\rangle$,
(b) R^c is a binary relation on $\mathcal{X}(L)$,
(c) $(S, T) \in R^c \iff \Box^{-1}(S) \subseteq T$.

Lemma 3.5. *The canonical tetravalent modal frame of a tetravalent modal algebra is a tetravalent modal frame.*

Proof. Taking into account the results established in [6], we only need to prove (K1)-(K4).

(K1): Let $S \in \mathcal{X}(L)$ and suppose that $x \in \Box^{-1}(S)$. Then $\Box x \in S$. Since $\Box x \leq x$, we have that $x \in S$. Therefore, $\Box^{-1}(S) \subseteq S$, i.e., $(S, S) \in R^c$.

(K2): Let $(P, F) \in (\leq^c \circ R^c \leq^c)$. Then there exist $T, S \in \mathcal{X}(L)$ such that $P \subseteq T$, $(T, S) \in R^c$, and $S \subseteq F$. From the last two statements, we have that $\Box^{-1}(T) \subseteq F$. Therefore, since $P \subseteq T$, we infer that $(P, F) \in R^c$.

(K3): Let $P, F \in \mathcal{X}(L)$ such that $(P, F) \in R^c$. Suppose that $P \not\subseteq F$ and $g^c(P) \not\subseteq F$. Then $P \cap g^c(P) \not\subseteq F$, because F is a prime filter. So there exists $a \in P \cap g^c(P)$ and $a \notin F$. Then $\sim a \notin P$. Since $\sim a \wedge a = a \wedge \sim \Box a$, and

$a \in P$, we have that $\sim \Box a \notin P$. Thus, $\Box a \in g^c(P)$. Since $\Box a \wedge \sim \Box a = 0$, $\sim \Box a \notin g(P)$, i.e., $\Box a \in P$. Hence, $a \in F$, which contradicts $(P, F) \in R^c$.

(K4): Let $P \in \mathcal{X}(L)$. Suppose that $(P, g^c(P)) \notin R^c$. Then there exists $a \in L$ such that $\Box a \in P$ and $a \notin g^c(P)$. Thus, $\Box a \wedge \sim a = 0 \in P$, which is a contradiction.

□

The following result is necessary for the proof of Lemma 3.7.

Lemma 3.6. *Let $\langle X, \leq, g, R\rangle$ be a tetravalent modal frame. Then, the set $\mathcal{C}(X)$ is closed under the operation $\Box^c$.*

Proof. We will prove that $\mathcal{C}(X)$ is closed under the operation $\Box^c$, i.e., for any $U \in \mathcal{C}(X)$, $[\leq]\Box^c(U) = \Box^c(U)$. The inclusion $\subseteq$ follows from the reflexivity of $\leq$. Assume that $x \in \Box^c(U)$. Let $y \in X$ such that $x \leq y$ and take any $z \in X$ with $(y, z) \in R$. Hence, from the reflexivity of $\leq$ and (K2) we infer that $(x, z) \in R$. So, $z \in U$ and therefore, $x \in [\leq]\Box^c(U)$. Thus, $\Box^c(U) \subseteq [\leq^c]\Box^c(U)$. □

Lemma 3.7. *The complex algebra of a tetravalent modal frame is a tetravalent modal algebra.*

Proof. By the results established in [6], $\mathcal{C}(X)$ is a De Morgan algebra. Additionally, from Lemma 3.7, $\mathcal{C}(X)$ is closed under the operation $\Box^c$. Therefore, it only remains to verify the axioms (t1) and (t2).

(t1): Suppose that $\sim U \cap \Box^c U \neq \emptyset$. Then, there exists $y \in \sim U$ such that $R(y) \subseteq U$. Since $g(y) \in R(y)$, we have that $g(y) \in U$, which is a contradiction. Therefore, $\sim U \cap \Box^c U = \emptyset$.

(t2): First, note that by (K1) it follows that $\Box^c U \subseteq U$. Then, we can infer that $\sim U \cap U \subseteq \sim \Box^c U \cap U$. The other inclusion follows from (K3). Indeed: Suppose that $x \in \sim \Box^c U \cap U$. Then, there exists $z \in R(g(x))$ such that $z \notin U$. From this, it follows that $g(x) \leq z$. This implies that $x \in \sim U$. Therefore, $x \in U \cap \sim U$, which completes the proof.

□

We now show that the embedding $h : L \longrightarrow \mathcal{C}(\mathcal{X}(L))$, defined in Subsection 1.1, preserves $\Box$, i.e.,

Lemma 3.8. *For any $a \in L$, $h(\Box a) = \Box^c(h(a))$.*

Proof. Let $F \in h(\Box a)$; then $\Box a \in F$. Suppose that $P \in \mathcal{X}(L)$ verifies that $(F, P) \in R^c$. Then, $\Box^{-1}(F) \subseteq P$ and thus $a \in P$. Therefore, $F \in \Box^c(h(a))$, from which we infer that $h(\Box a) \subseteq \Box^c(h(a))$. Conversely, assume that $F \in \Box^c(h(a))$. Then for every $P \in \mathcal{X}(L)$, $(F, P) \in R^c$ implies that $P \in h(a)$. Suppose that $\Box a \notin F$. Then $\Box^{-1}(F)$ is a filter and $a \notin \Box^{-1}(F)$. Hence, there exists $T \in \mathcal{X}(L)$ such that $a \notin T$ and $\Box^{-1}(F) \subseteq T$. This allows us to conclude that $(F, T) \in R^c$. From this, we have that $T \in h(a)$, so $a \in T$, which is a contradiction. Therefore, $h(\Box a) = \Box^c(h(a))$. Thus, by virtue of the results established in [6], the proof is complete. $\Box$

We now show that the order-embedding defined in Subsection 1.1 preserves the relation R, i.e.,

Lemma 3.9. *Let $\langle X, \leq, g, R\rangle$ be a tetravalent modal frame and let $x, y \in X$. Then*

$$(x, y) \in R \text{ if and only if } (k(x), k(y)) \in R^c.$$

Proof. Assume that $(x, y) \in R$ and suppose that $U \in \mathcal{C}(X)$ satisfies $\Box^c U \in k(x)$. Then it is easy to see that $y \in U$ and so, $(k(x), k(y)) \in R^c$. Conversely, let $x, y \in X$ such that $(k(x), k(y)) \in R^c$. Then $\Box^{c-1}(k(x)) \subseteq k(y)$. On the other hand, note that $[\leq](X \setminus (y]) \in \mathcal{C}(X)$ and $y \notin [\leq](X \setminus (y])$. Therefore, $[\leq](X \setminus (y]) \notin k(y)$ and so, $[\leq](X \setminus (y]) \notin \Box^{c-1}(k(x))$. Therefore, $\Box^c([\leq](X \setminus (y])) \notin k(x)$, which implies that $x \notin \Box^c([\leq](X \setminus (y]))$. Then there exists z such that $(x, z) \in R$ and $z \notin [\leq](X \setminus (y])$. From this last assertion, there exists w such that $z \leq w$ and $w \leq y$, which allows us to conclude that $z \leq y$. Hence, by virtue of the reflexivity of $\leq$ and (K2), $(x, y) \in R$ as required. $\Box$

Hence, we have a discrete duality between tetravalent modal frames and tetravalent modal algebras.

Theorem 3.10.

(a) *Every tetravalent modal algebra is embeddable into the complex algebra of its canonical frame.*

(b) *Every tetravalent modal frame is embeddable into the canonical frame of its complex algebra.*

4. Conclusions and Future Work

In this work, we have developed two discrete dualities for tetravalent modal algebras, extending the duality approach for De Morgan algebras. In particular, we presented the duality through Loureiro frames and the duality with tetravalent modal frames, highlighting how these structures can capture modal properties; in one case, we used binary relations that are useful for modal logics. These results extend existing representations and provide a formal framework that unifies both the algebraic properties and modal characteristics of these logical systems.

In future work, we will continue studying discrete dualities in De Morgan algebras with additional operations. It would also be interesting to establish a discrete duality for De Morgan algebras extended with an intuitionistic negation, as defined in [15].

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