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SOME RESULTS ON COMPLEX $(p, q)-$EXTENSION $\alpha-$CHEBYSHEV DIFFERENTIAL EQUATION FOR $|x| \leq 1$

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Abstract. In this paper, we define complex $(p, q)-$extension $\alpha-$Chebyshev differential equations on $|x| \leq 1$. Our consideration is focused on determining properties of generalized $\alpha-$Chebyshev polynomials of the first, second, third and Fourth kind, sparking interest in constructing a theory similar to the classic8al one and complex $(p, q)-$extension Chebyshev polynomials. We solve the complex $(p, q)-$extension $\alpha-$Chebyshev differential equations on $|x| \leq 1$.

Keywords: Complex $(p, q)-$extension $\alpha-$Chebyshev polynomials, Complex $(p, q)-$
extension $\alpha-$Chebyshev differential equations, Complex $(p, q)-$extension $\alpha-$Chebyshev differential equation wavelets.
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1. Introduction

Chebyshev polynomials appear in many areas of mathematics. In recent years, this interest has often arisen from outside the subject of orthogonal polynomials, after their connection with the class of analytic functions. We define $(p, q)-$extension $\alpha-$Chebyshev polynomials, and we obtain some results on them. These polynomials are extension of $q-$Chebyshev polynomials and complex $(p, q)-$extension Chebyshev.

In the following table, we have $\alpha-$Chebyshev polynomials on $|x| \leq 1$, where $x = cos\theta$ (see [3, 8]).

$Kinds$	$\alpha - Chebyshev\ functions$
$First - Kind$	$T_n^\alpha(x) = \cos{(n+\alpha)\theta}$
$Second - Kind$	$U_n^\alpha(x) = \frac{\sin(n+1-\alpha)(\theta)}{\sin(\theta)}$
$Third - Kind$	$V_n^\alpha(x) = \frac{\cos(n+\alpha)(\theta)}{\cos(\theta)}$
$Fourth\ Kind$	$W_n^\alpha(x) = \sin{(n+\alpha)\theta}$

We define complex $(p, q)-$extension $\alpha-$Chebyshev polynomials on $D = \{z : |z| < 1\}$ (see [1, 7, 9]).

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Definition 1.1. For $x = \cos\theta,\ \theta \in [0, 2\pi]\ n = 0, 1, 2, 3, \ldots$ and $p, q \in (-1, 1]$

$Kinds$	$(p,q) - extension\ \alpha - Chebyshev\ polynomials$
$First - Kind$	$T_n^{\alpha}(p,q,\theta) = \frac{1}{2}(p^{(n+\alpha)}e^{i(n+\alpha)\theta} + q^{(n+\alpha)}e^{-i(n+\alpha)\theta})$
$Second - Kind$	$U_n^{\alpha}(p,q,\theta) = \frac{p^{n+\alpha+1}e^{i(n+\alpha+1)\theta} - q^{n+\alpha+1}e^{-i(n+\alpha+1)\theta}}{pe^{i\theta} - qe^{-i\theta}}$
$Third - Kind$	$V_n^{\alpha}(p,q,\theta) = \frac{p^{n+\alpha+\frac{1}{2}}e^{i(n+\alpha+\frac{1}{2})\theta} + q^{n+\alpha+\frac{1}{2}}e^{-i(n+\alpha+\frac{1}{2})\theta}}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}}$
$Fourth - Kind$	$W_n^{\alpha}(p,q,\theta) = \frac{p^{n+\alpha+\frac{1}{2}}e^{i(n+\alpha+\frac{1}{2})\theta} - q^{n+\alpha+\frac{1}{2}}e^{-i(n+\alpha+\frac{1}{2})\theta}}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} - q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}}.$

Since $(sec\theta + tg\theta)(sec\theta - tg\theta) = 1$, if $\alpha = \frac{1}{2}$(It is said that pseudo Chebyshev. We can write some statements of $T_n^{\frac{1}{2}}(x), U_n^{\frac{1}{2}}(x), V_n^{\frac{1}{2}}(x), W_n^{\frac{1}{2}}(x)$ for $|x| \geq 1$. then

$$\begin{cases} T_0^{\frac{1}{2}}(x) = \frac{(sec\theta+tg\theta)^{\frac{1}{2}} + (sec\theta-tg\theta)^{\frac{1}{2}}}{2} = A^{\frac{1}{2}}, \\ T_1^{\frac{1}{2}}(x) = \frac{(sec\theta+tg\theta)^{1+\alpha} + (sec\theta-tg\theta)^{1+\alpha}}{2} = A^{\frac{1}{2}}(2sec\theta - 1) = A^{\frac{1}{2}}(2x-1), \\ T_n^{\frac{1}{2}}(x) = A^{\frac{1}{2}}((sec\theta+tg\theta)^n + (sec\theta-tg\theta)^n) - A^{n-\frac{1}{2}}, \end{cases} ,$$

$$\begin{cases} U_0^{\frac{1}{2}}(x) = \frac{(sec\theta+tg\theta)^{0+\frac{1}{2}+1} - (sec\theta-tg\theta)^{0+\frac{1}{2}+1}}{2tg\theta} = B^{\frac{1}{2}}, \\ U_1^{\frac{1}{2}}(x) = \frac{(sec\theta+tg\theta)^{1+\alpha+1} - (sec\theta-tg\theta)^{1+\alpha+1}}{2tg\theta} = B^{\frac{1}{2}}(2sec\theta + 1) = B^{\frac{1}{2}}(2x+1), \\ U_n^{\frac{1}{2}}(x) = B^{\frac{1}{2}}((sec\theta+tg\theta)^n + (sec\theta-tg\theta)^n) + B^{n-\frac{1}{2}}, \end{cases}$$

$$\begin{cases} V_0^{\frac{1}{2}}(x) = \frac{(sec\theta+tg\theta)^{0+\frac{1}{2}+\frac{1}{2}} - (sec\theta-tg\theta)^{0+\frac{1}{2}+\frac{1}{2}}}{(sec\theta+tg\theta)^{\frac{1}{2}} + (sec\theta-tg\theta)^{\frac{1}{2}}} = C^{\frac{1}{2}}, \\ 228V_1^{\frac{1}{2}}(x) = \frac{(sec\theta+tg\theta)^{1+\alpha+\frac{1}{2}} - (sec\theta-tg\theta)^{1+\frac{1}{2}+\frac{1}{2}}}{(sec\theta+tg\theta)^{\frac{1}{2}} + (sec\theta-tg\theta)^{\frac{1}{2}}} = C^{\frac{1}{2}}(2sec\theta - 1) = C^{\frac{1}{2}}(2x-1), \\ V_n^{\frac{1}{2}}(x) = C^{\frac{1}{2}}((sec\theta+tg\theta)^n + (sec\theta-tg\theta)^n) + C^{n-\frac{1}{2}}, \end{cases}$$
,

$$\begin{cases} W_0^{\frac{1}{2}}(x) = \frac{(sec\theta+tg\theta)^{0+\frac{1}{2}+\frac{1}{2}} - (sec\theta-tg\theta)^{0+\frac{1}{2}+\frac{1}{2}}}{(sec\theta+tg\theta)^{\frac{1}{2}} - (sec\theta-tg\theta)^{\frac{1}{2}}} = D^{\frac{1}{2}} \\ W_1^{\frac{1}{2}}(x) = \frac{(sec\theta+tg\theta)^{1+\alpha+\frac{1}{2}} - (sec\theta-tg\theta)^{1+\frac{1}{2}+\frac{1}{2}}}{(sec\theta+tg\theta)^{\frac{1}{2}} - (sec\theta-tg\theta)^{\frac{1}{2}}} = D^{\frac{1}{2}}(2sec\theta + 1) = D^{\frac{1}{2}}(2x+1) \\ W_n^{\frac{1}{2}}(x) = D^{\frac{1}{2}}((sec\theta+tg\theta)^n + (sec\theta-tg\theta)^n) + D^{n-\frac{1}{2}}. \end{cases} .$$

Note that in the following results, we need condition $p, q \in (-1, 1]$ and $pq = 1$.

Remark 1.2. The trigonometric polynomials $T_n^{\alpha}(p;q;\theta)$, $U_n^{\alpha}(p;q;\theta)$, $V_n^{\alpha}(p;q;\theta)$ and $W_n\alpha(p;q;\theta)$ satisfy the three-term recurrence relations for $n \geq 0$

(i) $T_{n+2}^{\alpha}(p;q;\theta)) = (pe^{i\theta} + qe^{-i\theta})T_{n+1}^{\alpha}(p,q;\theta) - pqT_n^{\alpha}(p;q;\theta)$,
$T_0^{\alpha}(p;q;\theta) = \frac{1}{2}(p^{i(\alpha)}e^{(\alpha)\theta} + q^{(\alpha)}e^{-i(\alpha)\theta})$
and
$T_1^{\alpha}(p;q;\theta) = \frac{1}{2}(p^{i(1+\alpha)}e^{(1+\alpha)\theta} + q^{(1+\alpha)}e^{-i(1+\alpha)\theta})$,

(ii) $U_{n+2}^{\alpha}(p;q;\theta)) = (pe^{i\theta} + qe^{-i\theta})U_{n+1}^{\alpha}(p;q;\theta) - pqU_n^{\alpha}(p;q;e^{i\theta})$,
$U_0^{\alpha}(p;q;\theta) = p^{\alpha}e^{i\alpha\theta} + q^{\alpha}e^{-i\alpha\theta}$
and

$U_1^{\alpha}(p;q;\theta) = p^{\alpha+1}e^{i(\alpha+1)\theta} + q^{\alpha+1}e^{-i(\alpha+1)\theta}$,

(iii)

$$V_{n+2}^{\alpha}(p;q;\theta)) = (pe^{i\theta} + qe^{-i\theta})V_{n+1}^{\alpha}(p;q;\theta) - pqV_n^{\alpha}(p;q;\theta),$$

$V_0^{\alpha}(p;q;i\theta) = p^{\alpha}e^{i\alpha\theta} + q^{\alpha}e^{-i\alpha\theta} - p^{\alpha-\frac{1}{2}}e^{i(\alpha-\frac{1}{2})\theta} - q^{\alpha-\frac{1}{2}}e^{-i(\alpha-\frac{1}{2})\theta}$
and
$V_1^{\alpha}(p;q;\theta) = p^{\alpha}e^{i\alpha\theta} + q^{\alpha}e^{-i\alpha\theta} - p^{\alpha-\frac{3}{2}}e^{i(\alpha-\frac{3}{2})\theta} - q^{\alpha-\frac{3}{2}}e^{-i(\alpha-\frac{3}{2})\theta}$,

(iv)

$$W_{n+2}^{\alpha}(p;q;\theta)) = (pe^{i\theta} + qe^{-i\theta})W_{n+1}^{\alpha}(p;q;\theta) - pqW_n^{\alpha}(p;q;\theta),$$

$W_0^{\alpha}(p;q;\theta) = p^{\alpha}e^{i\alpha\theta} - q^{\alpha}e^{-i\alpha\theta} - p^{\alpha-\frac{1}{2}}e^{i(\alpha-\frac{1}{2})\theta} + q^{\alpha-\frac{1}{2}}e^{-i(\alpha-\frac{1}{2})\theta}$
and
$W_1^{\alpha}(p;q;\theta) = p^{\alpha}e^{i\alpha\theta} - q^{\alpha}e^{-i\alpha\theta} - p^{\alpha-\frac{3}{2}}e^{i(\alpha-\frac{3}{2})\theta} + q^{\alpha-\frac{1}{2}}e^{-i(\alpha-\frac{1}{2})\theta}p^{\alpha}e^{i\alpha\theta} - q^{\alpha}e^{-i\alpha\theta} - p^{\alpha-\frac{3}{2}}e^{i(\alpha-\frac{3}{2})\theta} + q^{\alpha-\frac{1}{2}}e^{-i(\alpha-\frac{1}{2})\theta}$.

Proof. (i)

$$\begin{aligned} T_{n+2}^{\alpha}(x) &= \frac{1}{2}(p^{n+\alpha+2)}e^{i(n+\alpha+2)\theta} + q^{(n+\alpha+2)}e^{-i(n+\alpha+2)\theta}) \\ &= \frac{1}{2}(pe^{i\theta} + qe^{-i\theta})(p^{n+\alpha+1}e^{i(n+\alpha+1)\theta} + q^{(n+\alpha+1)}e^{-i(n+\alpha+1)\theta}) \\ &- \frac{1}{2}pq(p^{n+\alpha}e^{i(n+\alpha)\theta} + q^{n+\alpha}e^{-i(n+\alpha)\theta}) \\ &= (pe^{i\theta} + qe^{-i\theta})T_{n+1}^{\alpha}(q;e^{i\theta}) - pqT_n^{\alpha}(q;e^{i\theta}). \end{aligned}$$

(ii)

$$\begin{aligned} U_{n+2}^{\alpha}(x) &= \frac{p^{n+\alpha+2}e^{i(n+\alpha+2)\theta} - q^{n+\alpha+2}e^{-i(n+\alpha+2)\theta}}{pe^{i\theta} - qe^{-i\theta}} \\ &= \frac{1}{2}(pe^{i\theta} + qe^{-i\theta})\Big(\frac{p^{n+\alpha+1}e^{i(n+\alpha+1)\theta} - q^{n+\alpha+1}e^{-i(n+\alpha+1)\theta}}{pe^{i\theta} - qe^{-i\theta}}\Big) \\ &- \frac{1}{2}pq\Big(\frac{p^{n+\alpha+1}e^{i(n+\alpha+1)\theta} - q^{n+\alpha+1}e^{-i(n+\alpha+1)\theta}}{pe^{i\theta} - qe^{-i\theta}}\Big) \\ &= (pe^{i\theta} + qe^{-i\theta})U_{n+1}^{\alpha}(q;e^{i\theta}) - pqU_n^{\alpha}(q;e^{i\theta}). \end{aligned}$$

(iii) From Remark 1.1

$$\begin{aligned} V_{n+1}^{\alpha}(x) &= U_{n+1}^{\alpha}(x) - U_n^{\alpha}(x) \\ &= 2xU_n^{\alpha}(x) - U_{n-1}^{\alpha}(x) - 2xU_{n-1}^{\alpha}(x) + U_{n-2}^{\alpha}(x) \\ &= 2x[U_n^{\alpha}(x) - U_{n-1}^{\alpha}(x)] - [U_{n-1}^{\alpha}(x) - U_{n-2}^{\alpha}(x)] \\ &= 2xV_n^{\alpha}(x) - V_{n-1}^{\alpha}(x). \end{aligned}$$

(iv) From Remark 1.1

$$\begin{aligned} W_{n+1}^{\alpha}(x) &= U_{n+1}^{\alpha}(x) + U_n^{\alpha}(x) \\ &= 2xU_n^{\alpha}(x) - U_{n-1}^{\alpha}(x) + 2xU_{n-1}^{\alpha}(x) - U_{n-2}^{\alpha}(x) \\ &= 2x[U_n^{\alpha}(x) + U_{n-1}^{\alpha}(x)] - [U_{n-1}^{\alpha}(x) + U_{n-2}^{\alpha}(x)] \\ &= 2xW_n^{\alpha}(x) - W_{n-1}^{\alpha}(x). \end{aligned}$$

□

Remark 1.3. For $x = cos\theta$, $\theta \in [0, 2\pi]$, then the following statements are satisfying:

(i) $W_n^{\alpha}(p;q;\theta) = U_n^{\alpha}(p;q;\theta) + U_{n-1}^{\alpha}(p;q;\theta)$;
(ii) $V_n^{\alpha}(p;q;\theta) = U_n^{\alpha}(p;q;\theta) - U_{n-1}^{\alpha}(p;q;\theta)$;
(iii) $U_n^{\alpha}(p;q;\theta) = \frac{1}{2}(V_n^{\alpha}(p;q;\theta) + W_n^{\alpha}(p;q;\theta))$;
(vi) $V_n^{\alpha}(p;q;\theta) + V_{n-1}^{\alpha}(p;q;\theta) = 2T_n^{\alpha}(p;q;\theta)$;
(vi) $W_n^{\alpha}(p;q;\theta) + W_{n-1}^{\alpha}(p;q;\theta) = 2T_n^{\alpha}(p;q;\theta)$.

Proof. (i) Since $pq = 1$

$$\begin{aligned} & U_n^{\alpha}(p;q;\theta) - U_{n-1}^{\alpha}(p;q;\theta) \\ = \; & \frac{p^{n+\alpha+1}e^{i(n++\alpha+1)\theta} - q^{n+\alpha+1}e^{-i(n+\alpha+1)\theta}}{pe^{i\theta} - qe^{-i\theta}} \\ + \; & \frac{p^{n+\alpha}e^{i(n+\alpha)\theta} - q^{n+\alpha}e^{-i(n+\alpha)\theta}}{pe^{i\theta} - qe^{-i\theta}} \\ = \; & \frac{p^{n+\alpha+1}e^{i(n+\alpha+1)\theta} - q^{n+\alpha+1}e^{-i(n+\alpha+1)\theta} + p^{n}e^{i(n+\alpha)\theta} - q^{n}e^{-i(n+\alpha)\theta}}{pe^{i\theta} - qe^{-i\theta}} \\ = \; & \frac{(p^{n+\frac{1}{2}}e^{i(n+\frac{1}{2})\theta} - q^{n+\frac{1}{2}}e^{-i(n+\alpha+\frac{1}{2})\theta})(p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta})}{(p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta})(p^{\frac{1}{2}}e^{\frac{i}{2}\theta} - q^{\frac{1}{2}}e^{-\frac{i}{2}\theta})} \\ = \; & \frac{p^{n+\alpha+\frac{1}{2}}e^{i(n+\alpha+\frac{1}{2})\theta} - q^{n+\alpha+\frac{1}{2}}e^{-i(n+\frac{1}{2})\theta}}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} - q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}} \\ = \; & V_n^{\alpha}(p;q;\theta). \end{aligned}$$

(ii) Since $pq = 1$

$$
\begin{aligned}
& U_n^\alpha(p;q;\theta) + U_{n-1}^\alpha(p;q;\theta) \\
= & \frac{p^{n+\alpha+1}e^{i(n++\alpha+1)\theta} - q^{n+\alpha+1}e^{-i(n+\alpha+1)\theta}}{pe^{i\theta} - qe^{-i\theta}} \\
+ & \frac{p^{n+\alpha}e^{i(n+\alpha)\theta} - q^{n+\alpha}e^{-i(n+\alpha)\theta}}{pe^{i\theta} - qe^{-i\theta}} \\
= & \frac{p^{n+\alpha+1}e^{i(n+\alpha+1)\theta} - q^{n+\alpha+1}e^{-i(n+\alpha+1)\theta} + p^n e^{i(n+\alpha)\theta} - q^n e^{-i(n+\alpha)\theta}}{pe^{i\theta} - qe^{-i\theta}} \\
= & \frac{(p^{n+\frac{1}{2}}e^{i(n+\frac{1}{2})\theta} - q^{n+\frac{1}{2}}e^{-i(n+\alpha+\frac{1}{2})\theta})(p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta})}{(p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta})(p^{\frac{1}{2}}e^{\frac{i}{2}\theta} - q^{\frac{1}{2}}e^{-\frac{i}{2}\theta})} \\
= & \frac{p^{n+\alpha+\frac{1}{2}}e^{i(n+\alpha+\frac{1}{2})\theta} - q^{n+\alpha+\frac{1}{2}}e^{-i(n+\frac{1}{2})\theta}}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} - q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}} \\
= & W_n^\alpha(p;q;\theta).
\end{aligned}
$$

(iii) Since $pq = 1$, from (i), (ii) is trival.

(iv) Since $pq = 1$,

$$
\begin{aligned}
V_n^\alpha(p;q;\theta) + V_{n-1}^\alpha(p;q;\theta) &= \frac{p^{n+\alpha+\frac{1}{2}}e^{i(n+\alpha+\frac{1}{2})\theta} + q^{n+\alpha+\frac{1}{2}}e^{-i(n+\alpha+\frac{1}{2})\theta}}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}} \\
&+ \frac{p^{n+\alpha-\frac{1}{2}}e^{i(n+\alpha+\frac{1}{2})\theta} + q^{n+\alpha+\frac{1}{2}}e^{-i(n+\alpha+\frac{1}{2})\theta}}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}} \\
&= \frac{(p^{(n+\alpha)}e^{i(n+\alpha)\theta} + q^{(n+\alpha)}e^{-i(n+\alpha)\theta})(p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta})}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}} \\
&= p^{n+\alpha}e^{i(n+\alpha)\theta} + q^{n+\alpha}e^{-i(n+\alpha)\theta} \\
&= 2T_n^\alpha(p;q;\theta).
\end{aligned}
$$

(v) Since $pq = 1$

$$
\begin{aligned}
W_n^\alpha(p;q;\theta) + W_{n-1}^\alpha(p;q;\theta) &= \frac{p^{n+\alpha+\frac{1}{2}}e^{i(n+\alpha+\frac{1}{2})\theta} - q^{n+\alpha+\frac{1}{2}}e^{-i(n+\alpha+\frac{1}{2})\theta}}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} - q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}} \\
&+ \frac{p^{n+\alpha-\frac{1}{2}}e^{i(n+\alpha-\frac{1}{2})\theta} - q^{n+\alpha-\frac{1}{2}}e^{-i(n+\alpha+\frac{1}{2})\theta}}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} + q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}} \\
&= \frac{(p^{(n+\alpha)}e^{i(n+\alpha)\theta} + q^{(n+\alpha)}e^{-i(n+\alpha)\theta})(p^{\frac{1}{2}}e^{\frac{i}{2}\theta} - q^{\frac{1}{2}}e^{-\frac{i}{2}\theta})}{p^{\frac{1}{2}}e^{\frac{i}{2}\theta} - q^{\frac{1}{2}}e^{-\frac{i}{2}\theta}} \\
&= p^{n+\alpha}e^{i(n+\alpha)\theta} + q^{n+\alpha}e^{-i(n+\alpha)\theta} \\
&= 2T_n^\alpha(p;q;\theta).
\end{aligned}
$$

□

Lemma 1.4. *For $n \geq 1$, $p, q \in (-1, 1]$ and $\alpha > 0$ we have:*

(i)

$$\begin{aligned}
(& p^{n+\alpha+1}e^{i(n+\alpha+1)\theta}-q^{n+\alpha+1}e^{-i(n+\alpha+1)\theta}) \\
= & (pe^{i\theta}-qe^{-i\theta})(p^{n+\alpha}e^{i(n+\alpha)\theta}+q^{n+\alpha}e^{-i(n+\alpha)\theta}) \\
+ & pq(p^{n+\alpha-1}e^{i(n+\alpha-1)\theta}-q^{n+\alpha-1}e^{-i(n+\alpha-1)\theta}).
\end{aligned}$$

(ii)

$$\begin{aligned}
(& p^{n+\alpha+\frac{1}{2}}e^{i(n+\alpha+\frac{1}{2})\theta}+q^{n+\alpha+\frac{1}{2}}e^{-i(n+\alpha+\frac{1}{2})\theta}) \\
= & (p^{\frac{1}{2}}e^{\frac{i\theta}{2}}+q^{\frac{1}{2}}e^{\frac{-i\theta}{2}})(p^{n+\alpha}e^{i(n+\alpha)\theta}+q^{n+\alpha}e^{-i(n+\alpha)\theta}) \\
- & p^{\frac{1}{2}}q^{\frac{1}{2}}((p^{n+\alpha-\frac{1}{2}}e^{i(n+\alpha-\frac{1}{2})\theta}+q^{n+\alpha-\frac{1}{2}}e^{-i(+\alpha-\frac{1}{2}\theta}).
\end{aligned}$$

(iii)

$$\begin{aligned}
(& p^{n+\alpha+\frac{1}{2}}e^{i(n+\alpha+\frac{1}{2})\theta}-q^{n+\alpha+\frac{1}{2}}e^{-i(n++\alpha+\frac{1}{2})\theta}) \\
= & (p^{\frac{1}{2}}e^{\frac{i\theta}{2}}-q^{\frac{1}{2}}e^{\frac{-i\theta}{2}})(p^{n+\alpha}e^{i(n+\alpha)\theta}+q^{n+\alpha}e^{-i(n+\alpha)\theta}) \\
+ & p^{\frac{1}{2}}q^{\frac{1}{2}}((p^{n+\alpha-\frac{1}{2}}e^{i(n+\alpha-\frac{1}{2})\theta}-q^{n+\alpha-\frac{1}{2}}e^{-i(n-\frac{1}{2})\theta}).
\end{aligned}$$

Theorem 1.5. *The first kind $(p,q)-$extension $\alpha-$Chebyshev function for $|x|\leq 1$ is a solution for the first kind $(p,q)-$extension $\alpha-$Chebyshev differential equation $(1-x^2)y''-xy'-(n+\alpha)^2y=0$.*

Proof. i) We show that $y=\frac{1}{2}(p^{(n+\alpha)}e^{(n+\alpha)\theta}+q^{(n+\alpha)}e^{-i(n+\alpha)\theta})$ is a solution of $(p,q)-$extension $\alpha-$ Chebyshev differential equation Chebyshev differential equation of the first kind $(1-x^2)y''-xy'+n^2y=0$. This equation can be converted to a simpler form using the substitution $x= cos\theta$. Indeed, in this case, we have

$$\begin{aligned}
y' &= \frac{dy}{dx} \\
&= \frac{dy}{d\theta}\frac{d\theta}{dx} \\
&= \frac{dy}{d\theta}\frac{1}{\frac{dx}{d\theta}} \\
&= \frac{1}{2}(i(n+\alpha)p^{(n+\alpha)}e^{i(n+\alpha)\theta}-i(n+\alpha)q^{(n+\alpha)}e^{-i(n+\alpha)\theta})\frac{1}{-\sin\theta} \\
&= -\frac{i(n+\alpha)(p^{(n+\alpha)}e^{i(n+\alpha)}-q^{(n+\alpha)}e^{-i(n+\alpha)})}{2\sin\theta},
\end{aligned}$$

and

$$\begin{aligned} y'' &= \frac{d^2y}{dx^2} \\ &= -\frac{i(n+\alpha)}{2}\frac{d}{d\theta}[\frac{p^{(n+\alpha)}e^{i(n+\alpha)} - q^{(n+\alpha)}e^{-i(n+\alpha)}}{\sin\theta}]\frac{d\theta}{dx} \\ &= \frac{i(n+\alpha)}{2}[\frac{(i(n+\alpha)p^{(n+\alpha)}e^{i(n+\alpha)\theta} + q^{(n+\alpha)}e^{i(n+\alpha)\theta})\sin\theta}{\sin^3\theta}] \\ &- \frac{(p^{(n+\alpha)}e^{i(n+\alpha)} - q^{(n+\alpha)}e^{-i(n+\alpha)})(\cos\theta}{\sin^3\theta}] \\ &= \frac{i(n+\alpha)}{2}[\frac{i(n+\alpha)(p^{(n+\alpha)}e^{i(n+\alpha)\theta} + q^{(n+\alpha)}e^{-i(n+\alpha)\theta})\sin\theta}{\sin^3\theta}] \\ &- \frac{(p^{(n+\alpha)}e^{i(n+\alpha)} - q^{(n+\alpha)}e^{-i(n+\alpha)})(\cos\theta)}{\sin^3\theta}] \end{aligned}$$

Substituting the expressions of derivatives into the differential equation gives:

$$\begin{aligned} (& \quad 1-x^2)y'' - xy' + (n+\alpha)^2 y \\ = & \quad (\sin^2\theta)\frac{i(n+\alpha)}{2}[\frac{i(n+\alpha)(p^{(n+\alpha)}e^{i(n+\alpha)\theta} + q^{(n+\alpha)}e^{-(n+\alpha)n\theta})\sin\theta - (p^{(n+\alpha)}e^{i(n+\alpha)}}{\sin^3\theta} \\ - & \quad \frac{q^{(n+\alpha)}e^{-i(n+\alpha)})(\cos\theta)}{\sin^3\theta}] + \cos\theta\frac{i(n+\alpha)(p^{(n+\alpha)}e^{i(n+\alpha)} - q^n e^{-i(n+\alpha)})}{2\sin\theta} \\ + & \quad \frac{n^2}{2}(p^{(n+\alpha)}e^{i(n+\alpha)\theta} + q^{(n+\alpha)}e^{-i(n+\alpha)\theta}) = 0. \end{aligned}$$

□

2. Complex $(p, q)-$Extension $\alpha-$Chebyshev Wavelets differential equation on $|x| \leq 1$

In the section, we consider multi resolution analysis (see [4-6]) **(MRA)**.

Definition 2.1. Multiresolution Analysis: An **MRA** with scaling function ϕ is a collection of closed subspaces $\{V_j\}_{j\in Z}$ of $L^2((\mathbb{R})$, such that

(i) $V_j \subset V_{j+1}$;

(ii) $f(x) \in V_j \iff f(2x) \in V_{j+1}$

(iii) $\overline{\cup V_j} = L^2((\mathbb{R})$,

(iv) $\cap V_j = 08$

(v) There exists a function $\phi \in V_0$ such that the collection $\{\phi(x–k) : \ k \in \mathbb{Z}\}$ is a Riesz basis of V_0

The sequence of wavelet subspaces W_j of $L^2(\mathbb{R})$, are such that $V_j \perp W_j$, for all j and $V_{j+1} = V_j \bigoplus W_j$. Closure of $\bigoplus W_j$ is dense in $L^2(\mathbb{R})$ for L^2 norm.

Now we state Mallat's theorem which guarantees that in the presence of an orthogonal **MRA**, an orthonormal basis for $L^2(\mathbb{R})$ exists. These basis functions are fundamental in the theory of wavelets which helps us to develop advanced computational techniques.

Lemma 2.2. **[Mallat's Theorem]** *Given an orthogonal MRA with scaling function ϕ, there is a wavelet $\psi \in L^2(\mathbb{R})$ such that for each $j \in \mathbb{Z}$, the family $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for W_j. Hence the family $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Definition 2.3. Suppose $k \in \mathbb{N}$ (degree of multiresolution), $m \geq 0$, $n = 1, 2, \cdots, 2^{k-1}$. We define:

i) first kind complex $(p, q)-$extension $\alpha-$Chebyshev wavelets on $[0, L]$

$$T^{\alpha}_{n,m}(p, q, t) = \sqrt{\frac{2^{k+1}}{n}}T^{\alpha}_{m}(p; q; e^{i(\frac{2^k}{L}t-2n+1)})\boldsymbol{\chi}_{[\frac{(n-1)L}{2^{k-1}}, \frac{nL}{2^{k-1}})}(t).$$

ii) second kind complex $(p, q)-$extension$\alpha-$ Chebyshev wavelets on $[0, L]$

$$U^{\alpha}_{n,m}(p, q, t) = \sqrt{\frac{2^{k+1}}{n}}U^{\alpha}_{m}(p; q; e^{i(\frac{2^k}{L}t-2n+1)})\boldsymbol{\chi}_{[\frac{(n-1)L}{2^{k-1}}, \frac{nL}{2^{k-1}})}(t).$$

iii) third kind complex $(p, q)-$extension $\alpha-$Chebyshev wavelets on $[0, L]$

$$V^{\alpha}_{n,m}(p, q, t) = \sqrt{\frac{2^{k+1}}{n}}V^{\alpha}_{m}(p; q; e^{i(\frac{2^k}{L}t-2n+1))}\boldsymbol{\chi}_{[\frac{(n-1)LL}{2^{k-1}}, \frac{n}{2^{k-1}})}(t).$$

iv) fourth kind complex $(p, q)-$extension $\alpha-$Chebyshev wavelets on $[0, L]$

$$W^{\alpha}_{n,m}(p, q, t) = \sqrt{\frac{2^{k+1}}{n}}W^{\alpha}_{m}(p; q; e^{i(\frac{2^k}{L}t-2n+1)})\boldsymbol{\chi}_{[\frac{(n-1)L}{2^{k-1}}, \frac{nL}{2^{k-1}})}(t).$$

Theorem 2.4. *The first kind $(p, q)-$ extension $\alpha-$Chebyshev wavelet $T^{\alpha}_{n,m}(p, q, e^{it})$ is a solution of differential equation*

$$(1-(\frac{2^k}{L}t-2n+1)^2)\frac{d^2T^{\alpha}_{n,m}(p, q, t)}{dt^2}-(\frac{2^k}{L}t-2n+1))\frac{dT^{\alpha}_{n,m}(p, q, t)}{dt}+(n+\alpha)^2T^{\alpha}_{n,m}(p, q, t) = 0.$$

Proof. If $t \in (\frac{(n+\alpha-1)L}{2^{k-1}}, \frac{(n+\alpha)L}{2^{k-1}})$ and $y = T^{\alpha}_{m}(x) = \sqrt{\frac{n}{2^{k+1}}}T^{\alpha}_{n,m}$, where $x = \frac{2^k}{L}t - 2(n + \alpha) + 1$. Then

$$(1 - x^2)y'' - xy' + (n + \alpha)^2y = 0,$$

$$
\begin{aligned}
\frac{dT^{\alpha}_{n,m}(p,q,t)}{dt} &= \sqrt{\frac{2^{k+1}}{n}}\frac{dT^{\alpha}_{m}(p,q,t)}{dx}\frac{dx}{dt}. \\
&= \frac{2^k}{n}\sqrt{\frac{2^{k+1}}{n}}\frac{dT^{\alpha}_{m}(p,q,e^{it})}{dx}, \\
\frac{dT^{\alpha}_{m}(p,q,t)}{dx} &= \sqrt{\frac{n^3}{2^{\frac{3k}{2}+\frac{1}{2}}}}\sqrt{\frac{2}{\pi}}\frac{dT^{\alpha}_{n,m}(p,q,t)}{dt} \\
\frac{d^2T^{\alpha}_{n,m}(p,q,t)}{dt^2} &= 2^{\frac{3k}{2}}\sqrt{\frac{2}{\pi}}\frac{d^2T^{\alpha}_{m}(p,q,t)}{dx^2}\frac{dx}{dt}. \\
&= 2^{\frac{5k}{2}}\sqrt{\frac{2}{\pi}}\frac{d^2T^{\alpha}_{m}}{dx^2}(p,q,e^{it}) \\
\frac{d^2T^{\alpha}_{m}(x)}{dx^2} &= \frac{1}{2^{\frac{5k}{2}}\sqrt{\frac{2}{\pi}}}\frac{d^2T^{\alpha}_{n,m}(p,q,t)}{dt^2}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& (1-(\frac{2^k}{L}t-2(n+\alpha)+1)^2)\frac{1}{2^{\frac{5k}{2}}\sqrt{\frac{2}{\pi}}}\frac{d^2T^{\alpha}_{n,m}(p,q,t)}{dt^2} \\
- & (\frac{2^k}{L}t-2(n+\alpha)+1)\frac{1}{2^{\frac{3k}{2}}\sqrt{\frac{2}{\pi}}}\frac{d\Psi_{n,m}(p,q,t)}{dt}+\frac{(n+\alpha)^2}{2^{\frac{k}{2}}\sqrt{\frac{2}{\pi}}}T^{\alpha}_{n,m}(p,q,t) \\
= & (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+(n+\alpha)^2y \\
= & 0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& (1-(\frac{2^k}{L}t-2(n+\alpha)+1)^2)\frac{d^2T_{n,m}(p,q,t)}{dt^2}-(\frac{2^k}{L}t-2(n+\alpha)+1))\frac{dT_{n,m}(p,q,t)}{dt} \\
+ & (n+\alpha)^2T^{\alpha}_{n,m}(p,q,t)=0.
\end{aligned}
$$

□

3. conclusion

In this paper we define complex (p,q)-extension α-Chebyshev polynomials. If $\alpha=0$, it is complex (p,q)-extensiin Chebyshev polynomials. Therefore this concept is a extension of complex (p,q)-extension of Chebyshev polynomials. Also, we define $(p,q)-$extension $\alpha-$ Chebyshev differential equations, and show all complex (p,q)-extension α-Chebyshev polynomials are solutions of them. We can write there are these results for $|x|\geq 1$.

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