Shahid Bahonar University of Kerman
Journal of Mahani Mathematical Research
Print ISSN: 2251-7952 Online ISSN: 2645-4505

TOWARDS A GROUP THEORETICAL PROOF OF THE FROBENIUS THEOREM

M.R. Darafsheh

Article type: Research Article
(Received: 23 May 2025, Received in revised form 30 July 2025)
(Accepted: 30 August 2025, Published Online: 30 August 2025)

Abstract. The Frobenius finite group was defined by Frobenius more than 100 years ago to be a group with a non-trivial proper subgroup H whose intersection with all of its conjugates is the trivial group. In 1901, Frobenius proved that H has a normal complement K in the group, but in his proof, the character theory was used. Since then providing a character free proof was a challenging problem. Although the existence of K is proved by imposing extra conditions on H, but up to present time no general proof is known. In this note, we prove this theorem using elementary group theory by setting a certain condition on H.

Keywords: Frobenius group, complement, kernel.
2020 MSC: 20H20, 20F50

1. Introduction

Frobenius groups belong to an important class of groups that occur in many group theoretical investigations. The definition of this group is as follows:

Definition 1. Let G be a group and H a non-trivial proper subgroup of G. If H and each of its conjugates has only the identity element in common, then G is called a Frobenius group with Frobenius complement H. This means that $H \cap H^g = 1$, $\forall g \in G \backslash H$.

As an example of a Frobenius group, we may present the dihedral group of order $2n$, n odd, where a subgroup of order 2 or any of its conjugates is the Frobenius complement. Although infinite Frobenius exist, but here we are concerned with finite groups.

The mysterious fact about a Frobenius group is that H has a normal complement. In fact in 1901, Frobenius using the theory of characters of finite groups proved the following [2].

Theorem 2 (Frobenius). Let G be a finite Frobenius group. Then

$$K = (G \backslash \bigcup_{g \in G} H^g) \cup \{1\}$$

is a normal subgroup of G.

darafsheh@ut.ac.ir , ORCID: 0000-0002-5539-4989
https://doi.org/10.22103/jmmr.2025.25316.1804

Publisher: Shahid Bahonar University of Kerman
How to cite: M.R. Darafsheh, *Towards a group theoretical proof of the Frobenius theorem*, J. Mahani Math. Res. 2026; 15(1): 317-319.

K is called the Frobenius kernel of G.

So far no character free proof of this theorem is known, although several attempts have been done to prove it under special cases, such as if the complement is solvable or the complement has even order, or other conditions [1]. But a general proof that uses only elementary group theory is out of reach yet.

My propose to write this note is to present a different elementary proof of the Frobenius's theorem so that it stimulates interest among young mathematicians towards this problem.

2. Preliminaries and Proofs

Throughout, we assume G is a finite Frobenius group with complement H. The set K defined in Theorem 2 is only a subset of G and under a special case we will prove it is a subgroup, of course not using character theory. The terms normalizer and centralizer and conjugate of a subset S in a group G are standard and are denoted by $N(S)$, $C(S)$ and S^g respectively.

Lemma 3. $N(H) = H$.

Proof. It is clear that $H \leq N(H)$. Suppose $g \in N(H)$, then $H^g = H$. Therefore $H^g \cap H = H$, and if $g \in G \backslash H$, then $H = 1$, contradiction, proving $g \in H$, hence $N(H) \leq H$. We conclude $N(H) = H$. □

Lemma 4. $|K| = [G : H]$.

Proof. The number of distinct conjugates of H in G is $[G : N(H)] = [G : H]$. Therefore $|\bigcup_{g \in G} H^g| = (|H| - 1) \times [G : H] + 1$. Hence

$$|K| = |G| - ((|H| - 1)\frac{|G|}{|H|} + 1) + 1 = \frac{|G|}{|H|} = [G : H].$$

□

Lemma 5. Let $N \trianglelefteq G$, $G = NH$, $N \cap H = 1$. Then $N \subseteq K$.

Proof. Let $1 \neq n \in N$. Assume $n \notin K$. Then by the definition of K, we have $n \in H^g$, for some $g \in G$. Therefore, $n^{g^{-1}} \in H \cap N = 1$ implying $n^{g^{-1}} = 1$, hence $n = 1$, contradiction. □

Lemma 6. If $1 \neq x \in H$, then $C(\langle x \rangle) \leq H$.

Proof. Let $y \in C(\langle x \rangle)$. Then $yx = xy$, which implies $x = x^y \in H \cap H^y$. Since $x \neq 1$, we obtain $y \in H$. □

Lemma 7. $(|H|, [G : H]) = 1$.

Proof. Let p be a prime number and P be a Sylow p-subgroup of H. By elementary theorems of Sylow [5] or [1], P is contained a Sylow p-subgroup Q of G. By Lemma 6, we have $Z(Q) \leq C_G(P) \leq H$. Since $Z(Q)$ is non-trivial, take $1 \neq x \in Z(Q)$, and then use Lemma 6 to obtain $Q \leq C(\langle x \rangle) \leq H$.

Therefore Q is a Sylow p-subgroup of H which implies $P = Q$. Hence $|H|$ and $[G : H]$ are relatively prime. □

Next, we will use a theorem of Burnside whose proof can be found in [4].

Theorem 8 (Burnside). If G is a finite group and P is a Sylow p-subgroup of G such that $N(P) = C(P)$, then P has a normal complement in G, i. e. there is $N \trianglelefteq G$ such that $G = NP$, $N \cap P = 1$.

Next, we present our result.

Theorem 9. Let G be a finite Frobenius group with complement H. Assume that H is an abelian group. Then K is a normal subgroup of G.

Proof. By assumption $H \leq C_H(P)$, where P is a Sylow p-subgroup of H, and hence G (by proof of Lemma 7). But by Lemma 6, $C_G(P) \leq H$, hence $C_G(P) = H = N_G(H) = N_G(P)$. Now, by Burnside's Theorem, there exists a normal subgroup N_p of G such that $G = PN_p$. Set

$$N = \bigcap_{p \in \pi(H)} N_p.$$

Then N is a normal complement of H. By Lemma 5, $N \subseteq K$. Since

$$|N| = [G : H] = |K|,$$

we conclude that $|N| = |K|$. This implies that $K = N \trianglelefteq G$.

□

3. Acknowledgment

I would like to express my sincere gratitude to the referee for his careful reading of the manuscript and for providing insightful and constructive comments that improved Theorem 9.

References

[1] K. Corradi & E. Horvath, (1996). Steps towards on elementary proof of the Frobeniuse theorem, Comm. Alg. 24(7), 2285-2292.
[2] F. G. Frobenius, (1901). Uber auflosbare Groppen IV, S'ber. Akad. Wiss. Berlin, 1216-1230; Ges, Abh. III, 189-203.
[3] D. Gorenstein, (1968). Finite groups, Harper & Row, Publications, New York, Evanston, and New York.
[4] W. R. Scott, (1964). Group theory, Dover publishing Inc., New York.
[5] M. Suzuki, (1982). Group theory I, Springer-Verlay, Berlin, Heidelberg, New York.

Mohammad Reza Darafsheh
Orcid number: 0000-0002-5539-4989
School of mathematics, statistics and computer science
College of science
University of Tehran
Tehran, Iran
Email address: darafsheh@ut.ac.ir