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AMALGAMATION PROPERTY IN THE SUBVARIETIES OF GAUTAMA AND ALMOST GAUTAMA ALGEBRAS

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Abstract. Gautama algebras were introduced in 2022, as a common generalization of regular double Stone algebras and regular Kleene Stone algebras. Even more recently, Gautama algebras were further generalized to Almost Gautama algebras ($\mathbb{AG}$ for short). The main purpose of this paper is to investigate the Amalgamation Property (AP, for short) in the subvarieties of the variety $\mathbb{AG}$. In fact, we show that, of the eight non-trivial subvarieties of $\mathbb{AG}$, only four varieties, namely those of Boolean algebras, of regular double Stone algebras, of regular Kleene Stone algebras and of De Morgan Boolean algebras have the AP and the remaining four do not have the AP. We give several applications of this result; in particular, we examine the following properties for the subvarieties of $\mathbb{AG}$: transferability property (TP), having enough injectives (EI), Embedding Property, Bounded Obstruction Property and having a model companion.

Keywords: regular double Stone algebra, regular Kleene Stone algebra, Gautama algebra, Almost Gautama algebra, subdirectly irreducible algebra, simple algebra, discriminator variety, amalgamation property, transferability property (TP), having enough injectives (EI), Embedding Property, Bounded Obstruction Property, model companion.
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1. INTRODUCTION

Around the middle of the nineteenth century, Boole introduced (see [11, 12]) Boole's partial algebras in an attempt to cast the classical propositional logic into an algebraic setting. About ten years later, Jevons [33] gave a modification of those partial algebras, by replacing Boole's partial operations with total operations. Around the beginning of the twentieth century, those modified algebras came to be known as "Boolean algebras." Stone algebras–a generalization of Boolean algebras– were initiated in 1950's by Grätzer and Schmidt (see [26]) in an attempt to answer a problem of Stone. Double Stone algebras arose in 1970's by considering an expansion of a Stone algebra by adding the dual Stone operation. Soon thereafter, regular double Stone algebras were

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introduced by Varlet in 1972 (see [62]) to characterize those double Stone algebras which are congruence-regular, by means of a regularity condition which was a quasi-equation. Katriňák [39] studied this quasi-equation and showed that it is equivalent to an identity (R), thus showing that the class of regular double Stone algebras is a subvariety of the variety of double Stone algebras. By suitably modifying Katriňák's identity, Sankappanavar [57] introduced a notion of regularity in the variety of Kleene Stone algebras–actually in a much larger variety– from which it was clear that the regular Kleene Stone algebras also form a variety.

The definitions of regular double Stone algebras and regular Kleene Stone algebras and several results about these two classes are quite similar (see Section 2). This similarity has recently given rise to a new variety of algebras called "Gautama algebras" (see [60]), of which the varieties of regular double Stone algebras and regular Kleene Stone algebras are subvarieties. Even more recently, in [19], Gautama algebras are further generalized to Almost Gautama algebras ($\mathbb{AG}$ for short) by a slight weakening of one of their defining identities.

The main goal of this paper is to investigate the Amalgamation Property (AP, for short) in the subvarieties of the variety $\mathbb{AG}$. In fact, we show in our main result (Theorem 3.24) that, of the eight nontrivial subvarieties of $\mathbb{AG}$, only four varieties, namely those of Boolean algebras, of regular double Stone algebras, of regular Kleene Stone algebras and of De Morgan Boolean algebras have the AP and the remaining four do not. We give several applications of this result. More specifically, we examine such properties for the subvarieties of $\mathbb{AG}$ as transferability property (TP), having enough injectives (EI), Embedding Property, Bounded Obstruction Property and having a model companion.

The paper is organized as follows: In Section 2, we present some preliminary concepts and known results that will be needed in the rest of the paper. Section 3, the heart of this paper, gives a complete description of the subvarieties of the variety of Almost Stone algebras having the amalgamation property. The main theorem (Theorem 3.24) describes precisely which of the eight nontrivial subvarieties of the variety of Almost Gautama algebras have the Amalgamation Property. In section 4, we give applications of our main theorem, combined with some earlier known general results. In fact, we characterize the subvarieties of $\mathbb{AG}$ having the properties: (TP), (EI), Embedding property, Bounded Obstruction Property and the property of having a model companion.

2. **Preliminaries**

It is assumed here that the reader has had some familiarity with lattice theory and universal algebra (see [4, 14], for example). As such, for notions, notations and results assumed here, the reader can refer to these or other relevant books. It will be helpful, but not necessary, to have some familiarity with [19, 60].

Recall that an algebra $\mathbf{A} = \langle A, \vee, \wedge, ^c, 0, 1\rangle$ is a Boolean algebra if $\mathbf{A}$ is a complemented distributive lattice (with ^{c} as the complementation or the negation). Let $\mathbb{BA}$ denote the variety of Boolean algebras. It is well-known that $\mathbb{BA} = \mathbb{V}(\mathbf{2})$ (i.e. the variety generated by $\{\mathbf{2}\}$), where $\mathbf{2}$ denotes the 2-element Boolean algebra with the universe $\{0, 1\}$. In what follows, we will be casual in that any 2-element Boolean algebra in other languages will also be denoted by $\mathbf{2}$.

Among the many generalizations of Boolean algebres, we consider here the following three:
(1) Stone algebras, (2) dual Stone algebras and (3) Kleene algebras.

Stone algebras are a subvariety of the variety of pseudocomplemented lattices. We present here the following definition given in [58, Corollary 2.8].

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a **distributive pseudocomplemented lattice** (p-algebra, for short) if $\mathbf{A}$ satisfies the following:

(1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,
(2) the operation * satisfies the identities:
(a) $0^* \approx 1$,
(b) $1^* \approx 0$,
(c) $(x \vee y)^* \approx x^* \wedge y^*$,
(d) $(x \wedge y)^{**} \approx x^{**} \wedge y^{**}$,
(e) $x \leq x^{**}$,
(f) $x^* \wedge x^{**} \approx 0$.

Note that the identity (f) can be replaced by the identity: $x \wedge x^* \approx 0$.

A p-algebra $\mathbf{A}$ is a **Stone algebra** if $\mathbf{A}$ satisfies the identity:
(St) $x^* \vee x^{**} \approx 1$ (Stone identity).

Dual Stone algebras are, of course, defined dually.

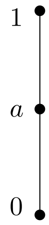
The variety of Kleene algebras is a subvariety of that of De Morgan algebras, which were first introduced by Moisil [46]) in 1935 (see also [8] and [38]).

An algebra $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a **De Morgan algebra** if
(1) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,
(2) $0' \approx 1$ and $1' \approx 0$,
(3) $(x \wedge y)' \approx x' \vee y'$ ($\wedge$-De Morgan law),
(4) $x'' \approx x$ (Involution).

A De Morgan algebra is a **Kleene algebra** if it satisfies:
(5) $x \wedge x' \leq y \vee y'$ (Kleene identity).

Let the 3-element chain (viewed as a bounded distributive lattice) be denoted by $\mathbf{3}$. The three operations, namely the pseudocomplement, the dual pseudocomplement and the De Morgan operation on the three-element chain are shown below in Figure 1.

3 :



*	0	a	1
	1	0	0

$^+$	0	a	1
	1	1	0

$'$	0	a	1
	1	a	0

Figure 1

Expansions of the language of bounded lattices by adding two unary operation symbols corresponding to any two of the above three unary operations give rise to the following three algebras on the 3-element chain: $\mathbf{3_{dblst}} = \langle 3, \vee, \wedge, ^*, ^+, 0, 1\rangle$, $\mathbf{3_{klst}} = \langle 3, \vee, \wedge, ^*, ', 0, 1\rangle$, and $\mathbf{3_{kldst}} = \langle 3, \wedge, \vee, ^+, ', 0, 1\rangle$. The algebra $\mathbf{3_{dblst}}$ is known as a "regular double Stone algebra" and $\mathbf{3_{klst}}$ as a "regular Kleene Stone algebra", while $\mathbf{3_{kldst}}$, being the dual of $\mathbf{3_{klst}}$, would not be of much interest to us in this paper.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ^+, 0, 1\rangle$ is a **regular double Stone algebra** if
(1) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a Stone algebra,
(2) $\langle A, \vee, \wedge, ^+, 0, 1\rangle$ is a dual Stone algebra,
(3) $\mathbf{A}$ satisfies the identity:
(R) $x \wedge x^+ \leq y \vee y^*$; or, equivalently, (R1) $x \wedge x^{+*+} \leq y \vee y^*$.

The variety of regular double Stone algebras is denoted by $\mathbb{RDBLS}$t. This variety was introduced in [39] and has later been investigated in many articles, see, for example, [1, 18, 59, 60].

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1\rangle$ is a **regular Kleene Stone algebra** if
(1) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a Stone algebra,
(2) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a Kleene algebra,
(3) $\mathbf{A}$ satisfies the identity:
(R1) $x \wedge x'^{*'} \leq y \vee y^*$ (Regularity).

The variety of regular Kleene Stone algebras is denoted by $\mathbb{RKLS}$t. This variety was introduced in [57] and has later been investigated by several authors– see, for example, [2, 17, 18, 20, 21, 59, 60].

The following two theorems list some of the known properties of the varieties $\mathbb{RDBLS}$t and $\mathbb{RKLS}$t.

Theorem 2.1. [59]

(i) $\mathbb{RDBLS}\text{t} = \mathbf{V}(\mathbf{3_{dblst}})$,
(ii) The variety $\mathbb{RDBLS}$t is a discriminator variety; and also $\mathbf{3_{dblst}}$ is quasiprimal; and $\mathbb{BA}$ is the only nontrivial proper subvariety of $\mathbb{RDBLS}$t.

Theorem 2.2. [59]

(i) $\mathbb{RKLS}\mathrm{t} = \mathbb{V}(\mathbf{3_{klst}})$,
(ii) $\mathbb{RKLS}\mathrm{t}$ is a discriminator variety, and $\mathbf{3_{klst}}$ is quasiprimal,

The similarities of $\mathbb{RDBLS}\mathrm{t}$ and $\mathbb{RKLS}\mathrm{t}$ led, recently, to the variety of Gautama algebras in [60], which were so named in honor and memory of Medhatithi Gautama and Aksapada Gautama, the founders of Indian Logic. To define Gautama algebras, we need the dual of the notion of an "upper quasi-De Morgan algebra" which was introduced in 1987. (We drop the word "upper.")

Quasi-De Morgan algebras arose as a subvariety of semi-De Morgan algebras which were introduced in [58] to unify pseudocomplemented lattices and De Morgan algebras. We need the dual of this notion here.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, ', 0, 1\rangle$ is a **dually quasi-De Morgan algebra** if the following conditions hold:
(a) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice,
(b) The operation $'$ is a dual quasi-De Morgan operation; that is, $'$ satisfies
(i) $0' \approx 1$ and $1' \approx 0$,
(ii) $(x \wedge y)' \approx x' \vee y'$,
(iii) $(x \vee y)'' \approx x'' \vee y''$,
(iv) $x'' \leq x$.
The variety of dually quasi-De Morgan algebras is denoted by $\mathbb{DQD}$.

We can now define "Gautama algebras."

Definition 2.1. [60] An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1\rangle$ is a **Gautama algebra** if the following conditions hold:
(a) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a Stone algebra,
(b) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually quasi-De Morgan algebra,
(c) $\mathbf{A}$ is regular; i.e., $\mathbf{A}$ satisfies:
(R1) $\quad x \wedge x'^{*\prime} \leq y \vee y^*$,
(d) $\mathbf{A}$ is star-regular; i.e., $\mathbf{A}$ satisfies the identity:
(*) $\quad x^{*\prime} \approx x^{**}$.

Let $\mathbb{G}$ denote the variety of Gautama algebras. It should be remarked here that, in the presence of (b), (c), and (a'): $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a p-algebra, it is known [**?**], but still unpublished, that the star-regular identity implies the Stone identity; thus the Stone identity is redundant in the definition of Gautama algebras. Clearly, $\mathbf{2}$, $\mathbf{3_{dblst}}$, $\mathbf{3_{klst}}$ are examples of algebras in $\mathbb{G}$.

The following theorem, proved in [60], gives a concrete description of the subdirectly irreducible algebras in the variety $\mathbb{G}$.

Theorem 2.3. [60] Let $\mathbf{A} \in \mathbb{G}$. Then the following are equivalent:
(1) $\mathbf{A}$ is simple;
(2) $\mathbf{A}$ is subdirectly irreducible;
(3) $\mathbf{A}$ is directly indecomposable;
(4) For every $x \in A$, $x \vee x^* = 1$ implies $x = 0$ or $x = 1$;
(5) $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}}\}$, up to isomorphism.

Soon after the chapter [60] appeared in print, we noticed that the star-regular identity implies the identity: $x^{*\prime\prime} \approx x^*$, and the algebra $\mathbf{4_{dmba}}$ (shown in Figure 2) satisfies this weaker identity but not the star-regular identity. These observations recently gave rise to a further generalization of Gautama algebras in [19], called "Almost Gautama algebras."

Definition 2.2. An algebra $\mathbf{A} = \langle A, \vee, \wedge, ^*, ', 0, 1\rangle$ is an **Almost Gautama algebra** if the following conditions hold:

(a) $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a Stone algebra,

(b) $\langle A, \vee, \wedge, ', 0, 1\rangle$ is a dually quasi-De Morgan algebra,

(c) $\mathbf{A}$ is regular. That is, $\mathbf{A}$ satisfies:
(R1) $x \wedge x'^{*\prime} \leq y \vee y^*$ (Regularity),

(d) $\mathbf{A}$ is Weak Star-Regular. That is, $\mathbf{A}$ satisfies the identity:
$(^*)_{\mathrm{w}}$ $x^{*\prime\prime} \approx x^*$, (weak star-regularity),

(e) $\mathbf{A}$ satisfies the identity:
(L1) $(x \wedge x'^*)'^* \approx x \wedge x'^*$.

Let $\mathbb{AG}$ denote the variety of Almost Gautama algebras. It is recently known (see [**?**]) that in the presence of (b), (c), (e) and (a'): $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a p-algebra, the weak star-regular identity and the Stone identity are equivalent; and hence one of the two is redundant in the above definition of Almost Gautama algebra.

Note that the varieties $\mathbb{BA}$ of Boolean algebras, $\mathbb{RDBLS}$t of regular double Stone algebras, and $\mathbb{RKLS}$t of regular Kleene Stone algebras and the variety $\mathbb{G}$ of Gautama algebras are all subvarieties of the variety $\mathbb{AG}$ of Almost Gautama algebras.

In Figure 2, we describe the 4-element algebra $\mathbf{4_{dmba}} := \langle\{0, b, b^*, 1\}, \vee, \wedge, ^*, ', 0, 1\rangle$, in which * is the Boolean complement (as $b \vee b^* = 1$ and $b \wedge b^* = 0$); and $0' = 1$, $1' = 0$, $b' = b$ and $b^{*\prime} = b^*$. It is easy to see that $\mathbf{4_{dmba}}$ is an Almost Gautama algebra. Observe that $\mathbf{4_{dmba}}$ is not a Gautama algebra (e.g., take $x := b$ in $(^*)$).

$\mathbf{4_{dmba}}$:

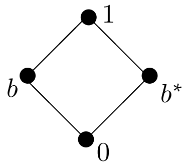


Figure 2

The following theorem, which gives an explicit description of subdirectly irreducible Almost Gautama algebras, is proved in [19].

Theorem 2.4. ([19, Theorem 3.12]) Let $\mathbf{A} \in \mathbb{AG}$. Then the following are equivalent:

(1) $\mathbf{A}$ is simple;

(2) $\mathbf{A}$ is subdirectly irreducible;

(3) $\mathbf{A}$ is directly indecomposable;

(4) $\mathbf{A}$ satisfies (SC): For every $x \in A$, $x \neq 1 \Rightarrow x \wedge x'^* = 0$;
(5) $\mathbf{A} \in \{\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{dmst}}, \mathbf{4_{dmba}}\}$, where $\mathbf{4_{dmba}}$ is the algebra in Figure 2.

The following corollary is also taken from [19, Corollary 3.33].

Corollary 2.5.

(1) The lattice of nontrivial subvarieties of $\mathbb{AG}$ is isomorphic to the eight-element Boolean lattice with $\mathbb{V}(\mathbf{2})$ (i.e., the variety of Boolean allgebras) as the least element. (The Hasse diagram of this lattice is given in Figure 3.)
(2) The lattice of nontrivial subvarieties of $\mathbb{G}$ is isomorphic to the 4-element Boolean lattice with $\mathbb{V}(\mathbf{2})$ as the least element (see Figure 3 also).

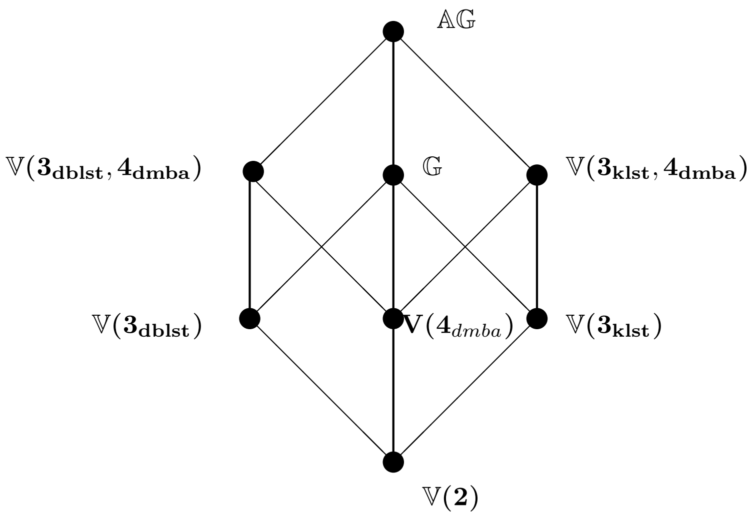


Figure 3

The following theorem was proved in [19, Corollary 5.3].

Theorem 2.3. *The variety* $\mathbb{AG}$ *is a discriminator variety.*

Definition 2.4. $\mathbf{A} \in \mathbb{V}$ has **the congruence extension property** (CEP), if, for every subalgebra $\mathbf{B}$ of $\mathbf{A}$, and $\theta \in \mathrm{Con}\ \mathbf{B}$, there is a $\phi \in \mathrm{Con}\ \mathbf{A}$ such that $\theta = \phi \cap B^2$. A variety V has CEP if all its members have CEP.

The following corollary is immediate from Theorem 2.3 and well-known results from universal algebra (see [14, Chapter 4] or [63]).

Corollary 2.6.

(1) The algebra $\mathbf{2}$ is primal,
(2) $\mathbf{3_{dblst}}$, $\mathbf{3_{dmst}}$, $\mathbf{4_{dmba}}$ are quasiprimal,
(3) All subvarieties of $\mathbb{AG}$ have CEP.

Equational Bases for subvarieties of $\mathbb{AG}$:

Equational bases, given below, for the subvarieties of the variety $\mathbb{AG}$ were obtained in [19]. We recall these equational bases below as they will be needed in what follows. In this subsection, let us abbreviate "defined, modulo $\mathbb{AG}$, by" to simply "defined by."

Theorem 2.7. [19]

(a) $\mathbb{V}(\mathbf{2})$ is defined by $x^* \approx x'$.
(b) $\mathbb{V}(\mathbf{3_{dblst}})$ is defined by the identity: $x \vee x' \approx 1$.
(c) $\mathbb{V}(\mathbf{3_{klst}})$ is defined by the identities: $x^{*\prime} \approx x^{**}$, and $x'' \approx x$.
(d) $\mathbb{V}(\mathbf{4_{dmba}})$ is defined by
  the identities: $x \vee x^* \approx 1$ and $x'' \approx x$.
(e) $\mathbb{G} = \mathbb{V}(\{\mathbf{3_{dblst}}, \mathbf{3_{klst}}\})$ is defined by the identity: $x^{*\prime} \approx x^{**}$.
(f) $\mathbb{V}(\{\mathbf{3_{dblst}}, \mathbf{4_{dmba}}\})$ is defined by the identity:
  (J) $x' \vee (y^* \vee z) \approx (x' \vee y)^* \vee (x' \vee z)$.
(g) $\mathbb{V}(\{\mathbf{3_{klst}}, \mathbf{4_{dmba}}\})$ is defined by the identity: $x'' \approx x$.

3. Amalgamation Property (AP) in the subvarieties of $\mathbb{AG}$

In this section we investigate the amalgamation property (AP, for short) in the subvarieties of $\mathbb{AG}$.

The interest in AP goes back to Schreier [61] and Neuman [50, 51]. In fact, AP appeared implicitly even earlier in the Galois theory of field extensions, since it is AP that lets us to consider all the extensions of a given base field as subfields of one large extension. AP appears in a universal algebraic setting in Fraïssé [24]. For early applications, see [34]. For the importance of the Amalgamation Property see Jónsson [35] and Grätzer [27]). For a more comprehensive history the reader is referred to [22] (see also [3, 5, 6, 9, 16, 22, 24, 25, 28, 29, 31, 32, 34–37, 40, 44, 50–55, 64]).

By a **diagram** in a class $\mathbb{K}$ of algebras we mean a quintuple $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C}\rangle$, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathbb{K}$ and $f : \mathbf{A} \mapsto \mathbf{B}$ and $g : \mathbf{A} \mapsto \mathbf{C}$ are embeddings. By an **amalgam** of this diagram in $\mathbb{K}$ we mean a triple $\langle f_1, g_1, \mathbf{D}\rangle$ with $\mathbf{D} \in \mathbb{K}$ and with embeddings $f_1 : \mathbf{B} \mapsto \mathbf{D}$ and $g_1 : \mathbf{C} \mapsto \mathbf{D}$ such that $f_1 f = g_1 g$. If such an amalgam exists for the diagram $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C}\rangle$, then we say that the diagram is **amalgamable** in $\mathbb{K}$. We say that $\mathbb{K}$ has the **Amalgamation Property**

(AP) if every diagram in $\mathbb{K}$ is amalgamable. Since f and g in a diagram are embeddings, we can, without loss of generality, simply think of a diagram $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C}\rangle$ as a triple $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}\rangle$ such that $\mathbf{A} \leq \mathbf{B} \cap \mathbf{C}$. Accordingly, we adopt this simplified notation in the definition of (AP).

We now examine the Amalgamation Property for non-trivial subvarieties of the variety $\mathbb{AG}$. For this purpose we need the following theorem from [28]. To state the theorem, we need one more definition.

An algebra $\mathbf{A}$ in a variety $\mathbb{V}$ is **hereditarily subdirectly irreducible** if every subalgebra of $\mathbf{A}$ is subdirectly irreducible.

Theorem 3.1. ([28]) Let $\mathbb{V}$ be a variety of algebras such that

(1) $\mathbb{V}$ has the Congruence Extension Property (CEP), and

(2) every subdirectly irreducible algebra in $\mathbb{V}$ is hereditarily subdirectly irreducible.

Then $\mathbb{V}$ satisfies the Amalgamation Property if and only if whenever $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are subdirectly irreducible algebras in $\mathbb{V}$ such that $\mathbf{A} \leq \mathbf{B} \cap \mathbf{C}$, the diagram $\langle \mathbf{A}, \mathbf{B}, \mathbf{C}\rangle$ is amalgamable in $\mathbb{V}$.

Let $\mathbb{DMBA}$ denote the variety $\mathbb{V}(\mathbf{4_{dmba}})$, whose members will be called "De Morgan Boolean algebras." Observe that it follows from Theorem 2.4 that the varieties $\mathbb{B}$, $\mathbb{RDBLS}$t, $\mathbb{RKLS}$t and $\mathbb{DMBA}$ are subvarieties of $\mathbb{AG}$. Note also that, since the variety $\mathbb{AG}$ is a discriminator variety by 2.3, it follows that every subvariety of $\mathbb{AG}$ has CEP and every subdirectly irreducible algebra in each subvariety of $\mathbb{AG}$ is hereditarily subdirectly irreducible. Hence the above theorem is applicable to all the subvarieties of $\mathbb{AG}$.

3.1. The varieties $\mathbb{B}$, $\mathbb{RDBLS}$t, $\mathbb{RKLS}$t and $\mathbb{DMBA}$.

Theorem 3.2. Let $\mathbb{V}$ be a nontrivial subvariety of $\mathbb{AG}$. Then

$\mathbb{V}$ has the Amalgamation Property if $\mathbb{V} \in \{\mathbb{BA}, \mathbb{RDBLS}\mathrm{t}, \mathbb{RKLS}\mathrm{t}, \mathbb{DMBA}\}$.

Proof. It is well-known that $\mathbb{BA}$ has the Amalgamation Property. It also trivially follows from Theorem 3.1 since $\mathbf{2}$ is the only subdirectly irreducible Boolean algebra. Since $\{\mathbf{2}, \mathbf{3_{dblst}}\}$, $\{\mathbf{2}, \mathbf{3_{klst}}\}$, $\{\mathbf{2}, \mathbf{3_{klst}}\}$ and $\{\mathbf{2}, \mathbf{4_{dmba}}\}$ are, respectively, the sets of subdirectly irreducible algebras in $\mathbb{RDBLS}$t, $\mathbb{RKLS}$t and $\mathbb{DMBA}$, it is also clear from Theorem 3.1 that the varieties $\mathbb{RDBLS}$t, $\mathbb{RKLS}$t and $\mathbb{DMBA}$ have the AP. □

3.2. The variety $\mathbb{G}$.

Theorem 3.3. The variety $\mathbb{G}$ does not have the Amalgamation Property.

Proof. In view of Theorem 3.1, it suffices to exhibit a diagram of subdirectly irreducible algebras of $\mathbb{G}$ which is not amalgamable in $\mathbb{G}$. From Theorem 2.3 we know that the algebras $\mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}}$ are subdirectly irreducible in $\mathbb{G}$ and $\mathbf{2}$ is a subalgebra of both the algebras $\mathbf{3_{dblst}}$ and $\mathbf{3_{klst}}$.

So, consider the diagram $\langle \mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}} \rangle$ of subdirectly irreducible algebras in $\mathbb{G}$. We claim that this diagram is not amalgamable in $\mathbb{G}$. Suppose our claim is false. Then there exists an algebra $\mathbf{A} \in \mathbb{G}$ such that both $\mathbf{3_{dblst}}$ and $\mathbf{3_{klst}}$ are subalgebras of $\mathbf{A}$. So, there are $a, b \in \mathbf{A}$ such that $0 < a < 1$, $a' = 1$, $a^* = 0$, $0 < b < 1$, $b' = b$ and $b^* = 0$. Now, by (R1), we have $(a \wedge a'^{*\prime}) \wedge (b \vee b^*) = a \wedge a'^{*\prime}$, which, as $a' = 1$ and $b^* = 0$, leads to

$$a \leq b. \tag{3.1}$$

Next, again using (R1), we get $(b \wedge b'^{*\prime}) \wedge (a \vee a^*) = b \wedge b'^{*\prime}$, from which, in view of $a^* = 0$, b'=b, and $b^* = 0$, we have

$$b \leq a. \tag{3.2}$$

From (3.1) and (3.2) it follows that $a = b$, which implies $b = 1$, as $a' = 1$ and $b' = b$, which is a contradiction since $b \neq 1$. Thus we conclude that the diagram $\langle \mathbf{2}, \mathbf{3_{dblst}}, \mathbf{3_{klst}} \rangle$ is not amalgamable, proving the theorem. □

Remark 3.1. The join of two discriminator varieties having AP need not have AP since $\mathbb{G}$ is the join of $\mathbb{RDBLS}$t, $\mathbb{RKLS}$t, each of which has AP, while $\mathbb{G}$ does not have AP, as shown in the above theorem This was also, independently, noticed by S. Burris (private communication).

3.3. The variety $\mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}})$.

Here we wish to show that $\mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}})$ does not have the AP, the proof of which is developed through the following lemmas, where we assume $\mathbb{V} := \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}})$, $\mathbf{A} \in \mathbb{V}$, and $x, y, a, b \in \mathbf{A}$ such that $0 < a < 1$, $a^* = 0$ and $a' = 1$, $0 < b < 1$, $b \vee b^* = 1$, $b' = b$, $b^{*\prime} = b^*$.

Lemma 3.4. $\mathbf{A}$ satisfies the following identities:
(a) $x' \vee x'^* \approx 1$.
(b) $x'^{**} \approx x'$.

Proof. By the defining identity (J) of the variety $\mathbb{V}$ (see (f) of Theorem 2.7), we have
$x' \vee (0^* \vee 0) = (x' \vee 0)^* \vee x' \vee 0$; so, $x' \vee 1 = x'^* \vee x'$, whence, $x' \vee x'^* = 1$, proving (a). In view of (a), we obtain $x' = x' \vee (x'^* \wedge x'^{**}) = (x' \vee x'^*) \wedge (x' \vee x'^{**}) = 1 \wedge (x' \vee x'^{**}) = x' \vee x'^{**} = x'^{**}$. □

Let (L1) denotes the identity : $x \wedge x'^* \approx (x \wedge x'^*)'^*$ and recall that $\mathbb{AG} \models$ (L1). Hence,$\mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}}) \models$ (L1).

Lemma 3.5. $\mathbf{A}$ satisfies the identity: $x'' \wedge x'^* \approx x \wedge x'^*$.

Proof.

$$\begin{aligned}
x \wedge x'^{*} &= (x \wedge x'^{*})'^{*} && \text{by (L1)}\\
&= x'^{*} \wedge x'^{*\prime *}\\
&= x'''^{*} \wedge x'''^{*\prime *}\\
&= (x'' \wedge x'''^{*})'^{*}\\
&= x'' \wedge x'''^{*} && \text{by (L1).}\\
&= x'' \wedge x'^{*},
\end{aligned}$$

proving the lemma. □

Lemma 3.6. $(a \vee x) \vee [(a \vee x) \wedge (a \vee x)'^{*}]^{*} = 1$.

Proof. As $x'' \geq (x'^{*} \wedge x'')$, we get from Lemma 3.4 (a) that $x'' \vee (x'^{*} \wedge x'')^{*} \geq x'' \vee x''^{*} = 1$; whence, $x \vee x'' \vee (x'^{*} \wedge x'')^{*} = x \vee 1$, implying

$$x \vee (x'^{*} \wedge x'')^{*} = 1, \tag{3.3}$$

as $x'' \leq x$. From (3.3) and Lemma 3.5, we get that $x \vee (x \wedge x'^{*})^{*} = 1$, from which, replacing x by $a \vee x$, we get

$$(a \vee x) \vee [(a \vee x) \wedge (a \vee x)'^{*}]^{*} = 1, \tag{3.4}$$

proving the lemma. □

Lemma 3.7. $b^{*} \geq (b^{*} \vee a)'$.

Proof. From Lemma 3.4 (a), we have $(x \vee y)' \vee (x \vee y)'^{*} = 1$, whence $x' \vee (x \vee y)' \vee (x \vee y)'^{*} = 1$. So, we conclude $x' \vee (x \vee y)'^{*} = 1$, as $x' \geq (x \vee y)'$, which (replacing x by x') implies $x'' \vee (x' \vee y)'^{*} = 1$. Hence we have

$$\mathbf{A} \models x \vee (x' \vee y)'^{*} \approx 1. \tag{3.5}$$

Then,

$$\begin{aligned}
b^{*} &= b^{*} \vee [(b^{*\prime} \vee a)'^{*} \wedge (b^{*\prime} \vee a)'^{**}]\\
&= [b^{*} \vee (b^{*\prime} \vee a)'^{*}] \wedge [b^{*} \vee (b^{*\prime} \vee a)'^{**}]\\
&= 1 \wedge [b^{*} \vee (b^{*\prime} \vee a)'^{**}] && \text{by (3.5)}\\
&= b^{*} \vee (b^{*\prime} \vee a)'^{**}\\
&= b^{*} \vee (b^{*} \vee a)' && \text{by Lemma 3.4 (b) and } b^{*\prime} = b^{*}.
\end{aligned}$$

completing the proof. □

Recall that an Almost Gautama algebra $\mathbf{A}$ has a Stone retract; so, as is well-known, it satisfies the identity: $(x \wedge y)^{*} \approx x^{*} \vee y^{*}$.

Lemma 3.8. $a \vee x \vee (a \vee x)' = 1$.

Proof. Now,
$[(a \vee y) \wedge x]^* = (a \vee y)^* \vee x^* = (a^* \wedge y^*) \vee x^* = (0 \wedge y^*) \vee x^* = x^*$. Hence, we have

$$[(a \vee y) \wedge x]^* = x^*. \tag{3.6}$$

Next,

$$\begin{aligned} a \vee x \vee (a \vee x)' &= a \vee x \vee (a \vee x)'^{**} && \text{by Lemma 3.4} \\ &= (a \vee x) \vee [(a \vee x) \wedge (a \vee x)'^{*}]^* && \text{by (3.6)} \\ &= 1 && \text{by Lemma 3.6,} \end{aligned}$$

arriving at the desired conclusion. □

Theorem 3.9. The variety $\mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}})$ does not have the Amalgamation Property.

Proof. Let $\mathbb{V} = \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}})$, and consider the diagram $\langle \mathbf{2}; \mathbf{3_{dblst}}, \mathbf{4_{dmba}} \rangle$ in $\mathbb{V}$. We claim that this diagram is not amalgamable in $\mathbb{V}$. Suppose our claim is false. Then there exists an algebra $\mathbf{A} \in \mathbb{V}$ such that both $\mathbf{3_{dblst}}$ and $\mathbf{4_{dmba}}$ are subalgebras of $\mathbf{A}$. Hence, there is an $a \in \mathbf{A}$ such that $0 < a < 1$, $a^* = 0$, $a' = 1$, and there exists a $b \in \mathbf{A}$ such that $0 < b < 1$, $b \vee b^* = 1$, $b' = b$, $b^{*\prime} = b^*$. From Lemma 3.8, we get $a \vee b^* \vee (a \vee b^*)' = 1$, which, by Lemma 3.7, further simplifies to

$$a \vee b^* = 1. \tag{3.7}$$

Also observe that $b \vee (a \vee b)' = b' \vee (b \vee a)' = [b \wedge (b \vee a)]' = b' = b$; thus, $b \vee (a \vee b)' = b$. Hence, it follows from Lemma 3.8 that $a \vee b = 1$. Then, $a \vee b^* = (a \vee b^*) \wedge (a \vee b) = a \vee (b \wedge b^*) = a$; hence,

$$a \vee b^* = a. \tag{3.8}$$

But then, (3.7) and (3.8) yield that $a = 1$. Thus, we have arrived at a contradiction which proves the claim. Thus the theorem is proved. □

3.4. The variety $\mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}})$.

In this subsection we will examine the variety $\mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}})$ with respect to (AP). Again, we need some lemmas.

Throughout this subsection, $\mathbb{V} := \mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}})$, $\mathbf{A} \in \mathbb{V}$.

Lemma 3.10. $\mathbf{A} \models x^{*\prime *} \wedge x^* = x^* \wedge x'$.

Proof. Let $x, y \in \mathbf{A}$. Then

$$\begin{aligned} x^* \wedge x' &= x''^{*} \wedge x' && \text{as } x'' = x \\ &= (x''^{*} \wedge x')'^{*} && \text{by (L1)} \\ &= x''^{*\prime *} \wedge x''^{*} \\ &= x^{*\prime *} \wedge x^*, \end{aligned}$$

which provs the lemma. □

Lemma 3.11. Let $a, b \in \mathbf{A}$ such that $a^* = 0$, $b' = b$ and $b^{*\prime} = b^*$. Then $b^* \wedge (b \wedge a)' = 0$.

Proof.

$$\begin{aligned}
0 &= b^* \wedge b \\
&= b^* \wedge b' && \text{since } b' = b \\
&= b^* \wedge b^{*\prime *} && \text{by Lemma 3.10} \\
&= b^* \wedge (b^* \vee a^*)^{\prime *} && \text{since } a^* = 0 \\
&= b^* \wedge (b \wedge a)^{*\prime *} && \text{since } \mathbf{A} \text{ has a Stone algebra-reduct} \\
&= b^* \wedge (b \wedge a)' && \text{by Lemma 3.10,}
\end{aligned}$$

proving the lemma. □

Theorem 3.12. The variety $\mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}})$ does not have the Amalgamation Property.

Proof. Let $\mathbb{V} = \mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}})$, and consider the diagram $\langle \mathbf{2}; \mathbf{3_{klst}}, \mathbf{4_{dmba}} \rangle$ in $\mathbb{V}$. We claim that this diagram is not amalgamable in $\mathbb{V}$. Suppose our claim is false. Then there exists an algebra $\mathbf{A} \in \mathbb{V}$ such that both $\mathbf{3_{klst}}$ and $\mathbf{4_{dmba}}$ are subalgebras of $\mathbf{A}$. So, there are $a, b \in \mathbf{A}$ such that $0 < a < 1$, $a^* = 0$, $a' = a$, and $0 < b < 1$, $b' = b$, $b^{*\prime} = b^*$, $b \vee b^* = 1$. Now,

$$\begin{aligned}
a \wedge b^* &= (b^* \wedge b) \vee (b^* \wedge a) \\
&= b^* \wedge (b \vee a) \\
&= b^* \wedge (b' \vee a') && \text{since } b' = b \text{ and } a' = a \\
&= b^* \wedge (b \wedge a)' \\
&= 0 && \text{by Lemma 3.11.}
\end{aligned}$$

Thus, we have

$$a \wedge b^* = 0. \tag{3.9}$$

Hence,

$$\begin{aligned}
b^* &= b^* \wedge 0^* \\
&= b^* \wedge (b^* \wedge a)^* && \text{by (3.9)} \\
&= b^* \wedge a^* && \text{since } ^* \text{ is a pseudocomplement} \\
&= 0 && \text{as } a^* = 0.
\end{aligned}$$

Thus, $b^* = 0$, which is a contradiction since $b \vee b^* = 1$, proving the claim, from which the conclusion follows. □

3.5. **The variety $\mathbb{AG}$.**

Here we wish to show that $\mathbb{AG}$ does not have the (AP). Throughout this subsection, $\mathbf{A} \in \mathbb{AG}$, and $a, b \in \mathbf{A}$ such that $a^* = 0$, $a' = 1$, $b' = b$, $b^{**} = b$, and $b^{*\prime} = b^*$. Hence, it is also immediate that $b'^{**} = b$.

Lemma 3.13. $\mathbf{A} \models (x \wedge x'^*) \vee (x \wedge x'^*)^* \approx 1$.

Proof. Let $x, y \in \mathbf{A}$. Then

$$\begin{aligned}(x \wedge x'^*) \vee (x \wedge x'^*)^* &= (x \wedge x'^*)'^* \vee (x \wedge x'^*)'^{**} && \text{by (L1)}\\ &= (x' \vee x'^{*\prime})^* \vee (x' \vee x'^{*\prime})^{**}\\ &= 1 && \text{by the Stone identity,}\end{aligned}$$

proving the lemma. ☐

Lemma 3.14. Let $x \in \mathbf{A}$. Then, $[(x \vee a) \wedge y]^* = y^*$.

Proof. $[(x \vee a) \wedge y]^* = [(x \vee a)^{**} \wedge y^{**}]^* = [(x^* \wedge a^*)^* \wedge y^{**}]^* = (1 \wedge y^{**})^* = y^*$, as $a^* = 0$. ☐

Lemma 3.15. Let $x \in \mathbf{A}$. Then, $x \vee a \vee (x \vee a)'^{**} = 1$.

Proof. $x \vee (x \wedge x'^*)^* \geq (x \wedge x'^*) \vee (x \wedge x'^*)^* = 1$ by Lemma 3.13. Hence,

$$\mathbf{A} \models x \vee (x \wedge x'^*)^* \approx 1. \tag{3.10}$$

Replacing x by $x \vee a$ in (3.10), we get $(x \vee a) \vee [(x \vee a) \wedge (x \vee a)'^*]^* = 1$, which, by Lemma 3.14, simplifies to $(x \vee a) \vee (x \vee a)'^{**} = 1$. ☐

Lemma 3.16. $\mathbf{A} \models y'' \vee y^* \approx (x' \vee x'^*)' \vee y'' \vee y^*$.

Proof. Let $x, y \in \mathbf{A}$. Then, from (R1) we get $y \vee y^* = (x \wedge x'^{*\prime}) \vee y \vee y^*$, from which we have $y'' \vee y^{*\prime\prime} = (x \wedge x'^{*\prime})'' \vee y'' \vee y^{*\prime\prime}$. Hence,

$$\begin{aligned}y'' \vee y^* &= y'' \vee y^{*\prime\prime} && \text{by the axiom: } x^{*\prime\prime} \approx x^*,\\ &= (x \wedge x'^{*\prime})'' \vee y'' \vee y^{*\prime\prime}\\ &= (x' \vee x'^{*\prime\prime})' \vee y'' \vee y^{*\prime\prime}\\ &= (x' \vee x'^*)' \vee y'' \vee y^* && \text{by the axiom: } x^{*\prime\prime} \approx x^*,\end{aligned}$$

proving the lemma. ☐

Lemma 3.17. $\mathbf{A} \models x' \vee x'^* \approx 1$.

Proof. From Lemma 3.16 we have $a'' \vee a^* = (x' \vee x'^*)' \vee a'' \vee a^*$, which implies $(x' \vee x'^*)' = 0$, since $a' = 1$ and $a^* = 0$. It follows that $x''' \vee x'^{*\prime\prime} = 0'$, whence $x' \vee x'^{*\prime\prime} = 1$, from which, in view of the axiom $x^{*\prime\prime} = x^*$, we get $x' \vee x'^* = 1$. ☐

Lemma 3.18. $x'^{**} = x'$.

Proof. Using Lemma 3.17 we have $x' \vee x'^{**} = (x' \vee x'^{**}) \wedge (x' \vee x'^{*}) = x' \vee (x'^{**} \wedge x'^{*}) = x'$. Thus $x'^{**} \leq x'$, from which we conclude $x'^{**} = x'$. □

Lemma 3.19. $x' \vee y' = (x'^{*} \wedge y'^{*})^{*}$.

Proof. By Lemma 3.18, we have $(x'^{*} \wedge y'^{*})^{*} = x'^{**} \vee y'^{**} = x' \vee y'$. □

Lemma 3.20. $(x \vee y)'' = (x''^{*} \wedge y''^{*})^{*}$.

Proof. Replacing x by x' and y by y' in Lemma 3.19, we get $x'' \vee y'' = (x''^{*} \wedge y''^{*})^{*}$, which implies $(x \vee y)'' = (x''^{*} \wedge y''^{*})^{*}$. □

Lemma 3.21. $a \vee x \vee x' = 1$.

Proof. By Lemma 3.15 and Lemma 3.18 we have $x \vee a \vee (x \vee a)' = 1$, which implies $x \vee a \vee x' = 1$, since $x' \geq (x \vee a)'$. □

Theorem 3.22. The variety $\mathbb{AG}$ does not have the Amalgamation Property.

Proof. Consider the diagram $\langle \mathbf{2}; \mathbf{3_{dblst}}, \mathbf{4_{dmba}} \rangle$ in $\mathbb{AG}$. We claim that this diagram is not amalgamable in $\mathbb{AG}$. Suppose our claim is false. Then there exists an algebra $\mathbf{A} \in \mathbb{AG}$ such that both $\mathbf{3_{klst}}$ and $\mathbf{4_{dmba}}$ are subalgebras of $\mathbf{A}$. So, there are $a, b \in \mathbf{A}$ such that $0 < a < 1$, $a^{*} = 0$, $a' = 1$, $0 < b < 1$, $b \vee b^{*} = 1$, $b' = b$, and $b^{*\prime} = b^{*}$. Hence it is clear that $b^{**} = b$. Now, from Lemma 3.21, we have $a \vee b \vee b' = 1$ and $a \vee b^{*} \vee b^{*\prime} = 1$, implying $a \vee b = 1$ and $a \vee b^{*} = 1$. Thus, $a = a \vee (b \wedge b^{*}) = (a \vee b) \wedge (a \vee b^{*}) = 1 \wedge 1 = 1$, whence $a = 1$, which is a contradiction since $a \neq 1$, proving the claim. It follows that the variety $\mathbb{AG}$ does not have the Amalgamation Property. □

We are ready to give our main result of the paper.

Theorem 3.23. Let $\mathbb{V}$ be a nontrivial subvariety of $\mathbb{AG}$. Then

(1) $\mathbb{V}$ has the Amalgamation Property if $\mathbb{V} \in \{\mathbb{BA}, \mathbb{RDBLS}\mathrm{t}, \mathbb{RKLS}\mathrm{t}, \mathbb{DMBA}\}$.
(2) $\mathbb{V}$ does not have the Amalgamation Property if
$$\mathbb{V} \in \{\mathbb{G}, \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}}), \mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}}), \mathbb{AG}\}.$$

Proof. Observe that (1) is just a restatement of Theorem 3.2, while (2) follows immediately from Theorems 3.3, 3.9, 3.12 and 3.22. □

The following remark is important.

Remark 3.2. The variety $\mathbb{AGH}$ of Almost Gautama Heyting algebras was also introduced in [19], where it was proved that $\mathbb{AGH}$ is term-equivalent to $\mathbb{AG}$ and their lattices of subvarieties are isomorphic. (For the definition of the variety $\mathbb{AGH}$ we refer the reader to [19].) Hence, as a consequence of Theorem 3.23, a theorem analogous to Theorem 3.23 for the variety $\mathbb{AGH}$ also holds. The actual statement of that theorem can be easily written down by the reader with the help of [19].

4. Applications

In this section we give several applications of Theorem 3.23 and of proofs of results in Section 3. We examine the following properties for the subvarieties of $\mathbb{AG}$: Transferability property (TP), having enough injectives (EI), Embedding Property, Bounded Obstruction Property and having a model companion.

Definition 4.1. Let $\mathbb{V}$ be a variety and let $A \in \mathbb{V}$. $\mathbb{V}$ has the **transferability property** (TP) or **transferable injection property** (TIP) if, for every $A, B, C \in \mathbb{V}$ such that $f : A \mapsto B$ an embedding and $g : A \to C$ a homomorphism, there exist D in V, a homomorphism $f_1 : B \to D$ and an embedding $g_1 : C \to D$ such that $f_1 f = g_1 g$.

The following result appears in Bacsich [5].

Theorem 4.1. Let $\mathbb{V}$ be a variety. Then $\mathbb{V}$ has (TP) if and only if V has AP and CEP.

Corollary 4.2. Let $\mathbb{V}$ be a nontrivial subvariety of $\mathbb{AG}$. Then
$\mathbb{V}$ has (TP) if and only if $\mathbb{V} \in \{\mathbb{BA}, \mathbb{RDBLS}t, \mathbb{RKLS}t, \mathbb{DMBA}\}$.

Proof. Apply Corollary 2.6, Theorem 3.23, and Theorem 4.1. □

Definition 4.2. Let $\mathbb{V}$ be a variety.

(1) **A is injective in $\mathbb{V}$, if for every B, C in $\mathbb{V}$, for every embedding $f : \mathbf{B} \mapsto \mathbf{C}$ and every homomorphism $g : \mathbf{B} \to \mathbf{A}$, there is a homomorphism $h : \mathbf{C} \to \mathbf{A}$ such that $hf = g$.**
(2) $\mathbb{V}$ has **enough injectives** (EI) if every algebra in $\mathbb{V}$ can be embedded in an injective algebra in $\mathbb{V}$.
(3) $\mathbb{V}$ is **residually small** (RS) if there exists a cardinal κ such that the size of every subdirectly irreducible algebra in $\mathbb{V}$ is $\leq \kappa$.

The following result is from Banaschewski [7]

Theorem 4.3. [7] A variety $\mathbb{V}$ has (EI) if and only if $\mathbb{V}$ has (TP) and is (RS).

Corollary 4.4. Let $\mathbb{V}$ be a nontrivial subvariety of $\mathbb{AG}$.

(a) If $\mathbb{V} \in \{\mathbb{V}(\mathbf{2}), \mathbb{V}(\mathbf{3_{dblst}}), \mathbb{V}(\mathbf{3_{klst}}), \mathbb{V}(\mathbf{4_{dmba}})\}$, then $\mathbb{V}$ has EI.
(b) If $\mathbb{V} \in \{\mathbb{G}, \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}}), \mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}}), \mathbb{AG}\}$, then $\mathbb{V}$ does not have EI.

Proof. It is clear that every subvariety of $\mathbb{AG}$ is RS. Then apply Corollary 4.2 and Theorem 4.3. □

4.1. Joint Embedding Property.

We will now examine which of the subvarieties of $\mathbb{AG}$ have the Joint Embedding Property.

Definition 4.3. We say that a variety $\mathbb{V}$ has the **Joint Embedding Property** if and only if for any two algebras $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{V}$, there exists an algebra $\mathbf{C}$ in $\mathbb{V}$ into which both $\mathbf{A}$ and $\mathbf{B}$ can be embedded.

Definition 4.4. A non-trivial algebra $\mathbf{M}$ in a variety $\mathbb{V}$ is said to be **$\mathbb{V}$-minimal** (or **$\mathbb{V}$-prime**) if $\mathbf{M}$ is, up to isomorphism, a subalgebra of every nontrivial algebra in $\mathbb{V}$.

Lemma 4.5. Let a variety $\mathbb{V}$ possess a $\mathbb{V}$-minimal algebra $\mathbf{M}$. Then $\mathbb{V}$ has the (AP) if and only if $\mathbb{V}$ has the Joint Embedding Property.

Proof. Suppose $\mathbb{V}$ satisfies the hypothesis and has the (AP). Let $\mathbf{A}$ and $\mathbf{B}$ be two non-trivial algebras in $\mathbb{V}$. Since $\mathbf{M}$ is a subalgebra of both $\mathbf{A}$ and $\mathbf{B}$, we can consider the diagram $\langle \mathbf{M}, \mathbf{A}, \mathbf{B} \rangle$. Then as $\mathbb{V}$ has (AP), there exists an algebra $\mathbf{C} \in \mathbb{K}$ such that $\mathbf{A}$ and $\mathbf{B}$ are subalgebras of $\mathbf{C}$, implying that $\mathbb{V}$ has the Joint Embedding Property. The converse is trivial. □

Corollary 4.6.

(a) If $\mathbb{V} \in \{\mathbb{V}(\mathbf{2}), \mathbb{V}(\mathbf{3_{dblst}}), \mathbb{V}(\mathbf{3_{klst}}), \mathbb{V}(\mathbf{4_{dmba}})\}$, then $\mathbb{V}$ has the Joint Embedding Property.

(b) If $\mathbb{V} \in \{\mathbb{G}, \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}}), \mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}}), \mathbb{AG}\}$, then $\mathbb{V}$ fails to have the Joint Embedding Property.

Proof. Since every subvariety $\mathbb{V}$ of $\mathbb{AG}$ has the algebra $\mathbf{2}$ as $\mathbb{V}$-minimal, the corollary is immediate from Theorem 3.23 and Lemma 4.5. □

4.2. **Amalgamation Classes.**

Recall that a diagram in a class $\mathbb{K}$ of algebras is a quintuple $\langle A, f, B, g, C \rangle$ with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \in \mathbb{K}$ and $f : \mathbf{A} \mapsto \mathbf{B}$ and $g : \mathbf{A} \mapsto \mathbf{C}$ embeddings, and an amalgam in $\mathbb{K}$ of this diagram is a triple $\langle f_1, g_1, \mathbf{D} \rangle$ with $\mathbf{D} \in \mathbb{K}$ and with $f_1 : \mathbf{B} \mapsto \mathbf{D}$ and $g_1 : C \mapsto D$ embeddings such that $f_1 f = g_1 g$. If such an amalgam exists, then we say that the diagram is *amalgamable* in $\mathbb{K}$. An algebra $\mathbf{A}$ is called an **amalgamation base** in a class $\mathbb{K}$ of algebras if every diagram $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C} \rangle$ is amalgamable in $\mathbb{K}$. Let $\mathrm{Amal}(\mathbb{K}) := \{\mathbf{A} : \mathbf{A}$ is an amalgamation base in $\mathbb{K}\}$. Then $\mathrm{Amal}(\mathbb{K})$ is called the **amalgamation class** of $\mathbb{K}$.

The following theorem is immediate from the proofs of Theorem 3.3, Theorem 3.9, Theorem 3.12 and Theorem 3.22.

Theorem 4.7. If $\mathbb{V} \in \{\mathbb{G}, \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}}), \mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}}), \mathbb{AG}\}$, then $\mathbf{2} \notin \mathit{Amal}(\mathbb{V})$.

We can now give a concrete description of $\mathrm{Amal}(\mathbb{V})$ for the varieties mentioned in the preceding theorem.

Theorem 4.8. Let $\mathbb{V} \in \{\mathbb{G}, \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}}), \mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}}), \mathbb{AG}\}$. Then $\mathrm{Amal}(\mathbb{V}) = \mathbb{T}$, where $\mathbb{T}$ is the trivial variety.

Proof. Let $\mathbf{A} \in \mathbb{V}$ with $|A| \geq 2$. We claim that $\mathbf{A} \notin \mathrm{Amal}(\mathbb{V})$. For, suppose $\mathbf{A} \in \mathrm{Amal}(\mathbb{V})$. Then, every diagram $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$, with $\mathbf{B}$ and $\mathbf{C}$ in $\mathbb{V}$, has an amalgam in $\mathbb{V}$. Let $\langle \mathbf{A}, \mathbf{B}_0, \mathbf{C}_0 \rangle$ be an arbitrary diagram in $\mathbb{V}$. Then it has an amalgam $\mathbf{D}_0 \in \mathbb{V}$. Since $\mathbf{2}$ is a subalgebra of $\mathbf{A}$, it follows that the algebra $\mathbf{D}_0$ is also an amalgam of the diagram $\langle \mathbf{2}, \mathbf{B}_0, \mathbf{C}_0 \rangle$, which is impossible, since $\mathbf{2} \notin \mathrm{Amal}(\mathbb{V})$ by Theorem 4.7. Thus the claim is true, implying that $\mathrm{Amal}(\mathbb{V}) = \mathbb{T}$. □

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Interestingly enough, the proof of the above theorem points to the following more general theorem.

Theorem 4.9. Let a variety $\mathbb{V}$ possess a $\mathbb{V}$-prime algebra $\mathbf{M}$ such that $\mathbf{M} \notin \mathrm{Amal}(\mathbb{V})$. Then $\mathrm{Amal}(\mathbb{V}) \in \{\mathbb{T}, \mathbb{V}\}$.

Proof. Suppose $\mathrm{Amal}(\mathbb{V}) \neq \mathbb{V})$. Then, replace $\mathbf{2}$ by $\mathbf{M}$ in the proof of the preceding theorem. □

.

Bergman [9] has proved that if $\mathbb{V}$ is a finitely generated discriminator variety, then $\mathrm{Amal}(\mathbb{V})$ is finitely axiomatizable (i.e., is definable by a finite set of first-order sentences.). This result leads us to the following open problem:

PROBLEM 1: if $\mathbb{V}$ is a discriminator variety generated by a finite set of finite algebras, is $\mathrm{Amal}(\mathbb{V})$ finitely axiomatizable?

4.3. Bounded Obstruction Property.

Albert and Burris [3] have introduced the notion of the bounded obstruction property and shown certain relatioships between the bounded obstructions, model companions, and amalgamation classes.

Definition 4.5. [3]

(1) Let $\mathbb{K}$ be an elementary class, and suppose that the diagram $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C} \rangle$ has no amalgam in $\mathbb{K}$. An **obstruction** is any subalgebra $\mathbf{C}'$ of $\mathbf{C}$ such that $(\mathbf{A}', f|_{\mathbf{A}'}, \mathbf{B}, g|_{\mathbf{A}'}, \mathbf{C}')$ has no amalgam in $\mathbb{K}$, where $\mathbf{A}' = g^{-1}(\mathbf{C}')$.

(2) Let $\mathbb{K}$ be a locally finite elementary class. **Amal(K)** has the **bounded obstruction property (BOP)** with respect to $\mathbb{K}$ if for every $k \in \omega$, there exists an $n \in \omega$ such that the following holds:
If $C \in Amal(\mathbb{K})$, $|\mathbf{B}| < k$ and the diagram $\langle \mathbf{A}, f, \mathbf{B}, g, \mathbf{C} \rangle$ has no amalgam in $\mathbb{K}$, then there is an obstruction $\mathbf{C}' \leq \mathbf{C}$ such that $|\mathbf{C}'| < n$.

Theorem 4.10. (Albert and Burris [3]) Let V be a locally finite variety. Then $Amal(V)$ satisfies the bounded obstruction property if and only if Amal(V) is an elementary class.

The following corollary is immediate from Theorem 4.10, Theorem 3.23 and Theorem **??**.

Corollary 4.11. Let $\mathbb{V}$ be a nontrivial subvariety of $\mathbb{AG}$.
If $\mathbb{V} \in \{\mathbb{G}, \mathbb{V}(\mathbf{3_{dblst}}, \mathbf{4_{dmba}}), \mathbb{V}(\mathbf{3_{klst}}, \mathbf{4_{dmba}}), \mathbb{AG}\}$, then $Amal(\mathbb{V})$ satisfies the bounded obstruction property.

4.4. **Model companions and model completions of the subvarieties of $\mathbb{AG}$.**

If $\mathbb{K}$ is a class of (first-order) structures, let $S(\mathbb{K})$ denote the class of all substructures of $\mathbb{K}$. If T is a set of (first-order) sentences then $\mathbf{M(T)}$ denotes the class of models of T, i.e. the class of all structures satisfying all the sentences in T. A (first-order) theory T is a deductively-closed consistent set of sentences in a f'trst-order language. Two theories T and T_1 (in the same language) are **mutually model-consistent** if $S(\mathbf{M(T)}) = S(\mathbf{M(T_1)})$, i.e. every model of T can be embedded in a model of T_1 and vice-versa. T_1 is **model-complete** if $\mathbf{A}, \mathbf{B} \in \mathbf{M}((\mathbf{T_1})$ and $\mathbf{A} \in S(\mathbf{B})$ imply $\mathbf{A}$ is an elementary substructure of $\mathbf{B}$. T_1 is a **model companion** of T if (i) T and T_1 are mutually model-consistent, and (ii) T_1 is model-complete. A class $\mathbb{K}$ has a **model companion** if Th($\mathbb{K}$) (i.e. the set all sentences true in $\mathbb{K}$) has a model companion. Let T and T_1 be (first-order) theories (in the same language). T_1 is **model-consistent relative to** T if for any model $\mathbf{A}$ of T, $T_1 \cup$ D($\mathbf{A}$) is consistent, where D($\mathbf{A}$) is the diagram of $\mathbf{A}$. T_1 is **model-complete relative to** T if for any model $\mathbf{A}$ of T, $T_1 \cup$ D($\mathbf{A}$) is complete. T_1 is the **model completion** of T if $T \subseteq T_1$ and T_1 is model-consistent and model-complete relative to T.

The following result is well-known (see, for example, [42]).

Theorem 4.12. ([42]) If a class $\mathbb{K}$ is locally finite, has a finite language and has the amalgamation property, then $\mathbb{K}$ has a model companion.

The following corollary is immediate from Theorem 4.12 and Theorem 3.23.

Corollary 4.13. Let $\mathbb{V}$ be a nontrivial subvariety of $\mathbb{AG}$. Then $\mathbb{V}$ has a model companion iff $\mathbb{V} \in \{\mathbb{BA}, \mathbb{RDBLS}\mathrm{t}, \mathbb{RKLS}\mathrm{t}, \mathbb{DMBA}\}$.

Theorem 4.14. [30, Proposition 2.8] A model-companion T* for a theory T is a complete theory if and only if T has the Joint Embedding Property.

Corollary 4.15. If $\mathbb{V} \in \{\mathbb{BA}, \mathbb{RDBLS}\mathrm{t}, \mathbb{RKLS}\mathrm{t}, \mathbb{DMBA}\}$, then the model companion of the Th($\mathbb{V}$) is a complete theory.

Proof. Let $\mathbb{V}$ be as given in the hypothesis. Then by Corollary 4.13 Th($\mathbb{V}$) has a model companion, say T_1. We know from Corollary 4.6 that V has the Joint Embedding Property. Hence by Theorem 4.14 T_1 is a complte theory. □

We conclude the paper by mentioning that a far-reaching generalization of Almost Gautama algebras, called “Quasi-Gautama algebras,” is under preparation in [20].

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