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SOME RESULTS ON GENERALIZATION $\alpha-$CHEBYSHEV WAVELETS

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ABSTRACT. In this paper, we introduce generalized formulae for well-known functions such as α-Chebyshev functions. We define $\alpha-$Chebyshev wavelets approximation and generalization $\alpha-$wavelet coapproximation. We show that if $\sum_{n=0}^{k}\sum_{n=0}^{\infty}|t_n|^2 L_{n,m}^{\alpha}$ is convergent, then generalization $\alpha-$Chebyshev wavelets approximation (generalization $\alpha-$ wavelets coapproximation) exists.

Keywords: Generalized $\alpha-$Chebyshev functions, Generalized $\alpha-$Chebyshev wavelets approximation, Generalized $\alpha-$wavelets coapproximation
2020 MSC: 41A65, 41A52, 46N10.

1. Introduction

We define $\alpha-$Chebyshev functions and found the recurrence relations for these functions

Definition 1.1. We can define some kinds of Chenyshev functions, where $x = cos\theta$ and $\alpha \geq 0$

The $\alpha-$ Chebyshev functions for $|x| \leq 1$ (see [1-3, 7]):

$Kinds$	$\alpha - Chebyshev\ functions$
$First - Kind\ \alpha - Chebyshev\ Function$	$T_n^{\alpha}(x) = cos(n+\alpha)\theta$
$Second - Kind\ \alpha - Chebyshev\ Function$	$U_n^{\alpha}(x) = \frac{sin(n+1-\alpha)\theta}{sin\theta}$
$Third - Kind\ \alpha - Chebyshev\ Function$	$V_n^{\alpha}(x) = \frac{cos(n+\alpha)\theta}{cos\theta}$
$Fourth\ Kind\ \alpha - Chebyshev\ Function$	$W_n^{\alpha}(x) = sin(n+\alpha)\theta$

Lemma 1.2. *For $n \geq 1$*

i) $T_{n+1}^{\alpha}(x) = 2xT_n^{\alpha}(x) - T_{n-1}^{\alpha}(x)$, $T_0^{\alpha}(x) = cos(\alpha cos x) = u_1(x)$ *and* $T_1^{\alpha}(x) = u_1(x)(2x-1)$,

ii) $U_{n+1}^{\alpha}(x) = 2xU_n^{\alpha}(x) - U_{n-1}^{\alpha}(x)$, $U_0^{\alpha}(x) = \frac{sin(1-\alpha)(cosx)}{sin(cosx)} = u_2(x)$ *and* $U_1^{\alpha}(x) = u_2(x)(2x+1)$,

iii) $V_{n+1}^{\alpha}(x) = 2xV_n^{\alpha}(x) - V_{n-1}^{\alpha}(x)$, $V_0^{\alpha}(x) = \frac{cos(\alpha cosx)}{cos(cosx)}u_3(x)$ *and* $V_1^{\alpha}(x) = u_3(x)(2x-1)$,

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iv) $W^{\alpha}_{n+1}(x) = 2xW^{\alpha}_{n}(x) - W^{\alpha}_{n-1}(x)$, $W^{\alpha}_{0}(x) = sin(\alpha cosx) = u_4(x)$ *and* $W^{\alpha}_{1}(x) = u_4(x)(2x+1)$.

Proof. i) We have $T^{\alpha}_{n}(x) = cos(n+\alpha)\theta$,

$$\begin{aligned} T^{\alpha}_{n+1}(x) + T^{\alpha}_{n-1}(x) &= cos(n+1+\alpha)\theta + cos(n-1+\alpha)\theta \\ &= 2cos\Big(\frac{n+1+\alpha+n-1+\alpha}{2}\Big)\theta cos\Big(\frac{n+1+\alpha-n+1-\alpha}{2}\Big)\theta \\ &= 2cos(n+\alpha)cos\theta. \\ &= 2xT^{\alpha}_{n}(x) \end{aligned}$$

The proof of parts ii), iii) and iv) are similar i). □

Lemma 1.3. *The first kind* $\alpha-$*Chebyshev functions, the second kind* $\alpha-$*Chebyshev functions, the third kind* $\alpha-$*Chebyshev and the fourth kind* $\alpha-$*Chebyshev functions, where* $sin2(\alpha\pi) = 0$ *are*

$$\begin{aligned} &\int_{-1}^{1} T^{2\alpha}_{n}(x)\frac{1}{\sqrt{1-x^2}}dx = \\ &\int_{-1}^{1} U^{2\alpha}_{n}(x)\sqrt{1-x^2}dx = \\ &\int_{-1}^{1} V^{2\alpha}_{n}(x)\sqrt{1-x^2}dx = \\ &\int_{-1}^{1} W^{2\alpha}_{n}(x)\frac{1}{\sqrt{1-x^2}}dx \\ &= \frac{\pi}{2}, \end{aligned}$$

and if $n \neq m$ *and* $n+m = 2k$, $k = 1,2,3,\ldots$

$$\begin{aligned} &\int_{-1}^{1} T^{\alpha}_{n}(x)T^{\alpha}_{m}(x)\frac{1}{\sqrt{1-x^2}}dx = \\ &\int_{-1}^{1} U^{\alpha}_{n}(x)U^{\alpha}_{m}(x)\frac{1}{\sqrt{1-x^2}}dx = \\ &\int_{-1}^{1} V^{\alpha}_{n}(x)V^{\alpha}_{m}(x)\frac{1}{\sqrt{1-x^2}}dx = \\ &\int_{-1}^{1} W^{\alpha}_{n}(x)W^{\alpha}_{m}(x)\frac{1}{\sqrt{1-x^2}}dx \\ &= 0. \end{aligned}$$

For $|x| \leq 1$ and $a,b,c,d \in \mathbb{R}$, generalized $\alpha-$Chebyshev polynomials $G^{\alpha}_{n}(x)$ is
defined by the recurrence relation

$$G^{\alpha}_{n+1}(x) = 2xG^{\alpha}_{n}(x) - G^{\alpha}_{n-1}(x);\ n \geq 1,$$

$$\begin{cases} G^{\alpha}_{0}(x) = au_1(x) + bu_2(x) + cu_3(x) + du_4(x) \\ G^{\alpha}_{1}(x) = (au_1(x) + bu_2(x) + cu_3(x) + du_4(x))(2x - a + b - c + d). \end{cases}.$$

We call each term $G_n^{\alpha}(x)$ as Generalized $\alpha-$Chebyshev polynomials for $n \geq 1$. We also indicate that this function can be transformed into the other kinds of Chebyshev polynomials for the special choices of a, b, c and d

If $a = 1$ and $b = c = d = 0$, then it turns into the first-kind $\alpha-$Chebyshev polynomial sequences are known as

$$\{T_n^{\alpha}(x)\} = \{u_1(x), u_1(x)(2x-1), \cdots\},$$

If $a = c = d = 0$ and $b = 1$, then it turns into the second kind $\alpha-$Chebyshev polynomial sequences are known as

$$\{U_n^{\alpha}(x)\} = \{u_2(x), u_2(x)(2x+1), \cdots\},$$

If $a = b = d = 0$ and $c = 1$, then it turns into the third kind $\alpha-$Chebyshev polynomial sequences are known as

$$\{V_n^{\alpha}(x)\} = \{u_3(x), u_3(x)(2x-1), \cdots\},$$

If $a = b = c = 0$ and $d = 1$, then it turns into the fourth kind Chebyshev polynomial sequences are known as

$$\{W_n^{\alpha}(x)\} = \{u_4(x), u_4(x)(2x+1), \cdots\}.$$

It is necessary to study multiresolution analysis and Mallat's Theorem for generalized $\alpha-$Chebyshev wavelets.

Definition 1.4. Multiresolution Analysis: An **MRA** with scaling function ϕ is a collection of closed subspaces $\{V_j\}_{j \in Z}$ of $L^2(\mathbb{R})$, such that

(i) $V_j \subset V_{j+1}$;
(ii) $f(x) \in V_j \iff f(2x) \in V_{j+1}$;
(iii) $\overline{\cup V_j} = L^2(\mathbb{R})$,
(iv) $\cap V_j = 0$;
(v) There exists a function $\phi \in V_0$ such that the collection $\{\phi(x–k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0.

The sequence of wavelet subspaces W_j of $L^2(\mathbb{R})$ is such that $V_j \perp W_j$, for all$j$ and $V_{j+1} = V_j \bigoplus W_j$. Closure of $\bigoplus W_j$ is dense in $L^2(\mathbb{R})$ for L^2 norm.

Now we state Mallat's theorem which guarantees that in the presence of an orthogonal **MRA**, an orthonormal basis for $L^2(\mathbb{R})$ exists.

Lemma 1.5. (Mallat's Theorem) *Given an orthogonal MRA with scaling function ϕ, there is a wavelet $\psi \in L^2(\mathbb{R})$ such that for each $j \in \mathbb{Z}$, the family $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ is an orthonormal basis for W_j. Hence the family $\{\psi_{j,k}\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Definition 1.6. (i) Let $P_n(f)$ be the orthogonal projection of $L^2([-1, 1])$ onto V_n. Then

$$P_n(f) = \sum_{-\infty}^{\infty} < f, \phi_{n,k} > \phi_{n,k}, \;\; n = 1, 2, 3, \cdots$$

(ii) The wavelet approximation of the Chebyshev polynomial is defined by

$$E_n(f) = \|f - P_n(f)\|_2 = \int_{-1}^{1} |f(t) - P_n(f)(t)|^2 dt = o(\phi(n)).$$

Definition 1.7. We define generalized $\alpha-$Chebyshev wavelets. Suppose $k \in \mathbb{N}$ (degree of multiresolution), $m \geq 0,\ n = 1, 2, \cdots, 2^k$ (see [4-6])

$$\Psi^{\alpha}_{n,m}(t) = \sqrt{\frac{2^{k+1}}{n}} G^{\alpha}_m(2^k t - 2n + 1) \boldsymbol{\chi}_{[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})}(t).$$

A function $f \in L^2[-1, 1)$ is expanded by generalized α-Chebyshev wavelets series as

$$f(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{\infty} c_{n,m} \Psi^{\alpha}_{n,m}(t),$$

where

$$c_{n,m} = \int_{-1}^{1} f(t) \Psi^{\alpha}_{n,m}(t) \omega^{\alpha}_{n,m}(t) dt,$$

and $\omega_{n,m}$ is the weight function of generalized $\alpha-$Chebyshev functions. Suppose

$$\int_{-1}^{1} \Psi^{\alpha}_{n,m}(x) \Psi^{\alpha}_{n,m}(x) \omega_{n,m}(x) dx = L^{\alpha}_{n,m},$$

2. Generalized α-Chebyshev wavelets approximation

Theorem 2.1. *Let $f \in L^2([-1, 1])$ be a continuous function and $f(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi^{\alpha}_{n,m}(t)$ and the series $\sum_{n=1}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L^{\alpha}_{n,m}$ be convergent. Then generalized $\alpha-$Chebyshev wavelet approximation f, for every M is the partial sums $s_{2^k, M-1}(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} t_{n,m} \Psi^{\alpha}_{n,m}(t)$, and $E_{2^k, l}(f) = o((\sum_{n=1}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L^{\alpha}_{n,m})^{\frac{1}{2}})$.*

Proof. We have

$$\begin{aligned}
\| \quad & f - s_{2^k,M-1}\|_2^2 \\
= \quad & \int_{-1}^{1} |\sum_{n=1}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\Psi_{n,m}^{\alpha}(t) \\
- \quad & \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} t_{n,m}\Psi_{n,m^{\alpha}}(t)|^2\omega_{n,m}(t)dt \\
= \quad & \int_{-1}^{1} |\sum_{n=1}^{2^k}\sum_{m=M}^{\infty} t_{n,m}\Psi_{n,m}^{\alpha}(t)|^2\omega_{n,m}(t)dt \\
\leq \quad & \sum_{n=1}^{2^k}\sum_{m=M}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{\alpha}(t)|^2\omega_{n,m}(t)dt \\
= \quad & \sum_{n=1}^{2^k}\sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}
\end{aligned}$$

Therefore $\|f - s_{M-1}\|_2 \leq (\sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha})^{\frac{1}{2}}$. That is

$$E_{2^k,M-1}(f) = o(\sum_{n=1}^{2^k}\sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha})^{\frac{1}{2}}),$$

□

Suppose $f(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\psi_{n,m}^{\alpha}(t)$, for $l \geq 1$, we put

$$f_1(x) = \sum_{n=1}^{2^k}\sum_{m=0}^{l} t_{n,m}\psi_{n,m}^{\alpha}{}^{+}(t),$$

and

$$f_2(x) = \sum_{n=1}^{2^k}\sum_{m=0}^{l} t_{n,m}\psi_{n,m}^{\alpha}{}^{-}(t),$$

Theorem 2.2. *Let $f(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\Psi_{m,n}^{\alpha}{}^{+}(t)$ be expanded in terms of generalized $\alpha-$Chebyshev wavelets. If $\sum_{n=1}^{2^k}\sum_{m=0}^{\infty} |t_{n,m}|^2$ is converge, then for every $l \geq 1$, generalized $\alpha-$Chebyshev wavelet approximation $E_{2^k,l}(t)$ of f is $f_1(t)$.*

$$E_{2^k,l}(f) = o((\sum_{n=1}^{2^k}\sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha})^{\frac{1}{2}}).$$

Proof.

$$\begin{aligned}
\|f-f_1\|_2^2 &= \int_{-1}^{1} |\sum_{n=1}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\Psi_{n,m}^{\alpha}{}^{+}(t) \\
&- \sum_{n=0}^{2^k}\sum_{m=0}^{l} t_{n,m}\Psi_{n,m}^{\alpha}{}^{+}(t)|^2\omega_{n,m}^{\alpha}(t)dt \\
&= \int_{-1}^{1} |\sum_{n=1}^{2^k}\sum_{m=l+1}^{\infty} t_{n,m}\Psi_{n,m}^{\alpha}{}^{+}(t)|^2\omega_{n,m}^{\alpha}(t)dt \\
&\leq \sum_{n=1}^{2^k}\sum_{m=l+1}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{\alpha}{}^{+}(t)|^2\omega_{n.m}^{\alpha}(t)dt \\
&\leq \sum_{n=1}^{2^k}\sum_{m=l+1}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{\alpha}(t)|^2\omega_{n.m}^{\alpha}(t)dt \\
&\leq \sum_{n=1}^{2^k}\sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}.
\end{aligned}$$

Therefore

$$\|f-f_1(t)\|_\infty \leq (\sum_{n=1}^{2^k}\sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha})^{\frac{1}{2}},$$

and therefore

$$E_{2^k,l}(f) = o(\sum_{n=1}^{2^k}\sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha})^{\frac{1}{2}}).$$

□

Theorem 2.3. *Let* $f(t) = \sum_{n=1}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\Psi_{m,n}^{\alpha}{}^{-}(t,s,p,q)$ *be expanded in terms of generalized* $-\alpha-$*Chebyshev wavelets. If* $\sum_{n=1}^{2^k}\sum_{m=0}^{\infty}|t_{n,m}|^2$ *is converge, then for every* $l \geq 1$*, generalized* $\alpha-$*Chebyshev wavelet approximation* $E_{2^k,l}(t)$ *of* f *is* $-f_2(t)$ *and*

$$E_{2^k,l}(f) = o((\sum_{n=1}^{2^k}\sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha})^{\frac{1}{2}}).$$

Proof. The proof is similar to Theorem 2.2. □

3. Generalized $\alpha-$wavelets coapproximation

In this section, we define generalized $\alpha-$wavelets coapproximation and obtain some results.

Definition 3.1. Suppose $W \subseteq L^2([-1.1])$ and $f \in L^2([-1,1])$ is a continuous function and $f(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi^{\alpha}_{n,m}(t)$. We say that the function $g \in L^2([-1,1])$ is generalized $\alpha-$wavelet coapproximation f concerning W, If

$$F_f(p) := \|g-p\|_2 - \|f-p\|_2 = o(\phi(n)),$$

for every $p \in W$.

If $F_f(p) \leq 0$ for every $p \in W$, then g is called best coapproximation for f. We put

$$\begin{aligned} W_1 &= \{\sum_{n=1}^{2^k} \sum_{m=0}^{\infty} c_{n,m} \Psi^{\alpha}_{n,m} : \ c_{n,m} \in \mathbb{R}\}, \\ W_2 &= \{\sum_{n=1}^{2^k} \sum_{m=0}^{\infty} c_{n,m} |\Psi^{\alpha}_{n,m}| : \ c_{n,m} \in \mathbb{R}\}, \end{aligned}$$

Theorem 3.2. *Let $f(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi^{\alpha}_{n,m}(t)$ be expanded in terms of generalized $\alpha-$Chebyshev wavelets. If the series $\sum_{n=1}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L^{\alpha}_{n,m}$ is convergent, then generalized α-wavelet coapproximation f with respect to W_1, for every $M \geq 0$ is the partial sums*

$$u_{2^k, M-1} = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} t_{n,m} \Psi^{\alpha}_{n,m},$$

and

$$F_f(p) = o(\sum_{n=1}^{2^k} \sum_{m=M}^{\infty} |t_{n,m}|^2 L^{\alpha}_{n,m}).$$

for every $p \in W_1$.

Proof. Suppose $p = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_m \Psi_{n,m}^{\alpha} \in W_1$.

$$\begin{aligned}
\|p - u_{2^{k-1},M-1}\|_2 &= \int_{-1}^{1} |\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty} c_{n,m}\Psi_{n,m}^{\alpha}(t) \\
&- \sum_{n=1}^{\infty}\sum_{m=0}^{M-1} t_{n,m}\Psi_{n,m}^{\alpha}(x)|^2 \omega_{n,m}^{\alpha}(t)dt \\
&= \int_{-1}^{1} |\sum_{n=1}^{2^{k}}\sum_{m=0}^{M-1} (c_{n,m} - t_{n,m})\Psi_{n,m}^{\alpha}(t) \\
&+ \sum_{n=1}^{\infty}\sum_{m=M}^{\infty} c_{n,m}\Psi_{n,m}^{\alpha}(t)|^2 \omega_{n,m}^{\alpha}(t)dt \\
&= \int_{-1}^{1} |\sum_{n=1}^{2^{k}}\sum_{m=0}^{M-1} (c_{n,m} - t_{n,m})\Psi_{n,m}^{\alpha}(t) \\
&+ \sum_{n=1}^{\infty}\sum_{m=M}^{\infty} (c_{n,m} - t_{n,m})\Psi_{n,m}^{\alpha}(t) \\
&+ \sum_{n=1}^{\infty}\sum_{m=M}^{\infty} t_{n,m}\Psi_{n,m}^{\alpha}(x)|^2 \omega_{n,m}^{\alpha}(t)dt \\
&\leq \int_{-1}^{1} |\sum_{n=1}^{2^{k}}\sum_{m=0}^{\infty} (t_{n,m} - c_{n,m})\Psi_{n,m}^{\alpha}(t))|^2 \omega_{n,m}^{\alpha}(t)dt \\
&+ \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{\alpha}(t)|^2 \omega_{n,m}^{\alpha}(t)dt \\
&\leq \|f - p\|_2 + \sum_{n=1}^{2^{k}}\sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}.
\end{aligned}$$

Therefore

$$\|p - u_{2^{k-1},M-1}\|_2 - \|f - p\|_2 \leq \sum_{n=1}^{2^{k}}\sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}.$$

That is

$$F_f(p) = o(\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}),$$

for every $p \in W_1$. □

Theorem 3.3. *Let* $f(t) = \sum_{n=l}^{2^k}\sum_{m=0}^{\infty} t_{n,m}|\Psi_{n,m}^{\alpha}|(t)$ *be expanded in terms of generalized* $\alpha-$*Chebyshev wavelet. If the series* $\sum_{n=1}^{2^k}\sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}$ *is* *convergent, Then generalized α-wavelet coapproximation $E_l(f)$ of f is*

$$u_{2^k,M-1}(t)=\sum_{n=1}^{2^k}\sum_{m=0}^{M-1}t_{n,m}|\Psi_{n,m}^{\alpha}|(t)$$

with respect to W_2
and

$$E_l(f)=o(\sum_{n=M}^{\infty}\sum_{m=0}^{\infty}|t_{n,m}|^2 l_{n,m}^{\alpha}).$$

Proof. Suppose $p=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_m|\Psi_{n,m}|\in W_2$.

$$\begin{aligned}
\|p-u_{2^{k-1},M-1}\|_2 &= \int_{-1}^{1}|\sum_{n=1}^{2^k}\sum_{m=0}^{\infty}c_{n,m}|\Psi_{n,m}^{\alpha}|(t)\\
&- \sum_{n=1}^{\infty}\sum_{m=0}^{M-1}t_{n,m}|\Psi_{n,m}^{\alpha}|(t)|^2\omega_{n,m}^{\alpha}(t)dt\\
&= \int_{-1}^{1}|\sum_{n=1}^{2^k}\sum_{m=0}^{M-1}(c_{n,m}-t_{n,m})|\Psi_{n,m}^{\alpha}|(t)\\
&+ \sum_{n=1}^{2^k}\sum_{m=M}^{\infty}c_{n,m}|\Psi_{n,m}^{\alpha}|(t,s,p,q)|^2\omega_{n,m}^{\alpha}(t)dt\\
&= \int_{-1}^{1}|\sum_{n=1}^{2^k}\sum_{m=0}^{M-1}(c_{n,m}-t_{n,m})|\Psi_{n,m}^{\alpha}|(t)\\
&+ \sum_{n=1}^{2^k}\sum_{m=M}^{\infty}(c_{n,m}-t_{n,m})||\Psi_{n,m}^{\alpha}||(t)\\
&+ \sum_{n=1}^{2^k}\sum_{m=M}^{\infty}t_{n,m}|\Psi_{n,m}|(t)|^2\omega_{n,m}^{\alpha}(t)dt\\
&\le \int_{-1}^{1}|\sum_{n=1}^{2^k}\sum_{m=0}^{\infty}(t_{n,m}-c_{n,m})\Psi_{n,m}^{\alpha}(t))|^2\omega_{n,m}^{\alpha}(t)dt\\
&+ \sum_{n=1}^{2^k}\sum_{m=M}^{\infty}|t_{n,m}|^2\int_{-1}^{1}|\Psi_{n,m}^{\alpha}(t)|^2\omega_{n,m}^{\alpha}(t)dt\\
&\le \|f-p\|_2+L\sum_{n=0}^{2^k}\sum_{m=M}^{\infty}|t_{n,m}|^2L_{n,m}^{\alpha}.
\end{aligned}$$

Therefore

$$\|p - u_{2^{k-1},M-1}\|_2 - \|f - p\|_2 \leq L \sum_{n=0}^{2^k} \sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}.$$

That is

$$F_f(p) = o(L \sum_{n=0}^{2^{k-1}} \sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}),$$

for every $p \in W_2$. □

Suppose $1 \leq M \leq 2^k$ is a fixed natural number and $f(t) = \sum_{n=M}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}(t)$, we put

$$f_1(t) = \sum_{n=M}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}^{-}(t),$$

and

$$f_2(t) = \sum_{n=M}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}^{+}(t),$$

then

$$f = f_1 - f_2.$$

Theorem 3.4. *Let $f(t) = \sum_{n=Ml}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}(t)$ be expanded in terms of generalized $\alpha-$Chebyshev wavelets. If the series $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}$ is convergent, Then generalized $\alpha-$wavelet coapproximation $E_M(f)$ of f_2 is f_1 with respect to any set $W \subset L^2([-1,1])$. and*

$$E_M(f) = o(\sum_{n=M}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}).$$

Proof. Suppose $p \in W_1$. We have

$$\begin{aligned} \|p - f_1\|_2 &= \|p - f_1 + f_2 - f_2\|_2 \\ &= \|p - f_2 - f\|_2 \\ &\leq \|f_2 - p\|_2 + \|f\|_2 \end{aligned}$$

and

$$\begin{aligned} F_f(p) &= \|p - f_1\|_2 - \|f_2 - p\|_2 \\ &\leq \|f\|_2 \\ &= \sum_{n=M}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha} \end{aligned}$$

□

Corollary 3.5. *Let* $f(t) = \sum_{n=M}^{\infty}\sum_{m=0}^{\infty} t_{n,m}\Psi_{n,m}(t)$ *be expanded in terms of generalized* $\alpha-$*Chebyshev wavelets. If the series* $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}$ *is convergent, Then generalized* $\alpha-$*wavelet coapproximation* $E_l(f)$ *of* f_1 *is* f_2 *with respect to* W
and

$$E_l(f) = o(\sum_{n=M}^{\infty}\sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{\alpha}).$$

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