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WEAKLY S-PRIME HYPERIDEALS

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Abstract. The purpose of this paper is to present a new expansion class, namely weakly n-ary S-prime hyperideals in Krasner (m,n)-hyperrings. In summary, we give an extension of n-ary S-prime hyperideals. Some results and examples are given to explain the structure of this new concept. Moreover, a version of Nakayama's Lemma is considered on commutative Krasner (m,n)-hyperrings.

Keywords: n-ary multiplicative set, Weakly n-ary S-prime hyperideal, Strongly weakly n-ary S-prime hyperideal.
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1. Introduction

In recent years, prime ideals and their expansions have gained significant attention in commutative algebra, drawing the attention of numerous authors. One of these extensions called S-prime ideals was introduced via a multiplicative subset of a commutative ring in [10]. Afterward, the idea of weakly S-prime ideals was investigated by Almahdi et al. in [1]. This study demonstrated that weakly S-prime ideals exhibits numerous properties similar to those of weakly prime ideals. Let A be a commutative ring, Q an ideal of A and $S \subseteq A$ a multiplicative set. The ideal Q is said to be weakly S-prime if there exists $s \in S$ such that, $0 \neq xy \in Q$ for $x, y \in A$ implies $sx \in Q$ or $sy \in Q$.

The concept of hyperstructures was introduced by the French mathematician F. Marty in 1934 at the 8th Congress of Scandinavian Mathematicians [12]. Marty's work laid the foundation for what would become the study of hyperstructures, which are algebraic structures defined by multi-valued operations known as hyperoperations. These structures generalize classical algebraic concepts by allowing the composition of two elements to yield a set of values rather than a single value. The study of hyperstructures is an active area of research, with ongoing investigations into their properties, classifications, and applications across different mathematical and scientific disciplines. Several books and many papers have been written now on hyperstructures [6,7,15,16,18]. The idea of (m,n)-hyperrings was proposed by Mirvakili and Davvaz in [14]. Moreover, these authors presented the notion of Krasner (m,n)-hyperring as a subclass of

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(m,n)-hyperrings in [13]. Let x_i^j denote the sequence $x_i, x_{i+1}, \cdots, x_j$ for $j > i$. Note that x_i^j is the empty symbol for $j < i$.

Definition 1.1. [13] The triple (A, f, g), or simply A, is a commutative Krasner (m,n)-hyperring with a scalar identity 1_A if

1. (A, f) is a canonical m-ary hypergroup,

2. (A, g) is a n-ary semigroup,

3. $g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \cdots, g(a_1^{i-1}, x_m, a_{i+1}^n))$ for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in A$ and $1 \leq i \leq n$,

4. $g(0, x_2^n) = g(x_2, 0, x_3^n) = \cdots = g(x_2^n, 0) = 0$ for each $x_2^n \in A$,

5. $f(a_1^n) = f(a_{\sigma(1)}^{\sigma(n)})$ for all $\sigma \in \mathbb{S}_n$, the group of all permutations of $\{1, \cdots, n\}$ and for each $a_1^n \in A$,

6. $g(x, 1_A^{(n-2)}) = x$ for each $x \in A$.

A non-empty subset B of A is called a subhyperring of A if (B, f, g) is a Krasner (m,n)-hyperring. Also, a non-empty subset Q of A is said to be a hyperideal if (Q, f) is an m-ary subhypergroup of (A, f) and $g(x_1^{i-1}, Q, x_{i+1}^n) \subseteq Q$, for each $1 \leq i \leq n$ and $x_1^n \in A$. Let $x \in A$. The hyperideal generated by x is denoted by $< x >$ and defined as $< x >= g(A, x, 1^{(n-2)}) = \{g(r, x, 1^{(n-2)}) \mid r \in A\}$ [2] . Also, let Q be a hyperideal of A. Then we define $(Q : x) = \{a \in A \mid g(a, x, 1_A^{(n-2)}) \in Q\}$. The important hyperideals of a commutative Krasner (m,n)-hyperring A such as maximal, n-ary prime and n-ary primary hyperideals were defined in [2]. A hyperideal M of A is maximal if for each hyperideal N in A, $M \subseteq N \subseteq A$ implies $N = M$ or $N = A$. Let $Q \neq A$ be a hyperideal of A. Q is called n-ary prime if $g(Q_1^n) \subseteq Q$ for hyperideals Q_1^n in A implies that $Q_i \subseteq Q$ for some $i \in \{1, \cdots, n\}$. Lemma 4.5 in [2] proves that Q is an n-ary prime hyperideal of A if $g(x_1^n) \in Q$ for all $x_1^n \in A$ implies $x_i \in Q$ for some $i \in \{1, \cdots, n\}$. The radical of Q, denoted by $rad(Q)$, is the intersection of all n-ary prime hyperideals of A containing Q. We define $rad(Q) = A$ if the set of all n-ary prime hyperideals containing Q is empty. Theorem 4.23 in [2] shows that

$$rad(Q) = \left\{ a \in A \mid \left\{ \begin{array}{ll} g(a^{(u)}, 1_A^{(n-u)}) \in Q, & u \leq n \\ g_{(l)}(a^{(u)}) \in Q & u > n, \ u = l(n-1)+1 \end{array} \right\} \right\}.$$

Furthermore, Q is called n-ary primary if $g(x_1^n) \in Q$ implies that $x_i \in Q$ or $g(x_1^{i-1}, 1_A, x_{i+1}^n) \in rad(Q)$ for some $i \in \{1, \cdots, n\}$. Theorem 4.28 in [2] indicates that if Q is an n-ary primary hyperideal in A, then $rad(Q)$ is an n-ary prime hyperideal of A. An element $x \in A$ is invertible if there exists $y \in A$ such that $1_A = g(x, y, 1_A^{(n-2)})$ [2]. In recent years, different types of hyperideals have been proposed in order to let us fully understand the structures of hyperrings in general. There are various ways to generalize the concept of a prime hyperideal in the context of Krasner (m,n)-hyperrings. Hila et al. in [11] studied an extention of prime hyperideals called (k,n)-absorbing hyperideals. Then, this notion was generalized to weakly (k,n)-absorbing hyperideals by Davvaz et al.

in [8]. Let $S \subseteq A$ be non-empty. If $g(s_1^n) \in S$ for all $s_1^n \in S$, then S is called n-ary multiplicative. The notion of S-prime hyperideals which is an extension of the prime hyperideals via an n-ary multiplicative subset of a Krasner (m,n)-hyperring was investigated in [4].

In this paper, motivated by the research work on weakly S-prime ideals, the notion of weakly n-ary S-prime hyperideals is introduced and investigated in a commutative Krasner (m,n)-hyperring A, where $S \subseteq A$ is a multiplicative set. Among many results in this paper, we prove that every n-ary prime hyperideals of A is a weakly n-ary S-prime hyperideal but the converse may not be always true in Example 2.2. It is shown that every weakly n-ary S-prime hyperideal of A is n-ary prime if and only if A is an n-ary hyperintegral domain and every S-prime hyperideal of A is n-ary prime in Theorem 2.7. We obtain that if Q is a strongly weakly n-ary S-prime hyperideal of A that is not n-ary S-prime, then $g(Q^{(n)}) = 0$ in Theorem 2.11. In theorem 2.15, we propose a type of Nakayama's Lemma for strongly weakly n-ary S-prime hyperideals. We indicate that a hyperideal Q satisfying $g(Q^{(n)}) = 0$ may not be a weakly S-prime hyperideal in Example 2.12. Moreover, we conclude that if Q is a strongly weakly n-ary S-prime hyperideal of A that is not n-ary S-prime, then $g(g(s, rad(0), 1_A^{(n-2)}), Q^{(n-1)}) = 0$ for some $s \in S$ in Theorem 2.18. Finally, we study the stability of weakly n-ary S-prime hyperideals in various hyperring-theoric constructions. Throughout this paper, we suppose that A is a commutative Krasner (m,n)-hyperring with a scalar identity 1_A.

2. weakly n-ary S-prime hyperideals

In [4], the notion of n-ary prime hyperideals was generalized to n-ary S-prime hyperideals via an n-ary multiplicative subset in Krasner (m,n)-hyperrings. Assume that Q is a hyperideal of a commutative Krasner (m,n)-hyperring A and $S \subseteq A$ is an n-ary multiplicative set such that $Q \cap S = \varnothing$. The hyperideal Q is called n-ary S-prime if there exists an $s \in S$ such that for every $a_1^n \in A$ with $g(a_1^n) \in Q$, $g(s, a_i, 1_A^{(n-2)}) \in Q$ for some $i \in \{1, \cdots, n\}$. Our purpose here is to introduce and study the notion of weakly n-ary S-prime hyperideals which constitutes a generalization of n-ary S-prime hyperideals. The weakly-version of n-ary S-prime hyperideals is defined as follows.

Definition 2.1. Assume that Q is a hyperideal of A and S is an n-ary multiplicative subset of A such that $Q \cap S = \varnothing$. We say that Q is a weakly n-ary S-prime hyperideal of A if there exists an $s \in S$ such that for all $a_1^n \in A$ if $0 \neq g(a_1^n) \in Q$, we have $g(s, a_i, 1_A^{(n-2)}) \in Q$ for some $i \in \{1, \cdots, n\}$.

Example 2.2. *The set $A = \{0, 1, 2, 3\}$ with following 2-hyperoperation "$\oplus$" is a canonical 2-ary hypergroup*

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	I	3	J
2	2	3	0	1
3	3	J	1	I

where $I = \{0, 1\}$ *and* $J = \{2, 3\}$. *Define a 4-ary operation* g *on* A *as* $g(a_1^4) = 2$ *if* $a_1^4 \in J$ *or* 0 *if otherwise. Then,* $(A, \oplus, g)$ *is a Krasner (2,4)-hyperring (see Example 4.8 in [2]). In the hyperring, the set* $S = \{2, 3\}$ *is 4-ary multiplicative and* $Q = \{0\}$ *is a weakly 4-ary* S*-prime hyperideal but it is not 4-ary prime, because* $g(1, 1, 2, 3) = 0 \in Q$ *while* $1, 2, 3 \notin Q$.

Example 2.3. *Consider the set A={0,1,2}. In [2], it is verified that* (A, f, g) *is a Krasner* $(3, 3)$*-hyperring, where 3-ary hyperoperation* f *and 3-ary operation* g *are defined as*

$f(0, 0, 0) = 0, \quad f(0, 0, 2) = 2, \quad f(0, 1, 1) = 1, \quad f(1, 1, 1) = 1, \quad f(2, 2, 2) = 2,$

$f(0, 0, 1) = 1, \quad f(0, 2, 2) = 2, \quad f(1, 1, 2) = f(1, 2, 2) = f(0, 1, 2) = A,$

$g(1, 1, 1) = 1, \quad g(1, 1, 2) = g(1, 2, 2) = g(2, 2, 2) = 2,$

and $g(0, a_1^2) = 0$ *for* $a_1^2 \in A$.

In the hyperring, the set $S = \{1\}$ *is 3-ary multiplicative and* $Q = \{0, 2\}$ *is a weakly 3-ary* S*-prime hyperideal of* A.

Proposition 2.4. *Let* $S \subseteq A$ *be an* n*-ary multiplicative set,* Q *a hyperideal of* A *with* $Q \cap S = \varnothing$ *and* Q_1^{n-1} *some hyperideals of* A *such that* $Q_j \cap S \neq \varnothing$ *for each* $j \in \{1, \cdots, n-1\}$. *If* Q *is a weakly* n*-ary* S*-prime hyperideal of* A*, then* $g(Q_1^{n-1}, Q)$ *is a weakly* n*-ary* S*-prime hyperideal of* A.

Proof. The proof is straightforward. □

Proposition 2.5. *Let* $S \subseteq A$ *be an* n*-ary multiplicative set and* Q *be a hyperideal of* A *with* $Q \cap S = \varnothing$. *If* $(Q : s)$ *is a weakly* n*-ary prime hyperideal of* A *for some* $s \in S$*, then* Q *is a weakly* n*-ary* S*-prime hyperideal of* A.

Proof. Let $(Q : s)$ be a weakly n-ary prime hyperideal of A for some $s \in S$. Assume that $0 \neq g(a_1^n) \in Q$ for $a_1^n \in A$. Then we have $0 \neq g(a_1^n) \in (Q : s)$ as $Q \subseteq (Q : s)$. Since $(Q : s)$ is a weakly n-ary prime hyperideal of A, we obtain $a_i \in (Q : s)$ for some $i \in \{1, \cdots, n\}$ which implies $g(s, a_i, 1_A^{(n-2)}) \in Q$. Consequently, Q is a weakly n-ary S-prime hyperideal of A. □

Now, we show that the converse of Proposition 2.5 is not true, in general.

Example 2.6. *Consider* $\mathbb{Z}_{12} = \{0, 1, 2, 3, \cdots, 11\}$. *In [11], Example 3.2 verifies that* $A = \mathbb{Z}_{12}/\mathbb{Z}_{12}^*$ *is a Krasner hyperring, where* $\mathbb{Z}_{12}^* = \{1, 5, 7, 11\}$ *is the multiplicative group of units of* $\mathbb{Z}_{12}$. *The set* $S = \{3\mathbb{Z}_{12}^*, 9\mathbb{Z}_{12}^*\}$ *is a 3-ary multiplicative subset of* A. *It is easily seen that* $Q = \langle 0\mathbb{Z}_{12}^* \rangle$ *is a weakly 3-ary* S*-prime hyperideal of* A. *However, the hyperideal* $(Q : s) = \{0\mathbb{Z}_{12}^*, 4\mathbb{Z}_{12}^*\}$ *for each* $s \in S$

is not a weakly 3-ary prime hyperideal of A. Because, $g((2\mathbb{Z}^*_{12})^{(2)}, 3\mathbb{Z}^*_{12}) \in (Q : s)$ *but neither* $g(2\mathbb{Z}^*_{12}, s, 1_A) \in (Q : s)$ *nor* $g(3\mathbb{Z}^*_{12}, s, 1_A) \in (Q : s)$ *for all* $s \in S$.

Theorem 2.7. *Let $S \subseteq A$ be an n-ary multiplicative set. Then every weakly n-ary S-prime hyperideal of A is n-ary prime if and only if A is an n-ary hyperintegral domain and every S-prime hyperideal of A is n-ary prime.*

Proof. The proof is trivial.

□

Proposition 2.8. *Let $S \subseteq T \subseteq A$ be two n-ary multiplicative sets such that for each $t \in T$, there exists $t' \in T$ with $g(t^{(n-1)}, t') \in S$. Then Q is a weakly n-ary T-prime hyperideal of A if and only if Q is a weakly n-ary S-prime hyperideal of A.*

Proof. ($\Longrightarrow$) Assume that $0 \neq g(a_1^n) \in Q$ for some $a_1^n \in A$. Since Q is a weakly n-ary T-prime hyperideal of A, there exists $t \in T$ such that $g(t, a_i, 1^{(n-2)}) \in Q$ for some $i \in \{1, \cdots, n\}$. By the hypothesis, there exists $t' \in T$ such that $g(t^{(n-1)}, t') \in S$. Let $s = g(t^{(n-1)}, t')$. Therefore $g(s, a_i, 1_A^{(n-2)}) = g(g(t^{(n-1)}, t'), a_i, 1_A^{(n-2)}) = g(g(t^{(n-2)}, t', 1_A), g(t, a_i, 1^{(n-2)}), 1_A^{(n-2)}) \in Q$, as needed.

($\Longleftarrow$) Let Q be a weakly n-ary S-prime hyperideal of A and $0 \neq g(a_1^n) \in Q$. Then there exists $s \in S \subseteq T$ such that $g(s, a_i, 1^{(n-2)}) \in Q$ for some $i \in \{1, \cdots, n\}$. This means that Q is a weakly n-ary T-prime hyperideal of A. □

The notion of Krasner (m, n)-hyperring of fractions was introduced and studied in [3].

Theorem 2.9. *Let Q be a hyperideal of A and $S \subseteq A$ be an n-ary multiplicative set such that $Q \cap S = \varnothing$ and $1_A \in S$. Then Q is a weakly n-ary S-prime hyperideal of A if and only if Q is a weakly n-ary $S^\star$-prime hyperideal, where $S^\star = \{a \in A \mid a/1_A$ is invertible in $S^{-1}A\}$.*

Proof. ($\Longrightarrow$) Since Q is a weakly n-ary S-prime hyperideal of A and $S^\star$ is an n-ary multiplicative subset of A containing S, it is proved.

($\Longleftarrow$) Assume that $t \in S^\star$. This means that $t/1_A$ is invertible in $S^{-1}A$. Therefore $G(t/1_A, x/s, 1_A/1_A{}^{(n-2)}) = g(t, x, 1_A^{(n-2)})/g(s, 1_A^{(n-1)}) = 1_A/1_A$ for some $x \in A$ and $s \in S$ which implies

$$\begin{aligned} 0 &\in g(s', f(g(t, x, 1_A^{(n-2)}), -g(s, 1_A^{(n-1)}), 0^{(m-2)}), 1_A^{(n-2)}) \\ &= f(g(s', t, x, 1_A^{(n-3)}), -g(s', s, 1_A^{(n-2)}), 0^{(m-2)}) \end{aligned}$$

for some $s' \in S$. Since $g(s', s, 1_A^{(n-2)}) \in S$, we get

$$g(s', t, x, 1_A^{(n-3)}) \in f(g(s', s, 1_A^{(n-2)}), 0^{(m-1)}) \subseteq S.$$

Assume that $s'' = g(s', x, 1_A^{(n-2)})$. Then we conclude that $s'' \in S^\star$ as

$$\begin{aligned} &G(g(s', x, 1_A^{(n-2)})/1_A, g(t, 1_A^{(n-1)})/g(s', x, t, 1_A^{(n-3)}), 1_A/1_A{}^{(n-2)}) \\ &\quad = g(s', x, t, 1_A^{(n-3)})/g(s', x, t, 1_A^{(n-3)}) \end{aligned}$$

$= 1_A/1_A$.
Hence we have $g(t^{(n-1)}, g(s''^{(n-1)}, g(t, s'', 1_A^{(n-2)}))) = g(g(t, s'', 1_A^{(n-2)})^n) \in S$. Put $t' = g(s''^{(n-1)}, g(t, s'', 1_A^{(n-2)}))$. So $t' \in S^\star$. Since $g(t^{(n-1)}, t') \in S$, we conclude that Q is a weakly n-ary S-prime hyperideal of A by Proposition 2.8. □

Recall from [8] that a hyperideal Q of A is called strongly weakly n-ary prime if $0 \neq g(Q_1^n) \subseteq Q$ for all hyperideals Q_1^n of A implies that $Q_i \subseteq Q$ for some $i \in \{1, \cdots, n\}$. Assume that $S \subseteq A$ is an n-ary multiplicative set satisfying $Q \cap S = \varnothing$. A hyperideal Q of A refers to a strongly weakly n-ary S-prime hyperideal if there exists an $s \in S$ such that for every hyperideals Q_1^n of A if $0 \neq g(Q_1^n) \subseteq Q$, we have $g(s, Q_i, 1_A^{(n-2)}) \subseteq Q$ for some $i \in \{1, \cdots, n\}$. In this case, it is said that Q is associated to s. It is clear that every strongly weakly n-ary S-prime hyperideal of A is a weakly n-ary S-prime hyperideal.

Theorem 2.10. *Assume that Q is a strongly weakly n-ary S-prime hyperideal of A such that Q is associated to s. If $g(a_1^n) = 0$ for $a_1^n \in A$ but $g(s, a_i, 1_A^{(n-2)}) \notin Q$ for all $i \in \{1, \cdots, n\}$, then $g(a_1, \cdots, \widehat{a_{k_1}}, \cdots, \widehat{a_{k_2}}, \cdots, \widehat{a_{k_v}}, \cdots, Q^{(v)}) = 0$ for each $k_1, \cdots, k_v \in \{1, \cdots, n\}$.*

Proof. We use the induction on v. Let $v = 1$. Assume that $g(a_1^{i-1}, Q, a_{i+1}^n) \neq 0$ for some $i \in \{1, \cdots, n\}$. Therefore we get $0 \neq g(a_1^{i-1}, a, a_{i+1}^n) \in Q$ for some $a \in Q$. So we conclude that

$$\begin{aligned}0 &\neq g(a_1^{i-1}, a, a_{i+1}^n)\\ &= f(g(a_1^n), g(a_1^{i-1}, a, a_{i+1}^n), 0^{(m-2)})\\ &= g(a_1^{i-1}, f(a, a_i, 0^{(m-2)}), a_{i+1}^n) \subseteq Q.\end{aligned}$$

Then $g(s, f(a, a_i, 0^{(m-2)}), 1_A^{(n-2)}) = f(g(s, a, 1_A^{(n-2)}), g(s, a_i, 1_A^{(n-2)}), 0^{(m-2)}) \subseteq Q$ which means $g(s, a_i, 1_A^{(n-2)}) \in Q$ or $g(s, a_j, 1_A^{(n-2)}) \in Q$ for some $j \in \{1, \cdots, \widehat{i}, \cdots, n\}$. It follows that $g(s, a_i, 1_A^{(n-2)}) \in Q$ for some $i \in \{1, \cdots, n\}$ which is impossible. Now, suppose that the claim is true for all positive integers that less than v. Suppose on the contrary that

$$g(a_1, \cdots, \widehat{a_{k_1}}, \cdots, \widehat{a_{k_2}}, \cdots, \widehat{a_{k_v}}, \cdots, Q^{(v)}) \neq 0$$

for some $k_1, \cdots, k_v \in \{1, \cdots, n\}$. Without loss of generality, we eliminate a_1^v. So we have $g(a_{v+1}, \cdots, a_n, Q^{(v)}) \neq 0$. Then there exist $x_1^v \in Q$ such that $0 \neq g(a_{v+1}, \cdots, a_n, x_1^v) \in Q$. By induction hypothesis, we conclude that $0 \neq g(f(a_1, x_1, 0^{(m-2)}), \cdots, f(a_v, x_v, 0^{(m-2)}), a_{v+1}^n) \subseteq Q$. By the hypothesis, we get $g(s, f(a_i, x_i, 0^{(m-2)}), 1_A^{(n-2)}) \subseteq Q$ for some $i \in \{1, \cdots, v\}$ or $g(s, a_j, 1_A^{(n-2)}) \in Q$ for some $j \in \{v+1, \cdots, n\}$. Therefore $g(s, a_i, 1_A^{(n-2)}) \in Q$ for some $i \in \{1, \cdots, n\}$ which is a contradiction. Hence we conclude that $g(a_1, \cdots, \widehat{a_{k_1}}, \cdots, \widehat{a_{k_2}}, \cdots, \widehat{a_{k_v}}, \cdots, Q^{(v)}) = 0$ for each $k_1, \cdots, k_u \in \{1, \cdots, n\}$. □

Theorem 2.11. *Let $S \subseteq A$ be an n-ary multiplicative set. If Q is a strongly weakly n-ary S-prime hyperideal of A that is not n-ary S-prime, then $g(Q^{(n)}) = 0$.*

Proof. Let Q be a strongly weakly n-ary S-prime hyperideal of A and Q is associated to s. Suppose on the contrary that $0 \neq g(Q^{(n)})$. We show that Q is an n-ary S-prime hyperideal of A. Let $g(a_1^n) \in Q$ for $a_1^n \in A$. If $0 \neq g(a_1^n) \in Q$, then we have $g(s, a_i, 1_A^{(n-2)}) \in Q$ for some $i \in \{1, \cdots, n\}$. Assume that $g(a_1^n) = 0$. From $0 \neq g(Q^{(n)})$, it follows that there exists $x_1^n \in Q$ such that $g(x_1^n) \neq 0$. By Theorem 2.10, we conclude that $0 \neq g(f(a_1, x_1, 0^{(m-2)}), \cdots, f(a_n, x_n, 0^{(m-2)}) \subseteq Q$. By the hypothesis, we get $f(g(s, a_i, 1_A^{(n-2)}), g(s, x_i, 1_A^{(n-2)}), 0^{(m-2)}) = g(s, f(a_i, x_i, 0^{(m-2)}) \subseteq Q$ for some $i \in \{1, \cdots, n\}$ which means $g(s, a_i, 1_A^{(n-2)}) \in Q$ as $g(s, x_i, 1_A^{(n-2)}) \in Q$. Thus Q is an n-ary S-prime hyperideal of A, a contradiction. Consequently, $g(Q^{(n)}) = 0$. □

Now, Example 2.12 shows that a hyperideal Q satisfying $g(Q^{(n)}) = 0$ may not be a weakly S-prime hyperideal.

Example 2.12. *Consider Krasner $(4, 3)$-hyperring $(\mathbb{Z}_{5^{5u}}, +, \cdot)$, where $+$ and $\cdot$ are usual addition and multiplication and $u > 4$. In the hyperring, $Q = \langle 5^u \rangle$ is not a weakly 5-ary S-prime hyperideal of $\mathbb{Z}_{5^{5u}}$, where $S = \{1_A\}$ since $5.5.5.5.5^{u-4} \in Q$ but $5, 5^{u-4} \notin Q$ while $Q^5 = 0$.*

In view of Theorem 2.11 and Theorem 3.9 in [4], we have Corollary 2.13.

Corollary 2.13. *Let $S \subseteq A$ be an n-ary multiplicative set and Q be a strongly weakly n-ary S-prime hyperideal of A. Then $Q \subseteq rad(0)$ or $g(s, rad(0), 1^{(n-2)}) \subseteq Q$ for some $s \in S$.*

Corollary 2.14 is a version of Theorem 4.4 in [8].

Corollary 2.14. *Let Q be a strongly weakly n-ary prime hyperideal of A but is not n-ary prime hyperideal. Then $g(Q^{(n)}) = 0$.*

Proof. By taking $S = \{1\}$ in Theorem 2.11, it is proved. □

Let R be a commutative ring, M a finitely generated R-module and I an ideal of R contained in the Jacobson radical of R. Then $IM = M$ implies $M = \{0\}$. This is usually called Nakayama's Lemma (for more details see [17]). In Theorem 2.15, we study a version of Nakayama's Lemma for strongly weakly n-ary S-prime hyperideals. In this regard, we need the definition of (m, n)-hypermodule (M, f', g') over Krasner (m, n)-hyperring (A, f, g). Suppose that M is a non-empty set and (A, f, g) is a commutative Krasner (m, n)-hyperring . If (M, f') is an m-ary hypergroup and the map

$$g' : \underbrace{A \times \cdots \times A}_{n-1} \times M \longrightarrow P^*(M)$$

satisfied the following conditions:

(1) $g'(x_1^{n-1}, f'(a_1^m)) = f'(g'(x_1^{n-1}, a_1), \cdots, g'(x_1^{n-1}, a_m))$
(2) $g'(x_1^{i-1}, f(y_1^m), x_{i+1}^{n-1}, a) = f'(g'(x_1^{i-1}, y_1, x_{i+1}^{n-1}, a), \cdots, g'(x_1^{i-1} y_m, x_{i+1}^{n-1}, a))$
(3) $g'(x_1^{i-1}, g(x_i^{i+n-1}), x_{i+m}^{n+m-2}, a) = g'(x_1^{n-1}, g'(x_m^{n+m-2}, a))$
(4) $0 = g'(x_1^{i-1}, 0, x_{i+1}^{n-1}, a)$,

for all $x_1^{n+m-2}, y_1^m \in A$ and $a_1^m, a \in M$ then the triple (M, f', g') is called an (m, n)-hypermodule over (A, f, g) [5].

Theorem 2.15. *Let (M, f', g') be an (m, n)-hypermodule over (A, f, g) and Q be a strongly weakly n-ary S-prime hyperideal of A that is not n-ary S-prime. If $M = g'(Q, 1_A^{(n-2)}, M)$, then $M = \{0\}$.*

Proof. Assume that Q is a strongly weakly n-ary S-prime hyperideal of A but is not n-ary S-prime and $M = g'(Q, 1_A^{(n-2)}, M)$. By Theorem 2.11, we conclude that $g'(g(Q^{(n)}), 1_A^{(n-2)}, M) = \{0\}$. Moreover, we have

$$\begin{aligned} g'(g(Q^{(n)}), 1_A^{(n-2)}, M) &= g'(g(Q^{(n-1)}, 1_A), 1_A^{(n-2)}, g'(Q, 1_A^{(n-2)}, M)) \\ &= g'(g(Q^{(n-1)}, 1_A), 1_A^{(n-2)}, M) \\ &= g'(g(Q^{(n-2)}, 1_A^{(2)}), 1_A^{(n-2)}, g'(Q, 1_A^{(n-2)}, M)) \\ &= g'(g(Q^{(n-2)}, 1_A^{(2)}), 1_A^{(n-2)}, M) \\ &= \cdots \\ &= g'(Q, 1_A^{(n-2)}, g'(Q, 1_A^{(n-2)}, M)) \\ &= g(Q, 1_A^{(n-2)}, M) \\ &= M. \end{aligned}$$

Then we conclude that $M = \{0\}$. □

Theorem 2.16 provides a useful characterization of the strongly weakly n-ary S-prime hyperideal property.

Theorem 2.16. *Let Q be a hyperideal of A and $S \subseteq A$ is an n-ary multiplicative set with $Q \cap S = \varnothing$. Then Q is a strongly weakly n-ary S-prime hyperideal of A if and only if there exists an element $s \in S$ such that for every $a \notin (Q : s)$, either $(Q : a) \subseteq (Q : s)$ or $(Q : a) = (0 : a)$.*

Proof. ($\Longrightarrow$) Assume that Q is a strongly weakly n-ary S-prime hyperideal of A such that Q is associated to s. Let $a \notin (Q : s)$ and $(Q : a) \neq (0 : a)$. Then there exists $x \in (Q : a)$ such that $g(x, a, 1_A^{(n-2)}) \neq 0$ as $(0 : a) \subseteq (Q : a)$. Since $0 \neq g(x, a, 1_A^{(n-2)}) \in Q$ and $g(s, a, 1_A^{(n-2)}) \notin Q$, we get $g(s, x, 1_A^{(n-2)}) \in Q$. Take any $b \in (Q : a)$. So $g(a, b, 1_A^{(n-2)}) \in Q$. Let $0 \neq g(a, b, 1_A^{(n-2)})$. Therefore $g(s, b, 1_A^{(n-2)}) \in Q$ which means $b \in (Q : s)$. If $0 = g(a, b, 1_A^{(n-2)})$, then

$$\begin{aligned} 0 &\neq g(a, x, 1_A^{(n-2)}) \\ &= f(g(a, x, 1_A^{(n-2)}), g(a, b, 1_A^{(n-2)}), 0^{(m-2)}) \\ &= g(a, f(x, b, 0^{(m-2)}), 1_A^{(n-2)}) \in Q. \end{aligned}$$

Since Q is a strongly weakly n-ary S-prime hyperideal of A and $a \notin (Q : s)$, we get $f(g(s, x, 1_A^{(n-2)}), g(s, b, 1_A^{(n-2)}), 0^{(m-2)}) = g(s, f(x, b, 0^{(m-2)}), 1_A^{(n-2)}) \subseteq Q$

which implies $g(s,b,1_A^{(n-2)}) \in Q$ as $g(s,x,1_A^{(n-2)}) \in Q$. This means that $b \in (Q:s)$ and so $(Q:a) \subseteq (Q:s)$.

($\Longleftarrow$) Let $0 \neq g(Q_1^n) \subseteq Q$ for hyperideals Q_1^n of A such that $g(s,Q_i,1_A^{(n-2)}) \nsubseteq Q$ for all $i \in \{1,\cdots,n\}$ and the element s mentioned in the hypothesis. Take any $a_i \in Q_i \backslash (Q:s)$ for $i \in \{1,\cdots,n\}$. Then $g(Q_1^{i-1},a_i,Q_{i+1}^n) \subseteq Q$ which means $g(Q_1^{i-1},1_A,Q_{i+1}^n) \subseteq (Q:a_i)$. Since $g(Q_1^{i-1},1_A,Q_{i+1}^n) \subseteq Q_j \nsubseteq (Q:s)$ for each $j \neq i$, we conclude that $g(Q_1^{i-1},1_A,Q_{i+1}^n) \subseteq (Q:a_i) = (0:a_i)$ which implies $g(Q_1^{i-1},a_i,Q_{i+1}^n) = 0$. Now, let $a_i \in Q_i \cap (Q:s)$ for some $i \in \{1,\cdots,n\}$ and $a_j \in Q_j$ for $j \in \{1,\cdots,\widehat{i},\cdots,n\}$. If $a_j \notin (Q:s)$, then we obtain $g(Q_1^{j-1},a_j,Q_{j+1}^n) = 0$. Let $a_j \in (Q:s)$. By the hypothesis, there exists $x_j \in Q_j$ such that $g(s,x_j,1_A^{(n-2)}) \notin Q$. This means $x_j \notin (Q:s)$ and so $f(a_j,x_j,0^{(m-2)}) \nsubseteq (Q:s)$. Therefore we get $g(f(a_1,x_1,0^{(m-2)}),\cdots,f(a_{i-1},x_{i-1},0^{(m-2)}),a_i,f(a_{i+1},x_{i+1},0^{(m-2)}),\cdots,f(a_n,x_n,0^{(m-2)})) = 0$. Then we conclude that $g(a_1^n) = 0$ and so $g(Q_1^n) = 0$ which is impossible. Thus Q is a strongly weakly n-ary S-prime hyperideal of A. □

Corollary 2.17. *Let Q be a hyperideal of A. Then Q is a strongly weakly n-ary prime hyperideal of A if and only if for every $a \notin Q$, either $(Q:a) = Q$ or $(Q:a) = (0:a)$.*

Proof. By taking $S = \{1\}$ in Theorem 2.16, it is proved. □

Theorem 2.18. *Let Q be a hyperideal of A and $S \subseteq A$ is an n-ary multiplicative set. If Q is a strongly weakly n-ary S-prime hyperideal of A that is not n-ary S-prime, then $g(g(s,rad(0),1_A^{(n-2)}),Q^{(n-1)}) = 0$ for some $s \in S$.*

Proof. Suppose that Q is a strongly weakly n-ary S-prime hyperideal of A. Then there exists an element $s \in S$ such that for every $a \notin (Q:s)$, either $(Q:a) \subseteq (Q:s)$ or $(Q:a) = (0:a)$, by Theorem 2.16. Take any $x \in rad(0)$. If $x \in (Q:s)$, then $g(s,x,1_A^{(n-2)}) \in Q$ and so $g(g(s,rad(0),1_A^{(n-2)}),Q^{(n-1)}) = 0$ by Theorem 2.11. Now, let $x \notin (Q:s)$. This implies that $(Q:x) \subseteq (Q:s)$ or $(Q:x) = (0:x)$. The first case leads to the following contradiction. Since $x \in rad(0)$, there exists $t \in \mathbb{N}$ such that $g(x^{(t)},1_A^{(n-t)}) = 0$ for $t \leq n$ or $g_{(l)}(x^{(t)}) = 0$ for $t = l(n-1)+1$. Assume that t is a minimal integer satisfying the possibilities. If $g(x^{(t)},1_A^{(n-t)}) = 0$ for $t \leq n$, then $g(x^{(t-1)},1_A^{(n-t+1)}) \in (Q:x) \subseteq (Q:s)$ which means $g(g(x^{(t-1)},1_A^{(n-t+1)}),s,1_A^{(n-2)}) = g(x^{(t-1)},s,1_A^{(n-t)}) \in Q$. If $0 \neq g(x^{(t-1)},s,1_A^{(n-t)})$, then we get $g(s,x,1_A^{(n-2)}) \in Q$ which is impossible. Therefore $g(x^{(t-1)},s,1_A^{(n-t)}) = 0$. Assume that u is a minimal integer satisfying $g(s,x^u,1_A^{(n-2)}) = 0$. Assume that $g(g(s,x,1_A^{(n-2)}),q_1^{n-1}) \neq 0$ for some $q_1^{n-1} \in Q$. Since Q is a strongly weakly n-ary S-prime hyperideal of A and $0 \neq g(g(s,x,1_A^{(n-2)}),q_1^{n-1}) = g(g(s,x,1_A^{(n-2)}),f(x^{u-1},g(q_1^{n-1},1_A),0^{(m-2)}),1_A^{(n-2)})$

$\subseteq Q$, we get $g(s, f(x^{u-1}, g(q_1^{n-1}, 1_A), 0^{(m-2)}), 1_A^{(n-2)}) \subseteq Q$. Since $g(s, q_1^{n-1}) \in Q$, we have $0 \neq g(s, x^{(u-1)}, 1_A^{(n-u)}) \in Q$ which implies $g(s, x, 1_A^{(n-2)}) \in Q$ which is a contradiction. If $g_{(l)}(x^{(t)}) = 0$ for $t = l(n-1)+1$, then by using a similar argument, we get a contradiction. In the second case, we conclude that $g(x, Q, 1_A^{(n-2)}) = 0$ as $Q \subseteq (Q : x)$. Thus $g(s, g(x, Q, 1_A^{(n-2)}), Q^{(n-2)}) = g(g(s, x, 1_A^{(n-2)}), Q^{(n-1)}) = 0$ and so $g(g(s, rad(0), 1_A^{(n-2)}), Q^{(n-1)}) = 0$. □

Corollary 2.19. *Let Q_1 and Q_2 be two hyperideals of A and $S \subseteq A$ is an n-ary multiplicative set. If Q_1 and Q_2 are strongly weakly n-ary S-prime hyperideals of A that are not n-ary S-prime, then we have $g(g(s, Q_1, 1_A^{(n-2)}), Q_2^{(n-1)}) = g(g(s, Q_2, 1_A^{(n-2)}), Q_1^{(n-1)}) = 0$ for some $s \in S$.*

Proof. Let Q_1 and Q_2 be two strongly weakly n-ary S-prime hyperideals of A that are not n-ary S-prime. Then we get $Q_1, Q_2 \subseteq rad(0)$ by Theorem 2.11. Therefore we have

$$\begin{aligned} g(g(s, Q_1, 1_A^{(n-2)}), Q_2^{(n-1)}) &= g(Q_1, g(s, Q_2^{(n-1)}), 1_A^{(n-2)}) \\ &\subseteq g(rad(0), g(s, Q_2^{(n-1)}), 1_A^{(n-2)}) \\ &= g(g(s, rad(0), 1_A^{(n-2)}), Q_2^{(n-1)}) = 0, \end{aligned}$$

by Theorem 2.18. Similarly, we can conclude that $g(g(s, Q_2, 1_A^{(n-2)}), Q_1^{(n-1)}) = 0$ □

Corollary 2.20. *Let Q be a strongly weakly n-ary prime hyperideal of A but is not n-ary prime hyperideal. Then $g(rad(0), Q^{(n-1)}) = 0$.*

Proof. By taking $S = \{1\}$ in Theorem 2.18, it is proved. □

3. Stability of weakly n-ary S-prime hyperideals

In this section, we examine the stability of weakly n-ary S-prime hyperideals in various hyperring-theoric constructions.

Theorem 3.1. *Let $S \subseteq A$ be an n-ary multiplicative set with $1_A \in S$. If Q is a weakly n-ary S-prime hyperideal of A, then $S^{-1}Q$ is a weakly n-ary prime hyperideal of $S^{-1}A$.*

Proof. Let $0 \neq G(a_1/s_1, \cdots, a_n/s_n) \in S^{-1}Q$ for $a_1/s_1, \cdots, a_n/s_n \in S^{-1}A$. This means that $g(a_1^n)/g(s_1^n) \in S^{-1}Q$ and so $g(t, g(a_1^n), 1_A^{(n-2)}) \in Q$ for some $t \in S$. Since $0 \neq g(g(t, a_1, 1^{(n-2)}), a_2^n) \in Q$ and Q is a weakly n-ary S-prime hyperideal of A, there exists $s \in S$ such that $g(s, g(t, a_1, 1_A^{(n-2)}), 1_A^{(n-2)}) = g(s, t, a_1, 1_A^{(n-3)}) \in Q$ or $g(s, a_i, 1_A^{(n-2)}) \in Q$ for some $i \in \{2, \cdots, n\}$. Therefore

$$\begin{aligned} G(a_1/s_1, 1_A/1_A{}^{(n-1)}) &= g(a_1, 1_A^{(n-1)})/g(s_1, 1_A^{(n-1)}) \\ &= g(s, t, a_1, 1_A^{(n-3)})/g(s, t, s_1, 1_A^{(n-2)}) \\ &\in S^{-1}Q \end{aligned}$$

or

$$G(a_i/s_i, 1_A/1_A{}^{(n-1)}) = g(a_i, 1_A^{(n-1)})/g(s_i, 1_A^{(n-1)}))$$

$$= g(s, a_i, 1_A^{(n-2)})/g(s, s_i, 1_A^{(n-2)})$$
$$\in S^{-1}Q$$

for some $i \in \{2, \cdots, n\}$. Hence $S^{-1}Q$ is a weakly n-ary prime hyperideal of $S^{-1}A$. □

Example 3.2. *Consider the set $A = \{a, b, c, d, e, f\}$. In [9], Example 3.8, shows that A is a Krasner $(2, 2)$-hyperring with the hyperoperation $\boxplus$ and the multiplication $\boxdot$ defined as follows:*

$\boxplus$	a	b	c	d	e	f
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$
b	$\{b\}$	$\{a, b\}$	$\{d\}$	$\{c, d\}$	$\{f\}$	$\{e, f\}$
c	$\{c\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{a, c, e\}$	$\{b, d, f\}$
d	$\{d\}$	$\{c, d\}$	$\{d\}$	$\{c, d\}$	$\{b, d, f\}$	A
e	$\{e\}$	$\{f\}$	$\{a, c, e\}$	$\{b, d, f\}$	$\{e\}$	$\{f\}$
f	$\{f\}$	$\{e, f\}$	$\{b, d, f\}$	A	$\{f\}$	$\{e, f\}$

and

$\boxdot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	b	a	b	a	b
c	a	a	c	c	e	e
d	a	b	c	d	e	f
e	a	a	e	e	c	c
f	a	b	e	f	c	d

In the hyperring, $S = \{b, d\}$ is a 2-ary multiplicative set. It is easy to check that $Q = \{a, c, e\}$ is a weakly 2-ary S-prime hyperideal of A. Then $S^{-1}Q$ is a weakly 2-ary prime hyperideal of A by Theorem 3.1.

Theorem 3.3. *Let $S \subseteq A$ be an n-ary multiplicative set with $1_A \in S$ and Q be a hyperideal of A with $Q \cap S = \varnothing$. If $S^{-1}Q$ is a weakly n-ary prime hyperideal of $S^{-1}A$ and $S^{-1}Q \cap A = (Q : s)$ for some $s \in S$, then Q is a weakly n-ary S-prime hyperideal of A*

Proof. Let $S^{-1}Q$ be a weakly n-ary prime hyperideal of $S^{-1}A$ and $S^{-1}Q \cap A = (Q : s)$ for some $s \in S$. Suppose that $0 \neq g(a_1^n) \in Q$ for some $a_1^n \in A$. Then we have $0 \neq G(a_1/1_A, \cdots, a_n/1_A) \in S^{-1}Q$. Since $S^{-1}Q$ is a weakly n-ary prime hyperideal of $S^{-1}A$, we get $a_i/1 \in S^{-1}Q$ for some $i \in \{1, \cdots, n\}$ which means $g(t, a_i, 1_A^{(n-2)}) \in Q$ for some $t \in S$. Therefore $a_i = g(t, a_i, 1_A^{(n-2)})/g(t, 1_A^{(n-1)}) \in S^{-1}Q$. This means $a_i \in (Q : s)$. Therefore we have $g(s, a_i, 1_A^{(n-2)}) \in Q$. This shows that I is a weaky n-ary S-prime hyperideal of A. □

Example 3.4. *In Example 3.2, consider the hyperideal $Q = \{a, b\}$ in A and the 2-ary multiplicative subset $S = \{c, d\}$ of A. Since $S^{-1}Q = \{0_{S^{-1}A}\}$ is a weakly n-ary prime hyperideal of $S^{-1}A$ and $S^{-1}Q \cap A = (Q : c) = \{a\}$, we conclude that Q is a weakly 2-ary S-prime hyperideal of A by Theorem 3.3.*

Theorem 3.5. *Let $S \subseteq A$ be an n-ary multiplicative set with $1_A \in S$ and Q be a hyperideal of A with $Q \cap S = \varnothing$. If there exists $t \in S$ satisfying $(Q : s) \subseteq (Q : t)$ for all $s \in S$ and $S^{-1}Q$ is a weakly n-ary prime hyperideal of $S^{-1}A$, then Q is a weakly n-ary S-prime hyperideal of A.*

Proof. Assume that there exists $t \in S$ satisfying $(Q : s) \subseteq (Q : t)$ for all $s \in S$. Let $0 \neq g(a_1^n) \in Q$ for $a_1^n \in A$. Therefore we get $0 \neq G(a_1/1_A, \cdots, a_n/1_A) \in S^{-1}Q$. It follows that $a_i/1 \in S^{-1}Q$ for some $i \in \{1, \cdots, n\}$ as $S^{-1}Q$ is a weakly n-ary prime hyperideal of $S^{-1}A$. Hence $g(s, a_i, 1_A^{(n-2)}) \in Q$ for some $s \in S$ which implies $a_i \in (Q : s) \subseteq (Q : t)$ and so $g(t, a_i, 1_A^{(n-2)}) \in Q$. Consequently, Q is a weakly n-ary S-prime hyperideal of A. □

Theorem 3.6. *Let $S \subseteq A$ be an n-ary multiplicative set with $1_A \in S$. If A is an n-ary hyperintegral domain and $S^{-1}A$ is a hyperfield, then $\langle 0 \rangle$ is the only weakly n-ary S-prime hyperideal of A.*

Proof. Assume that $Q \neq 0$ is a weakly n-ary S-prime hyperideal of A. Take $a \in Q \backslash \{0\}$. So there exists $x \in A \backslash \{0\}$ and $s \in S$ that $G(a/1_A, x/s, 1_A/1_A{}^{(n-2)}) = g(a, x, 1_A^{(n-2)})/g(s, 1_A^{(n-1)}) = 1_A/1_A$ as $S^{-1}A$ is a hyperfield. Then there exists $t \in S$ such that $0 \in g(t, f(g(a, x, 1_A^{(n-2)}), -g(s, 1_A^{(n-1)}), 0^{(m-2)}), 1_A^{(n-2)}) = f(g(t, a, x, 1_A^{(n-3)}), -g(t, s, 1_A^{(n-2)}), 0^{(m-2)})$. Therefore we get $g(t, a, x, 1_A^{(n-3)}) \in f(g(t, s, 1_A^{(n-2)}), 0^{(m-1)}) \subseteq S$. Since $0 \neq g(t, a, x, 1_A^{(n-3)}) \in Q$, we get $Q \cap S \neq \varnothing$ which is impossible. Thus $\langle 0 \rangle$ is the only weakly n-ary S-prime hyperideal of A. □

Recall from [13] that a mapping $h : A_1 \longrightarrow A_2$ is called a homomorphism, where (A_1, f_1, g_1) and (A_2, f_2, g_2) are commutative Krasner (m, n)-hyperrings if for all $a_1^m, b_1^n \in A_1$ we have

(i) $h(f_1(a_1, \cdots, a_m)) = f_2(h(a_1), \cdots, h(a_m))$,

(ii) $h(g_1(b_1, \cdots, b_n)) = g_2(h(b_1), \cdots, h(b_n))$

(iii) $h(1_{A_1}) = 1_{A_2}$.

Theorem 3.7. *Let (A_1, f_1, g_1) and (A_2, f_2, g_2) be two commutative Krasner (m, n)-hyperrings, $h : A_1 \longrightarrow A_2$ a monomorphism and $S \subseteq A_1$ an n-ary multiplicative set. If Q_2 is a weakly n-ary $h(S)$-prime hyperideal of A_2, then $h^{-1}(Q_2)$ is a weakly n-ary S-prime hyperideal of A_1.*

Proof. Assume that Q_2 is a weakly n-ary $h(S)$-prime hyperideal of A_2. Then there exists $s \in S$ such that for all $b_1^n \in A_2$ with $0 \neq g_2(b_1^n) \in Q_2$, we have $g_2(h(s), b_i, 1_{A_2}^{(n-2)}) \in Q_2$ for some $i \in \{1, \cdots, n\}$. Put $Q_1 = h^{-1}(Q_2)$. It is easy to see that $Q_1 \cap S = \varnothing$. Let $0 \neq g_1(a_1^n) \in Q_1$ for $a_1^n \in A_1$. Then $0 \neq h(g_1(a_1^n)) = g_2(h(a_1), ..., h(a_n)) \in Q_2$ as h is a monomorphism. So, we have $g_2(h(s), h(a_i), 1_{A_2}^{(n-2)}) = h(g_1(s, a_i, 1_{A_1}^{(n-2)})) \in Q_2$ for some $i \in \{1, \cdots, n\}$ which implies $g_1(s, a_i, 1_{A_1}^{(n-2)}) \in h^{-1}(Q_2) = Q_1$. Consequently, $h^{-1}(Q_2)$ is a weakly n-ary S-prime hyperideal of A_1. □

Corollary 3.8. *Let $S \subseteq A_1$ be an n-ary multiplicative set. If A_1 is a subhyperring of A_2 and Q_2 is a weakly n-ary S-prime hyperideal of A_2, then $Q_2 \cap A_1$ is a weakly n-ary S-prime hyperideal of A_1.*

Proof. Consider the monomorphism $h : A_1 \longrightarrow A_2$, defined by $h(a) = a$. Since $h^{-1}(Q_2) = Q_2 \cap A_1$, we conclude that $Q_2 \cap A_1$ is a weakly n-ary S-prime hyperideal of A_1, by Theorem 3.7. □

Let P be a hyperideal of a Krasner (m, n)-hyperring (A, f, g). Then the set

$$A/P = \{f(a_1^{i-1}, P, a_{i+1}^m) \mid a_1^{i-1}, a_{i+1}^m \in A\}$$

endowed with m-ary hyperoperation f and n-ary operation g which for all $a_{11}^{1m}, \cdots, a_{m1}^{mm} \in A$

$$f\Big(f(a_{11}^{1(i-1)}, P, a_{1(i+1)}^{1m}), \cdots, f(a_{m1}^{m(i-1)}, P, a_{m(i+1)}^{mm})\Big)$$
$$= f\Big(f(a_{11}^{m1}), \cdots, f(a_{1(i-1)}^{m(i-1)}), P, f(a_{1(i+1)}^{m(i+1)}), \cdots, f(a_{1m}^{mm})\Big)$$

and for all $a_{11}^{1m}, \cdots, a_{n1}^{nm} \in A$

$$g\Big(f(a_{11}^{1(i-1)}, P, a_{1(i+1)}^{1m}), \cdots, f(a_{n1}^{n(i-1)}, P, a_{n(i+1)}^{nm})\Big)$$
$$= f\Big(g(a_{11}^{n1}), \cdots, g(a_{1(i-1)}^{n(i-1)}), P, g(a_{1(i+1)}^{n(i+1)}), \cdots, f(a_{1m}^{nm})\Big)$$

construct a Krasner (m, n)-hyperring, and $(A/P, f, g)$ is called the quotient Krasner (m, n)-hyperring of A by P [2]. In Theorem 3.9, we determine when the hyperideal Q/P is a weakly n-ary $\bar{S}$-prime hyperideal in A/P, where $\bar{S} = \{f(s_1^{i-1}, P, s_{i+1}^n) \mid s_1^{i-1}, s_{i+1}^n \in S\}$.

Theorem 3.9. *Let P and Q be two hyperideals of A with $P \subseteq Q$ and $S \subseteq A$ be an n-ary multiplicative set such that $1_A \in S$, $P \cap S = \varnothing$ and $Q/P \cap \bar{S} = \varnothing$.*

(i) *If Q is a strongly weakly n-ary S-prime hyperideal of A, then Q/P is a weakly n-ary $\bar{S}$-prime hyperideal of A/P.*

(ii) *If P is a weakly n-ary S-prime hyperideal of A and Q/P is a weakly n-ary $\bar{S}$-prime hyperideal of A/P, then Q is a weakly n-ary S-prime hyperideal of A.*

Proof. (i) Let Q be a strongly weakly n-ary S-prime hyperideal of A. Then there exists some $s \in S$ such that if $0 \neq g(a_1^n) \in Q$ for $a_1^n \in A$, then we have $g(s, a_i, 1^{(n-2)}) \in Q$ for some $i \in \{1, \cdots, n\}$. Let

$$0_{A/P} \neq g\Big(f(a_{11}^{1(i-1)}, P, a_{1(i+1)}^{1m}), \cdots, f(a_{n1}^{n(i-1)}, P, a_{n(i+1)}^{nm})\Big) \in Q/P$$

for some $f(a_{11}^{1(i-1)}, P, a_{1(i+1)}^{1m}), \cdots, f(a_{n1}^{n(i-1)}, P, a_{n(i+1)}^{nm}) \in A/P$.
This implies

$$f\Big(g(a_{11}^{n1}), \cdots, g(a_{1(i-1)}^{n(i-1)}), P, g(a_{1(i+1)}^{n(i+1)}), \cdots, g(a_{1m}^{nm})\Big) \in Q/P.$$

Then

$$f\bigg(g(a_{11}^{n1}),\cdots,g(a_{1(i-1)}^{n(i-1)}),0,g(a_{1(i+1)}^{n(i+1)}),\cdots,g(a_{1m}^{nm})\bigg)\subseteq Q$$

which means

$$g\bigg(f(a_{11}^{1(i-1)},0,a_{1(i+1)}^{1m}),\cdots,f(a_{n1}^{n(i-1)},0,a_{n(i+1)}^{nm})\bigg)\subseteq Q.$$

Since $0_A\neq g(f(a_{11}^{1(i-1)},0,a_{1(i+1)}^{1m}),\cdots,f(a_{n1}^{n(i-1)},0,a_{n(i+1)}^{nm}))\subseteq Q$ and Q is a strongly weakly n-ary S-prime hyperideal of A, then we conclude that

$$g\bigg(s,f(a_{j1}^{j(i-1)},0,a_{j(i+1)}^{jm}),1_A^{n-2}\bigg)\subseteq Q.$$

for some $j\in\{1,\cdots,n\}$. Hence

$$f\bigg(g(s,f(a_{j1}^{j(i-1)},0,a_{j(i+1)}^{jm}),1_A^{n-2}),P,0^{(m-2)}\bigg)\in Q/P$$

and so

$$f\bigg(g(g(s,1_A^{(n-2)}),f(a_{j1}^{j(i-1)},0,a_{j(i+1)}^{jm}),1_A^{n-2}),P,0^{(m-2)}\bigg)\in Q/P.$$

Thus we get

$$g\bigg(f(s,P,1_A^{(n-2)}),f(a_{j1}^{j(i-1)},P,a_{j(i+1)}^{jm}),1_{A/P}^{(n-2)}\bigg)\in Q/P.$$

Consequently, Q/P is a weakly n-ary $\bar{S}$-prime hyperideal of A/P.

(ii) Let $0\neq g(a_1^n)\in Q$ for $a_1^n\in A$. If $g(f(a_1,P,0^{(m-2)})\cdots,f(a_n,P,0^{(m-2)}))=0_{A/P}$, then $f(g(a_1^n),P,0^{(m-2)})=0_{A/P}$ and so $0_A\neq g(a_1^n)\in P$. Since P is a weakly n-ary S-prime hyperideal of A, there exitsts $s\in S$ such that $g(s,a_i,1_A^{(n-2)})\in P\subseteq Q$ for some $i\in\{1,\cdots,n\}$. Now, we assume that $g(f(a_1,P,0^{(m-2)})\cdots,f(a_n,P,0^{(m-2)}))\neq 0_{A/P}$. Since Q/P is a weakly n-ary $\bar{S}$-prime hyperideal of A/P, there exist $s_1^{i-1},s_{i+1}^n\in S$ such that

$$g(f(s_1^{i-1},P,s_{i+1}^n),f(a_j,P,0^{(m-2)}),1_{A/P}^{(n-2)}))\in Q/P$$

for some $j\in\{1,\cdots,n\}$. This means $f(g(s_j,a_j,1_A^{(n-2)}),P,0^{(m-2)})\in Q/P$ which implies $g(s_j,a_j,1_A^{(n-2)})\in Q$. Thus, Q is a weakly n-ary S-prime hyperideal of A. □

4. Conclusion

Over the years, various kinds of hyperideals have been defined and analysed in order to let us fully comprehend the structures of hyperrings in general. In this paper, the idea of weakly n-ary S-prime hyperideals was proposed in Krasner (m,n)-hyperring A. This notion as a generalization of (weakly) prime hyperideals was defined via an n-ary multiplicative subset S of A. We have obtained numerous findings that provide a comprehensive explanation for this structure. For instance, we indicated that if Q is a strongly weakly n-ary S-prime hyperideal of A that is not n-ary S-prime, then $g(Q^{(n)}) = 0$. However, it was shown that a hyperideal Q satisfying $g(Q^{(n)}) = 0$ may not be a weakly S-prime hyperideal. Moreover, a type of Nakayama's Lemma was presented for a strongly weakly n-ary S-prime hyperideal of A. This study can be continued in several directions, such as:

(i) to define ϕ-S-prime hyperideals.
(ii) to introduce S-J-hyperideals, where J is the intersection of all maximal hyperideals of A.
(iii) to propose graded weakly S-prime hyperideals.

5. Future work

As a future work, we intend to analyse similar notions in the context of (m,n)-hypermodules.

6. Data Availability Statement

Not applicable.

7. Aknowledgement

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8. Conflict of interest

The authors declare no con ict of interest.

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