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CHARACTERIZATION OF D-RECURRENT FINSLER SURFACES

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Abstract. In this paper, we study the class of two-dimensional Douglas-recurrent Finsler metrics, which we refer to as D-recurrent Finsler surfaces for simplicity. First, we derive the necessary and sufficient condition under which a Finsler surface is D-recurrent. Then, we show that a Kropina surface is D-recurrent if and only if it is a Douglas surface.

Keywords: Douglas curvature, D-recurrent metric, Berwald curvature.
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1. Introduction

Let (M, F) be a Finsler manifold. A curve $\rho = \rho(t)$ is called a geodesic of Finsler metric F if and only if its coordinates $(\rho^i = \rho^i(t))$ satisfy the equation $\ddot{\rho}^i + 2\,G^i(\rho, \dot{\rho}) = 0$, where $G^i = G^i(x, y)$ are the spray coefficients of F. Among the Finsler metrics, Berwald metrics bear the closest resemblance to Riemannian ones and are characterized by the fact that their spray coefficients are quadratic in the direction. Then, the spray coefficients of a Berwald metric are given by

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k.$$

There is an interesting generalization of Berwald metrics, namely, Douglas metrics. A Finsler metric F is said to be a Douglas metric if its geodesics coincide with the geodesics of an affine connection up to orientation preserving reparametrization. The notion of Douglas curvature was proposed by Bácsó and Matsumoto as a natural extension of Berwald curvature [3]. Indeed, F is called a Douglas metric if

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k + P(x, y)y^i,$$

where $P = P(x, y)$ is a scalar function on TM. For more progress about the Douglas metrics, see [7], [17] and [18].

The concept of recurrent Riemannian manifolds was first introduced by Ruse in 1949 [12,13], who denoted an n-dimensional recurrent Riemannian manifold

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by K_n. He constructed an example of a manifold to show that harmonic manifolds need not be symmetric. Since then, manifolds with recurrent curvature have garnered attention as an important area of study. A Riemannian manifold $(M, \mathbf{g})$ is said to be recurrent if its Riemannian curvature tensor satisfies $\nabla R = \Phi R$, where ∇ indicates the Levi-Civita connection of $\mathbf{g}$, $\mathbf{R}$ denotes the Riemannian curvature of $\mathbf{g}$, and Φ is a 1-form.

In [10], Faraji-Tayebi-Najafi studied recurrent Finsler metrics. They showed that a recurrent Finsler metric of non-zero isotropic flag curvature is a Landsberg metric. In [2], Atashafrouz-Najafi introduced the concept of D-recurrent Finsler metrics. A Finsler space (M, F) is said to be D-recurrent if the Douglas tensor of F satisfies

$$D^i{}_{jkl|m} y^m = \Phi(x, y) D^i{}_{jkl}, \tag{1}$$

where "$|$" denotes the horizontal covariant derivative with respect to the Berwald connection of F, and $\Phi = \Phi(x, y)$ is a smooth scalar function on TM_0 satisfying $\Phi(x, \lambda y) = \lambda\Phi(x, y)$. They proved that a generalized Douglas-Weyl Randers metric is a Douglas metric provided that it is D-recurrent.

Recurrent Finsler spaces represent a natural extension of Riemannian geometry, enriched by direction-dependent characteristics that make them particularly relevant in physical contexts where anisotropy is essential. Defined through recurrence conditions on their curvature tensors involving non-zero covariant vector fields, these spaces exhibit complex geometric behavior. Their structure provides a powerful mathematical tool for modeling direction-sensitive phenomena in physics, such as anisotropic gravitational fields, early-universe cosmology, and the geometry underlying gauge field interactions [1].

Finsler surfaces play a unique role in Finsler geometry. Their curvature behavior differs markedly from that of higher-dimensional cases. Specifically, they are C-reducible, meaning that their Cartan torsion can be expressed in terms of the mean Cartan torsion. In dimensions $n \geq 3$, only Randers and Kropina metrics are C-reducible. Furthermore, Finsler surfaces exhibit scalar flag curvature, meaning the flag curvature is independent of the choice of flag [16]. Due to these unique properties, L. Berwald introduced what is now known as the Berwald frame to facilitate their study. In this paper, by using the Berwald frame, we study the Douglas curvature of Finsler surfaces. Then, we find the necessary and sufficient condition under which a Finsler surface is D-recurrent. More precisely, we prove the following.

Theorem 1.1. *Let (M, F) be a two-dimensional Finsler space. Then F is a D-recurrent metric if and only if the following holds*

$$\begin{aligned}&\Big[F^{-1}(2\mu' - \mu^2 F) - 4\lambda' + 2\lambda\mu F + 2\Phi(2\lambda - \mu F^{-1})\Big] I_l \\ &\quad + \Big[2(\lambda_{|l} - \lambda' y_l F^{-2} - \lambda'_{,l}) + 2\Phi(\lambda y_l F^{-2} + \lambda_{,l})\Big] = 0,\end{aligned} \tag{2}$$

where $\mu := -2I_{,1}/I$, $\mu' := \mu_{|i}y^i$, $\lambda := I_2/(3F)$, $\lambda' := \lambda_{|i}y^i$ *and* $I = I(x,y)$ *denotes the main scalar of* F.

To find explicit examples of D-recurrent metrics, one may consider the class of (α,β)-metrics. An (α,β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, where $s = \beta/\alpha$, $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a positive-definite Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. If $\phi = 1 + s$, the resulting metric is known as a Randers metric defined as $F(x,y) = \alpha(x,y) + \beta(x,y)$, where the norm of β is controlled by the Riemannian metric α, namely, $||\beta||_\alpha < 1$. Randers's metrics were first introduced by G. Randers in his work on general relativity in four-dimensional manifolds [11]. He regarded these metrics not as Finsler metrics, but as affinely connected Riemannian metrics [11]. In [2], Atashafrouz-Najafi showed that a Randers metric $F = \alpha + \beta$ is D-recurrent if and only if $d\beta$ is nearly recurrent. Indeed, they proved the following.

Theorem A. ([2]) *Let* $F = \alpha + \beta$ *be a Randers metric on n-dimensional manifold* M. *Then* F *is a D-recurrent metric if and only if* β *satisfies the following*

$$s_{ij;k} + s_{ik;j} = \Phi_k s_{ij} + \Phi_j s_{ik}, \tag{3}$$

where ";" denotes the covariant derivation with respect to the Levi-Civita connection of α, *and* $\Phi_i := \partial\Phi/\partial y^i$.

The history of the Kropina metric goes back to Hilbert's fourth problem. Indeed, Kropina metrics were introduced by L. Berwald in his study of Finsler surfaces with rectilinear extremals and further explored by V. K. Kropina in the context of projective Finsler surfaces [8]. In [5], Hashiguchi-Hōjō-Matsumoto studied two-dimensional Landsberg metrics, and the metric obtained by Kropina was named the Kropina metric by them. These metrics are among the simplest nontrivial Finsler metrics and find applications in various physical theories including electron optics, dissipative systems, and irreversible thermodynamics (see [6] and [19]). While both Randers and Kropina metrics are C-reducible [9], only Randers metrics are regular; Kropina metrics are singular.

In [10], Faraji-Tayebi-Najafi found the necessary and sufficient condition under which a Kropina metric is Ricci-recurrent, provided that its one-form is closed and conformal. They concluded that for every Ricci-recurrent Kropina metric $F = \alpha^2/\beta$ of Berwald-type, the Riemannian metric α is Ricci-recurrent. In this paper, we study the Douglas curvature of Kropina surfaces. Then, we prove the following result.

Theorem 1.2. *Let* $(M, F = \alpha^2/\beta)$ *be a two-dimensional Kropina space. Then* F *is a D-recurrent metric if and only if it is a Douglas metric.*

2. Preliminaries

Let M be an n-dimensional manifold, $TM = \bigcup_{x\in M} T_xM$ its tangent bundle and $TM_0 := TM - \{0\}$ the slit tangent bundle. A Finsler structure on M is a function $F : TM \to [0, \infty)$ satisfying the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of tangent bundle, i.e., $F(x, \lambda y) = \lambda F(x, y)$, $\forall\lambda > 0$; (iii) The following quadratic form $\mathbf{g}_y$: $T_xM \times T_xM \to \mathbb{R}$ is positively defined on TM_0

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Then the pair (M, F) is called a Finsler manifold.

To quantify the non-Euclidean characteristics of $F_x := F|_{T_xM}$, $x \in M$, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. Also, the mean Cartan torsion $\mathbf{I}_y : T_xM \to \mathbb{R}$ defined by

$$\mathbf{I}_y(u) := \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$.

Let $c = c(t)$ be a C^∞ curve and $U(t) = U^i(t)\partial/\partial x^i|_{c(t)}$ be a vector field along c. Define the covariant derivative of $U(t)$ along c by

$$D_{\dot c}U(t) := \Big\{\frac{dU^i}{dt}(t) + U^j(t)\frac{\partial G^i}{\partial y^j}\big(c(t), \dot c(t)\big)\Big\}\xi\Big|_{c(t)}.$$

$U(t)$ is said to be linearly parallel if $D_{\dot c}U(t) = 0$.

For a vector $y \in T_xM$, define

$$\mathbf{L}_y(u, v, w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot\sigma(t)}\big(U(t), V(t), W(t)\big)\Big]|_{t=0},$$

where $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot\sigma(0) = y$, and $U = U(t)$, $V = V(t)$, $W = W(t)$ are linearly parallel vector fields along σ with $U(0) = u, V(0) = v, W(0) = w$. We call $\mathbf{L}$ the Landsberg curvature. The Landsberg curvature measures the rate of change of the Cartan torsion along geodesics.

For a Finsler manifold (M, F), a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4} g^{il} \Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \Big].$$

Here, $G^i = G^i(x, y)$ are called spray coefficients. For $y \in T_xM_0$, let us define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl} u^i v^j w^k \partial/\partial x^i|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

-$\mathbf{B}$ is called the Berwald curvature. F is called a Berwald metric if $\mathbf{B} = \mathbf{0}$.

Taking a trace of Berwald curvature $\mathbf{B}$ gives us the mean of Berwald curvature $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$ defined by

$$\mathbf{E}_y(u, v) := \frac{1}{2} \sum_{i=1}^{n} g^{ij}(y) \mathbf{g}_y\big(\mathbf{B}_y(u, v, \partial_i), \partial_j\big).$$

F is called a weakly Berwald metric if $\mathbf{E} = \mathbf{0}$.

For $y \in T_xM_0$, define $\mathbf{D}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{D}_y(u, v, w) := D^i_{jkl}(y) u^i v^j w^k \partial/\partial x^i|_x$, where

$$D^i{}_{jkl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \Big[G^i - \frac{2}{n+1} \frac{\partial G^m}{\partial y^m} y^i \Big].$$

$\mathbf{D}$ is called the Douglas curvature. F is called a Douglas metric if $\mathbf{D} = \mathbf{0}$. Also, F is said a D-recurrent metric its Douglas tensor satisfies $D^i{}_{jkl|m} y^m = \Phi(x, y) D^i{}_{jkl}$, where $\Phi = \Phi(x, y)$ is a 1-homogeneous smooth scalar function on TM_0.

Example 2.1. *A Finsler metric F satisfying $F_{x^k} = F F_{y^k}$ is called a Funk metric. The standard Funk metric on the Euclidean unit ball $B^2(1)$ is denoted by Θ and defined by*

$$(4)\quad \Theta(x, y) := \frac{\sqrt{|y|^2 - (|x|^2 |y|^2 - <x, y>^2)}}{1 - |x|^2} + \frac{<x, y>}{1 - |x|^2}, \quad y \in T_x\mathbb{B}^2(1) \simeq \mathbb{R}^2,$$

where $<,>$ and $|.|$ denote the Euclidean inner product and norm on $\mathbb{R}^2$, respectively. The spray coefficients of F are given by $G^i = \frac{1}{2} F y^i$. Note that F is not Riemannian in general. Thus, Funk metric is a D-recurrent metric which is not Berwaldian, namely, $\mathbf{B} \neq 0$.

Example 2.2. *Let*

$$F := |y| + \frac{<x, y>}{\sqrt{1 + |x|^2}}, \quad y \in T_x\mathbb{R}^n \simeq \mathbb{R}^2$$

where $|.|$ and $<,>$ denote the Euclidian norm and inner product on $\mathbb{R}^2$, respectively. F is indeed a Randers metric on the whole of $\mathbb{R}^2$. The spray coefficients are in the form $G^i = Py^i$, where

$$P = c\left(|y| - \frac{< x, y >}{\sqrt{1+|x|^2}}\right),$$

where

$$c = \frac{1}{2}(\sqrt{1+|x|^2}).$$

Then F is a D-recurrent Finsler metric.

For a vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ which is defined by $\mathbf{R}_y(u) := R^i_k(y)u^k\partial/\partial x^i$, where

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y \in TM_0}$ is called the Riemann curvature [14].

For a flag $\Pi := \mathrm{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(\Pi, y)$ is defined by

$$\mathbf{K}(x, y, \Pi) := \frac{\mathbf{g}_y\big(u, \mathbf{R}_y(u)\big)}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}.$$

The flag curvature $\mathbf{K}(x, y, \Pi)$ is a function of tangent planes $\Pi = \mathrm{span}\{y, v\} \subset T_xM$. A Finsler manifold (M, F) is said to have scalar flag curvature if $\mathbf{K}(\Pi_x, y) = \mathbf{K}(x, y)$ is only a function of $(x, y) \in TM_0$.

3. Two-Dimensional Finsler Metrics

The special and useful Berwald frame was introduced and developed by Berwald [4]. Let (M, F) be a two-dimensional Finsler manifold. We study two dimensional Finsler space and define a local field of an orthonormal frame (ℓ^i, m^i) called the Berwald frame, where $\ell^i = y^i/F(y)$, m^i is the unit vector with $\ell_i m^i = 0$, $\ell_i = g_{ij}\ell^i$ and g_{ij} is defined by $g_{ij} = \ell_i\ell_j + m_i m_j$. The Berwald curvature of Finsler surfaces is given by

$$B^i{}_{jkl} = F^{-1}(-2I_{,1}\ell^i + I_2 m^i)m_j m_k m_l, \tag{5}$$

where $I = I(x, y)$ is 0-homogeneous function called the main scalar of Finsler metric and $I_2 = I_{,2} + I_{,1|2}$ (see page 689 in [1]). On the other hand, the Cartan tensor of F is in the following form

$$C_{ijk} = F^{-1} I m_i m_j m_k. \tag{6}$$

By (5) and (6), we have

$$B^i{}_{jkl} = -\frac{2I_{,1}}{I}C_{jkl}\ell^i + \frac{I_2}{3F}\Big\{h_{jk}h^i_l + h_{kl}h^i_j + h_{lj}h^i_k\Big\}. \tag{7}$$

Let us define

$$\mu := -\frac{2I_{,1}}{I}, \qquad \lambda := \frac{I_2}{3F}.$$

One can see that μ and λ are homogeneous functions on TM of degrees 0 and -1 with respect to y, respectively. Then for a Finsler surface, the Berwald curvature (7) can be written as follows

$$B^i{}_{jkl} = \mu C_{jkl}\ell^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{8}$$

Taking a trace of (8) and using $C_{jkm}\ell^m = 0$ yields

$$E_{jk} = \frac{3}{2}\lambda h_{jk}. \tag{9}$$

Also, contracting (8) with y_i implies that

$$L_{jkl} + \frac{\mu}{2}FC_{jkl} = 0. \tag{10}$$

Putting (9) and (10) in (8) gives us

$$B^i{}_{jkl} = -\frac{2}{F}L_{jkl}\ell^i + \frac{2}{3}\lambda\Big\{h^i_j E_{kl} + h^i_k E_{jl} + h^i_l E_{jk}\Big\}. \tag{11}$$

By (11), we conclude the following.

Corollary 3.1. *Let (M, F) be a Finsler surface. Then $\mathbf{B} = 0$ if an only if $\mathbf{E} = 0$ and $\mathbf{L} = 0$.*

Proof of Theorem 1.1: The following holds

$$h_{ij,k} = 2C_{ijk} - F^{-2}(y_j h_{ik} + y_i h_{jk}). \tag{12}$$

Taking a vertical derivation of (9), and using (12) yields

$$2E_{jk,l} = 3\lambda_{,l}h_{jk} + 3\lambda\Big\{2C_{jkl} - F^{-2}(y_k h_{jl} + y_j h_{kl})\Big\}. \tag{13}$$

The Douglas tensor is given by

$$D^i{}_{jkl} = B^i{}_{jkl} - \frac{2}{3}\Big\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\Big\}. \tag{14}$$

Putting (11) and (13) in (14) yields

$$\begin{aligned} D^i{}_{jkl} &= -2\{F^{-2}L_{jkl} + \lambda C_{jkl}\}y^i - (\lambda y_l F^{-2} + \lambda_{,l})h_{jk}y^i \\ &= \Big[-2(F^{-2}L_{jkl} + \lambda C_{jkl}) - (\lambda y_l F^{-2} + \lambda_{,l})h_{jk}\Big]y^i. \end{aligned} \tag{15}$$

We can rewrite (15) as follows

$$D^i{}_{jkl} = \big[V_{jkl} + W_{jkl}\big]y^i, \tag{16}$$

where

$$V_{jkl} := -2(F^{-2}L_{jkl} + \lambda C_{jkl}), \quad W_{jkl} := -(\lambda y_l F^{-2} + \lambda_{,l})h_{jk}. \tag{17}$$

Taking a horizontal derivation of (16) along Finslerian geodesics implies that

$$D^i{}_{jkl|s}y^s = \big[V_{jkl|s} + W_{jkl|s}\big]y^s y^i, \tag{18}$$

Since F is a D-recurrent Finsler metric, then by (1), (16) and (18), we get

$$[V_{jkl|s}+W_{jkl|s}]y^s y^i=\Phi(x,y)D^i{}_{jkl}=\Phi[V_{jkl}+W_{jkl}]y^i \tag{19}$$

equivalently

$$[V_{jkl|s}+W_{jkl|s}]y^s=\Phi[V_{jkl}+W_{jkl}]. \tag{20}$$

By (10), we get

$$L_{jkl|s}y^s=-\frac{1}{2}F(\mu' C_{jkl}+\mu L_{jkl})=-\frac{1}{4}F(2\mu'-\mu^2F)C_{jkl}. \tag{21}$$

Using (21), one can get

$$\begin{aligned} V_{jkl|s}y^s &= -2(F^{-2}L_{jkl|s}+\lambda_{|s}C_{jkl}+\lambda C_{jkl|s})y^s\\ &= -2F^{-2}L_{jkl|s}y^s-2\lambda' C_{jkl}-2\lambda L_{jkl}\\ &= \frac{1}{2}\Big[F^{-1}(2\mu'-\mu^2F)-4\lambda'+2\lambda\mu F\Big]C_{jkl}, \end{aligned} \tag{22}$$

where $\lambda' := \lambda_{|s}y^s$ and $\mu' := \mu_{|s}y^s$. The following holds

$$h_{jk|s}=g_{jk|s}=-2L_{jks},\quad \lambda_{,l|s}=\lambda_{|s,l}.$$

Taking a horizontal derivation of (17) and using the above relation, give us the following

$$\begin{aligned} W_{jkl|s}y^s &= -(\lambda' y_l F^{-2}+\lambda_{,l|s}y^s)h_{jk}-(\lambda y_l F^{-2}+\lambda_{,l})h_{jk|s}y^s\\ &= -(\lambda' y_l F^{-2}+\lambda_{,l|s}y^s)h_{jk}\\ &= -(\lambda' y_l F^{-2}+\lambda_{|s,l}y^s)h_{jk}, \end{aligned} \tag{23}$$

The following holds

$$\lambda_{|s,l}y^s=\lambda'_{,l}-\lambda_{|l}. \tag{24}$$

By (23) and (24), we get

$$W_{jkl|s}y^s=(\lambda_{|l}-\lambda' y_l F^{-2}-\lambda'_{,l})h_{jk}, \tag{25}$$

Putting (22) and (25) in (20) yields

$$\begin{aligned} &\Big[F^{-1}(2\mu'-\mu^2F)-4\lambda'+2\lambda\mu F+2\Phi(2\lambda-\mu F^{-1})\Big]C_{jkl}\\ &+\Big[2(\lambda_{|l}-\lambda' y_l F^{-2}-\lambda'_{,l})+2\Phi(\lambda y_l F^{-2}+\lambda_{,l})\Big]h_{jk}=0. \end{aligned} \tag{26}$$

Contracting (26) with g^{jk} implies (2). □

For weakly Berwald metrics, we conclude the following.

Corollary 3.2. *Let (M,F) be a two-dimensional weakly Berwald space. Then F is a D-recurrent metric if and only if the following ODE holds*

$$2\mu'-(\mu F+2\Phi)\mu=0. \tag{27}$$

In particular if $\Phi = c$ $(c \in \mathbb{R})$, then μ is given by

$$\mu(t) = \frac{2ce^{2t}}{1 - e^{ct}}. \tag{28}$$

Proof. By (9), $\mathbf{E} = 0$ if and only if $\lambda = 0$. Putting it in (2) yields

$$\Big[F^{-1}(2\mu' - \mu^2 F) - 2\Phi\mu F^{-1})\Big] I_l = 0. \tag{29}$$

According to assumption, F is not Riemannian. Then (29) gives us (27). Moreover, if $\Phi = c$, on a Finslerian geodesic $\sigma = \sigma(t)$, we get

$$2\mu' = \mu^2 + 2c\mu. \tag{30}$$

Solving (30) gives us (28). □

Also, for two-dimensional Landsberg metrics, we have the following.

Corollary 3.3. *Let (M, F) be a two-dimensional Landsberg space. Then F is a D-recurrent metric if and only if the following holds*

$$2\big(\lambda\Phi - \lambda'\big) I_l + \Big[(\lambda_{|l} - \lambda' y_l F^{-2} - \lambda'_{,l}) + (\lambda y_l F^{-2} + \lambda_{,l})\Phi\Big] = 0. \tag{31}$$

Proof. By (10), $\mathbf{L} = 0$ if and only if $\mu = 0$. Putting it in (2) yields (31). □

4. D-Recurrent Finsler Surfaces

In this section, we are going to prove Theorem 1.2. First, we remark that in [15], Tayebi and Eslami studied the Douglas curvature of Finsler surfaces and established the following result.

Lemma 4.1. ([15]) *The Douglas curvature of any Finsler surface (M, F) satisfies*

$$D^i{}_{jkl|s} y^s = \mathfrak{D}_{jkl} y^i, \tag{32}$$

where

$$\mathfrak{D}_{jkl} = \frac{1}{3}\Big\{6\boldsymbol{K}C_{jkl} + 2\big(g_{kl}\boldsymbol{K}_{\cdot j} + g_{kj}\boldsymbol{K}_{\cdot l} + g_{jl}\boldsymbol{K}_{\cdot k}\big) + y_j\boldsymbol{K}_{\cdot k\cdot l} + y_k\boldsymbol{K}_{\cdot j\cdot l} + y_l\boldsymbol{K}_{\cdot k\cdot j} - 2E_{jk\cdot l|m}y^m\Big\}.$$

Here, $\boldsymbol{K} = \boldsymbol{K}(x, y)$ denotes the flag curvature of F, $\boldsymbol{K}_{\cdot j} := \boldsymbol{K}_{y^j}$ and $\boldsymbol{K}_{\cdot j\cdot k} := \boldsymbol{K}_{y^j y^k}$.

Lemma 4.2. *The Douglas curvature of any D-recurrent Finsler surface satisfies following*

$$D^p{}_{jkl} = \mathbb{D}_{jkl} y^p, \tag{33}$$

where $\mathbb{D}_{jkl} = F^{-2} y_i D^i_{jkl}$.

Proof. Contracting (32) with h^p_i implies that

$$h^p_i D^i_{jkl|m} y^m = 0, \tag{34}$$

where $h^p_i = \delta^p_i - F^{-2} y_i y^p$. According to the assumption, F is D-recurrent. By (1) and (34) we have

$$\Phi h^p_i D^i_{jkl} = 0,$$

Thus we get $h^p_i D^i_{jkl} = 0$ which is equal to (33). □

Remark 4.3. There is another version of (32) in Lemma 4.1. Indeed, according to (15) we have

$$D^i{}_{jkl} = -\Big[2(F^{-2}L_{jkl} + \lambda C_{jkl}) + (\lambda y_l F^{-2} + \lambda_{,l})h_{jk}\Big] y^i. \tag{35}$$

Taking horizontal derivation of (35) yields

$$\begin{aligned} D^i{}_{jkl|s} y^s = -\Big[&2(F^{-2}L_{jkl|s} y^s + \lambda' C_{jkl} + \lambda L_{jkl}) + (\lambda' y_l F^{-2} + \lambda_{,l|s} y^s) h_{jk} \\ &-2(\lambda y_l F^{-2} + \lambda_{,l}) L_{jks}\Big] y^i. \end{aligned} \tag{36}$$

Let $F = \alpha^2/\beta$ be a Kropina metric on an n-dimensional manifold M, where $\alpha(x,y) := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta(x,y) := b_i(x)y^i$ a 1-form on M. Let us define $b_{i;j}$ by $b_{i;j}\theta^j := db_i - b_j \theta^j_i$, where $\theta^i := dx^i$ and $\theta^j_i := \Gamma^j_{ik} dx^k$ denote the Levi-Civita connection forms of α. Put

$$\begin{aligned} r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}), \quad r_{i0} := r_{ij} y^j, \quad r_{00} := r_{ij} y^i y^j, \\ r_j := b^i r_{ij}, \quad s_{i0} := s_{ij} y^j, \quad s_j := b^i s_{ij}, \quad s^i{}_j = a^{im} s_{mj}, \\ s^i{}_0 = s^i{}_j y^j, \quad r_0 := r_j y^j, \quad s_0 := s_j y^j. \end{aligned}$$

Proof of Theorem 1.2: The spray coefficients G^i of the Kropina metric $F = \alpha^2/\beta$ are given by the following

$$G^i = \bar{G}^i - \frac{1}{2\beta}\alpha^2 s^i{}_0 + \frac{1}{2\beta b^2}(\beta r_{00} + \alpha^2 s_0) b^i - \frac{1}{\alpha^2 b^2}(\beta r_{00} + \alpha^2 s_0) y^i. \tag{37}$$

where $\bar{G}^i$ are the spray coefficients α. Let us define

$$\Pi^i := G^i - \frac{1}{3}\Big(\frac{\partial G^m}{\partial y^m}\Big) y^i, \qquad \bar{\Pi}^i := \bar{G}^i - \frac{1}{3}\Big(\frac{\partial \bar{G}^m}{\partial y^m}\Big) y^i. \tag{38}$$

By (37) and (38) we get

$$\Pi^i = \bar{\Pi}^i - \frac{1}{2\beta}\alpha^2 s^i{}_0 + \frac{1}{2b^2\beta}(\alpha^2 s_0 + \beta r_{00}) b^i - \frac{1}{3b^2}(s_0 + r_0) y^i. \tag{39}$$

Let us put

$$A_{ij} := \frac{1}{\beta b^2}a_{ij} + \frac{\alpha^2}{b^2\beta^3}b_ib_j - \frac{1}{b^2\beta^2}\Big(b_jy_i + b_iy_j\Big),$$
$$B_{lkj} := \frac{2}{b^2\beta^3}b_jb_ky_l - \frac{1}{b^2\beta^2}a_{lk}b_j - \frac{\alpha^2}{b^2\beta^4}b_jb_kb_l.$$

Then, the of Douglas curvature of $F = \alpha^2/\beta$ is given by

$$\begin{aligned} D^i{}_{jkl} = (s_0b^i - s^i{}_0b^2)(B_{lkj} + B_{jlk} + B_{kjl}) + (s_jb^i - b^2s^i{}_j)A_{lk} \\ +(s_kb^i - b^2s^i{}_k)A_{lj} + (s_lb^i - b^2s^i{}_l)A_{kj}. \end{aligned} \tag{40}$$

By using (33) and (40) we have

$$\begin{aligned} \mathbb{D}_{jkl}y^i = (B_{jlk} + B_{kjl} + B_{lkj})(s_0b^i - s^i{}_0b^2) + A_{lk}(s_jb^i - s^i{}_jb^2) \\ +A_{lj}(s_kb^i - s^i{}_kb^2) + A_{kj}(s_lb^i - s^i{}_lb^2). \end{aligned} \tag{41}$$

Multiplying (41) with y_i yields

$$\begin{aligned} \mathbb{D}_{jkl} = \alpha^{-2}\left(B_{lkj} + B_{jlk} + B_{kjl}\right)s_0 + \alpha^{-2}A_{lk}(s_j\beta - s_{0j}b^2) \\ +\alpha^{-2}A_{lj}(s_k\beta - s_{0k}b^2) + \alpha^{-2}A_{kj}(s_l\beta - s_{0l}b^2). \end{aligned} \tag{42}$$

By substituting (42) in to (41) we have

$$\begin{aligned} &\frac{1}{b^2\beta^4}\Big[2\beta(b_jb_ky_l + b_kb_ly_j + b_lb_jy_k) - \beta^2(a_{lk}b_j + a_{jl}b_k + a_{kj}b_l) \\ &\quad -3b_jb_kb_l\alpha^2\Big]\Big[(\beta\alpha^{-2}y_i - b_i)s_0 - b^2s_{i0}\Big] \\ &+\frac{1}{b^2\beta^3}\Big[a_{lk}\beta^2 - (b_ly_k + b_ky_l)\beta + b_lb_k\alpha^2\Big]\Big[(\beta\alpha^{-2}s_j - b^2\alpha^{-2}s_{0j})y_i \\ &\quad -b_is_j + b^2s_{ij}\Big] + \frac{1}{b^2\beta^3}\Big[a_{lj}\beta^2 - (b_ly_j + b_jy_l)\beta \\ &\quad +b_lb_j\alpha^2\Big]\Big[\alpha^{-2}(\beta s_k - b^2s_{0k})y_i - b_is_k + b^2s_{ik}\Big] \\ &+\frac{1}{b^2\beta^3}\Big[a_{kj}\beta^2 - (b_ky_j + b_jy_k)\beta + b_kb_j\alpha^2\Big]\Big[\alpha^{-2}(\beta s_l - b^2s_{0l})y_i \\ &\quad -b_is_l + b^2s_{il}\Big] = 0. \end{aligned} \tag{43}$$

Contracting (43) with a^{lk}, we obtain

$$\begin{aligned} &\frac{1}{b^2\alpha^2\beta^4}\Big[2(1-b^2)\beta^2y_iy_j - (2\beta^2 + 3b^2\alpha^2)\beta y_ib_j + 2\alpha^2\beta^2b_ib_j \\ &\quad -2\alpha^2\beta b_iy_j + 3b^2\alpha^4b_ib_j\Big]s_0 + \frac{1}{\beta^4}\Big[2\beta y_j - (2\beta^2 + 3b^2\alpha^2)b_j\Big]s_{i0} \\ &+\frac{1}{b^2\alpha^2\beta^2}\Big[(2\beta^2 + b^2\alpha^2)y_i - 2\beta\alpha^2b_i\Big]s_j - \frac{1}{\beta^3\alpha^2}\Big[2\beta^2 + b^2\alpha^2\Big]y_is_{0j} \\ &\quad +\frac{1}{\beta^3}\Big[2\beta^2 + b^2\alpha^2\Big]s_{ij} - \frac{2}{\beta^2}s_iy_j = 0. \end{aligned} \tag{44}$$

Multiplying (44) with $b^2\alpha^2\beta^2 b^i b^j$ implies that

$$\Big\{(1-3b^2)\beta^2 - 2b^2\alpha^2\Big\}s_0 = 0. \tag{45}$$

By (45), we get

$$s_i = 0. \tag{46}$$

Putting (46) in (44) and contracting the result with $\alpha^2\beta^4$ yields

$$\alpha^2(2y_j\beta - 2b_j\beta^2 - 3b_j\alpha^2 b^2)s_{i0} + \beta(2\beta^2 + b^2\alpha^2)(\alpha^2 s_{ij} - y_i s_{0j}) = 0. \tag{47}$$

Contracting (47) with b^i gives us the following

$$(2\beta^2 + b^2\alpha^2)s_{0j} = 0. \tag{48}$$

By (48), one can obtain $s_{ij} = 0$. Putting $s_{ij} = 0$ in (40) implies that F is a Douglas metric. □

5. Conclusion

Recurrent Finsler spaces extend the classical idea of recurrence from Riemannian geometry to Finsler geometry by linking the curvature tensor to a non-vanishing covariant vector field. In these spaces, the curvature exhibits a directional pattern of repetition, reflecting consistent geometric behavior along certain vector fields. Unlike Riemannian spaces, where the length of a curve depends solely on its path, Finsler geometry defines length as a function of both position and direction, making the recurrence conditions more intricate and physically meaningful. Therefore the study of the class of D-recurrent Finsler spaces will enhance our understanding of the geometric meaning of Douglas metrics.

6. Conflict of Interest

There is not any conflict of interest.

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