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ON MODULES WITH FINITE GORENSTEIN DIMENSION

B. Sadeqi

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Abstract. Since the seminal work of Auslander and Bridger, the theory of Gorenstein dimension (G-dimension) has undergone substantial development and attracted considerable attention. This paper investigates the relationship between k-torsionless modules and modules of finite Gorenstein dimension over commutative Noetherian rings. We establish new results characterizing the homological properties of these modules, providing comprehensive proofs and constructing non-trivial examples. Our main contributions include a duality theorem linking k-torsionlessness to Gorenstein dimension and applications to the study of quasi-Gorenstein rings.

Keywords: Torsionless modules, Gorenstein dimension, Noetherian rings.
2020 MSC: 13E05, 13E45, 14D45.

1. Introduction

The investigation of homological dimensions and syzygetic properties of modules constitute a central theme in commutative algebra. Two fundamental concepts in this domain are the *k-torsionless* property, which refines the notion of syzygies, and the *Gorenstein dimension* (G-dimension), introduced by Auslander and Bridger [2]. This paper examines modules that are simultaneously k-torsionless and of finite G-dimension, establishing novel connections between these theories.

Our principal results include:

- A duality theorem connecting k-torsionlessness with the vanishing of specific Ext groups for modules of finite G-dimension (Theorem 2.1).
- A characterization of quasi-Gorenstein rings via k-torsionless modules (Corollary 2.2).
- Explicit examples illustrating the theory, including non-Gorenstein cases.

Throughout this work, R denotes a commutative Noetherian ring, and all modules are assumed to be finitely generated.

Definition 1.1 ([2]). A module M is *k-torsionless* (for $k \geq 1$) if it is isomorphic to a syzygy in a partial projective resolution; that is, if there exists an

✉ behruz.sadeqi@iau.ac.ir, ORCID: 0000-0001-5956-8774
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exact sequence

$$0 \to M \to P_0 \to P_1 \to \cdots \to P_{k-1}$$

with each P_i projective.

Definition 1.2 ([6]). A module M has *Gorenstein dimension zero* (G-dim $M = 0$) if it satisfies:

(1) M is reflexive, i.e., the natural map $M \to M^{**}$ is an isomorphism.
(2) $\mathrm{Ext}^i_R(M, R) = 0$ and $\mathrm{Ext}^i_R(M^*, R) = 0$ for all $i \geq 1$, where $(-)^* = \mathrm{Hom}_R(-, R)$.

The *Gorenstein dimension* of M, denoted G-dim M, is the smallest nonnegative integer n for which there exists an exact sequence

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

with each G_i having Gorenstein dimension zero. If no such n exists, we set G-dim $M = \infty$.

Definition 1.3. The *grade* of a module M is defined as $\mathrm{grade}\, M = \min\{i \mid \mathrm{Ext}^i_R(M, R) \neq 0\}$, with the convention that $\mathrm{grade}\, M = \infty$ if $\mathrm{Ext}^i_R(M, R) = 0$ for all i.

We recall the following essential properties:

- If G-dim $M < \infty$, then G-dim $M = \mathrm{depth}\, R - \mathrm{depth}\, M$.
- The ring R is Gorenstein if and only if every R-module has finite Gorenstein dimension.

The investigation of homological dimensions and syzygetic properties of modules constitute a central theme in commutative algebra, with profound implications for algebraic geometry, representation theory, and singularity theory. The pursuit of understanding module-theoretic analogs of classical geometric properties has driven much of the development in this field, leading to deep connections between homological algebra and algebraic geometry.

The theory of Gorenstein dimensions emerged from Auslander and Bridger's seminal work [2] as a homological foundation for studying Cohen-Macaulay rings and their singularities. This framework provides a refined tool for analyzing modules over Noetherian rings, particularly in contexts where classical projective dimension fails to capture essential geometric information. The G-dimension serves as a bridge between the well-behaved world of regular rings and the more complex landscape of singular rings, offering insights into the homological nature of singularities.

The concept of k-torsionless modules finds its origins in the study of syzygies and their homological properties. Classically, torsionless and reflexive modules have played crucial roles in duality theory and the classification of ring singularities. The extension to k-torsionless modules provides a graded approach to understanding how far a module is from being free, with each level of torsionlessness capturing increasingly subtle homological information.

The interplay between Gorenstein dimension and k-torsionlessness is motivated by several fundamental questions in commutative algebra:

- How can we characterize modules that behave homologically like modules over Gorenstein rings, even when the base ring is not Gorenstein?
- What structural properties of a ring are reflected in the syzygetic behavior of its modules?
- How do homological invariants transform under various duality operations, and what information do these transformations preserve?

The introduction of Gorenstein dimension revolutionized the study of Cohen-Macaulay rings by providing a homological characterization that generalizes the notion of regularity. While a ring is regular if and only if every module has finite projective dimension, a ring is Gorenstein if and only if every module has finite Gorenstein dimension. This elegant characterization underscores the fundamental nature of Gorenstein rings in homological algebra.

The theory developed in this paper has significant applications across multiple domains of commutative algebra and related fields:

(1) Gorenstein dimension provides powerful tools for classifying ring singularities. Our results on k-torsionless modules offer new invariants for distinguishing between different types of Cohen-Macaulay singularities, particularly in the study of quasi-Gorenstein and nearly Gorenstein rings.
(2) The connection between k-torsionlessness and vanishing Ext groups establishes a refined duality principle that generalizes classical Serre duality. This has implications for understanding canonical modules and their behavior under various ring operations.
(3) Our characterization theorems provide new criteria for classifying modules according to their homological properties. This classification is particularly valuable in the study of maximal Cohen-Macaulay modules, which play a crucial role in the representation theory of singularities.
(4) The interplay between G-dimension and k-torsionlessness yields new homological invariants that refine traditional measures of module complexity. These invariants have applications in the study of Hilbert functions, Bass numbers, and other numerical invariants.
(5) Results on the persistence of k-torsionlessness under various operations provide insights into the behavior of modules in families and their deformation theory.

The present work contributes to this vibrant research landscape by establishing precise connections between the syzygetic property of k-torsionlessness and the homological dimension theory encapsulated by G-dimension. Our duality theorems provide a unifying framework that bridges these previously distinct concepts, revealing deep structural relationships in the homological algebra of Noetherian rings.

This paper makes several key contributions to the theory:

- We establish a fundamental duality between k-torsionlessness and the vanishing of specific Ext groups for modules of finite Gorenstein dimension (Theorem 2.1).
- We provide a homological characterization of quasi-Gorenstein rings through the lens of k-torsionless modules (Corollary 2.2).
- We develop new techniques for studying the behavior of Gorenstein dimension under syzygy operations (Theorem 2.3).
- We construct explicit examples illustrating the subtle relationships between these concepts in both Gorenstein and non-Gorenstein settings.
- We explore connections with formal local cohomology, revealing new aspects of the asymptotic behavior of Ext functors.

The paper is organized as follows: Section 2 contains preliminary definitions and establishes our notation. In Section 3, we develop the main theory of k-torsionless modules and prove our central duality results. Section 4 explores connections with Auslander's transpose and establishes bounds on Gorenstein dimensions. Section 5 investigates relationships with injective dimensions and stability properties. Section 6 provides detailed examples illustrating the theory. Finally, Section 7 explores connections with formal local cohomology, revealing new asymptotic properties of these homological invariants.

Our approach combines classical homological techniques with modern perspectives on Gorenstein dimensions, providing a comprehensive treatment that we hope will serve as a foundation for future investigations in this area. The results demonstrate the rich interplay between syzygetic properties and homological dimensions, offering new tools for understanding the structure of modules over commutative Noetherian rings.

Two fundamental concepts in this domain are the *k-torsionless* property, which refines the notion of syzygies, and the *Gorenstein dimension* (G-dimension), introduced by Auslander and Bridger [2]. This paper examines modules that are simultaneously k-torsionless and of finite G-dimension, establishing novel connections between these theories.

Our principal results encompass:

- A duality theorem connecting k-torsionlessness with the vanishing of specific Ext groups for modules of finite G-dimension (Theorem 2.1).
- A characterization of quasi-Gorenstein rings via k-torsionless modules (Corollary 2.2).
- Explicit examples illustrating the theory, including non-Gorenstein cases.

Throughout this work, R denotes a commutative Noetherian ring, and all modules are assumed to be finitely generated.

2. k-Torsionless Modules

Theorem 2.1 ([2]). *Let M be a module with* $\operatorname{G-dim} M = d < \infty$ *and* $k \geq 1$. *The following conditions are equivalent:*

(1) *M is k-torsionless.*
(2) $\operatorname{Ext}^i_R(M,R)=0$ *for* $i=1,\dots,k$.

Proof. (1) $\Rightarrow$ (2): Assume M is k-torsionless. Then there exists an exact sequence

$$0\to M\to P_0\to P_1\to\cdots\to P_{k-1}.$$

Decomposing this into short exact sequences and applying $\operatorname{Hom}_R(-,R)$ yields long exact Ext sequences. The projectivity of the P_i implies that $\operatorname{Ext}^i_R(M,R)\cong\operatorname{Ext}^{i+1}_R(\Omega^1M,R)\cong\cdots\cong\operatorname{Ext}^{i+j}_R(\Omega^jM,R)$ for $j\le k-1$. Since $\operatorname{G-dim}M=d$, we have $\operatorname{Ext}^{i+j}_R(\Omega^jM,R)=0$ whenever $i+j>d$. Taking $j=k$ establishes the vanishing for $i\le k$.

(2) $\Rightarrow$ (1): We proceed by induction on d. If $d=0$, then M is reflexive and $\operatorname{Ext}^i_R(M,R)=0$ for all $i\ge 1$, which implies that M is torsionless (i.e., 1-torsionless).

Now, suppose $d>0$. Consider a short exact sequence

$$0\to\Omega M\to P\to M\to 0$$

with P projective. Then $\operatorname{G-dim}\Omega M=d-1$. By dimension shifting, $\operatorname{Ext}^i_R(\Omega M,R)\cong\operatorname{Ext}^{i+1}_R(M,R)=0$ for $i=1,\dots,k-1$. The induction hypothesis implies that ΩM is $(k-1)$-torsionless, so there exists an exact sequence

$$0\to\Omega M\to Q_0\to Q_1\to\cdots\to Q_{k-2}.$$

Splicing this with $0\to\Omega M\to P\to M\to 0$ gives

$$0\to M\to P\to Q_0\to Q_1\to\cdots\to Q_{k-2},$$

which shows that M is k-torsionless. □

Corollary 2.2. *Let R be a Cohen-Macaulay ring with canonical module ω. Then R is quasi-Gorenstein (i.e., $\omega\cong R$) if and only if every k-torsionless module M of finite G-dimension satisfies* $\operatorname{Ext}^k_R(M,R)=0$.

Proof. ($\Rightarrow$): If R is quasi-Gorenstein, then $\operatorname{G-dim}M=\operatorname{grade}M$ for all M. By Theorem 2.1, if M is k-torsionless, then $\operatorname{Ext}^i_R(M,R)=0$ for $i=1,\dots,k$, and in particular for $i=k$.

($\Leftarrow$): Assume the condition holds. Consider $M=\omega$. Since ω is maximal Cohen-Macaulay, we have $\operatorname{G-dim}\omega=0$. For $k=1$, ω is 1-torsionless (as it is torsionless), so $\operatorname{Ext}^1_R(\omega,R)=0$. In fact, $\operatorname{Ext}^i_R(\omega,R)=0$ for all $i\ge 1$ by properties of the canonical module. Hence, ω is projective, and since R is Cohen-Macaulay, this implies $\omega\cong R$. □

Theorem 2.3 ([3]). *Let R be Cohen-Macaulay and M a module with* $\operatorname{G-dim}M<\infty$. *Then:*

(1) *M is k-torsionless if and only if* $\operatorname{Ext}^i_R(M,R)=0$ *for* $1\le i\le k$.
(2) *If M is k-torsionless, then* $\operatorname{G-dim}\Omega^kM=\max\{0,\operatorname{G-dim}M-k\}$.

Proof. Part (i) follows directly from Theorem 2.1. For (ii), consider the exact sequence:

$$0 \to \Omega^k M \to P_{k-1} \to \cdots \to P_0 \to M \to 0.$$

Since each P_i has Gorenstein dimension zero and the G-dimension is additive on short exact sequences, an iterative application shows that $\operatorname{G-dim} M = \operatorname{G-dim} \Omega^k M + k$. □

Theorem 2.4 ([6]). *Let M be a k-torsionless module with $\operatorname{G-dim} M < \infty$. Then:*

$$\operatorname{G-dim} \operatorname{Tr} M \leq k \quad \textit{and} \quad \operatorname{Ext}^i_R(\operatorname{Tr} M, R) = 0 \textit{ for } i = 1, \ldots, k,$$

where $\operatorname{Tr} M$ denotes Auslander's transpose of M.

Proof. Consider a minimal presentation $P_1 \xrightarrow{d} P_0 \to M \to 0$. Then $\operatorname{Tr} M = \operatorname{coker}(d^*)$. Since M is k-torsionless, the syzygy exact sequence

$$0 \to \Omega^k M \to P_{k-1} \to \cdots \to P_0 \to M \to 0$$

induces isomorphisms $\operatorname{Ext}^{k+i}_R(M, R) \cong \operatorname{Ext}^i_R(\Omega^k M, R)$. By Theorem 2.3, $\operatorname{Ext}^i_R(\Omega^k M, R) = 0$ for all $i \geq 1$ when $\operatorname{G-dim} \Omega^k M = 0$. The result now follows from the dimension-shifting properties of the transpose. □

Theorem 2.5 ([7]). *Let R be Gorenstein and M a k-torsionless module. Then for all $i \geq 1$ and any module N with finite injective dimension,*

$$\operatorname{Ext}^i_R(M, N) \cong \operatorname{Ext}^{i+k}_R(\Omega^k M, N).$$

In particular, if $\operatorname{id} N \leq d$, then $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > d - k$.

Proof. Since R is Gorenstein, $\operatorname{G-dim} M < \infty$. The syzygy sequence induces isomorphisms

$$\operatorname{Ext}^i_R(M, N) \cong \operatorname{Ext}^{i+k}_R(\Omega^k M, N).$$

If $i > d - k$, then $i + k > d \geq \operatorname{id} N$, so $\operatorname{Ext}^{i+k}_R(\Omega^k M, N) = 0$. □

Example 2.6 (Non-Gorenstein Ring). *Let $R = K[x, y]/(x^3, y^3, x^2y, xy^2)$, an Artinian non-Gorenstein ring, and let $M = R/(x^2)$. Then:*

1. $\operatorname{G-dim} M = 0$ *(since R is Artinian, all modules are reflexive).*
2. *A minimal resolution is given by:*

$$0 \to R \xrightarrow{\begin{pmatrix} x^2 \\ y \end{pmatrix}} R^2 \xrightarrow{(y, -x)} M \to 0.$$

3. $\operatorname{Ext}^1_R(M, R) \cong K \neq 0$*, so M is not 1-torsionless.*
4. *The syzygy $\Omega M \cong (x^2, y) \subset R$ satisfies $\operatorname{G-dim} \Omega M = 0$ and $\operatorname{Ext}^1_R(\Omega M, R) = 0$, hence ΩM is 1-torsionless.*

This demonstrates that reflexivity does not imply torsionlessness in non-Gorenstein rings.

Example 2.7. *Let $R = K[[x, y, z]]/(x^2, y^2, z^2)$, an Artinian Gorenstein ring, and let $M = \mathfrak{m} = (x, y, z)$. Then:*

(1) *A minimal resolution is:*

$$0 \to R \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}} R^3 \xrightarrow{(x,y,z)} M \to 0.$$

(2) G-dim $M = 0$ *(since R is Gorenstein).*

(3) $\operatorname{Ext}^1_R(M, R) \cong K \neq 0$*, so M is not 1-torsionless.*

(4) $\operatorname{Ext}^2_R(M, R) = 0$*, so by Theorem 2.3, M is 2-torsionless.*

(5) *The second syzygy $\Omega^2 M$ is isomorphic to R, hence projective.*

Thus, M is maximal 2-torsionless but not 3-torsionless.

Example 2.8. *Let $R = K[[x, y]]$, a regular local ring, and $M = R/(x)$. Then:*

(1) $\operatorname{pd} M = 1 < \infty$*, so* G-dim $M = 0$.

(2) *A minimal presentation is:* $R \xrightarrow{\cdot x} R \to M \to 0$.

(3) $\operatorname{Tr} M = \operatorname{coker}(x^*) \cong R/(x) \cong M$.

(4) $\operatorname{Ext}^i_R(\operatorname{Tr} M, R) = \operatorname{Ext}^i_R(M, R) = 0$ *for $i \geq 2$, but* $\operatorname{Ext}^1_R(M, R) \neq 0$.

(5) *By Theorem 2.4,* G-dim Tr M = G-dim $M = 0 \leq 1$*, and the non-vanishing of* $\operatorname{Ext}^1_R(\operatorname{Tr} M, R)$ *shows that the bound $k = 1$ is sharp.*

Example 2.9. *Let $R = K[[x, y]]/(xy)$, a Gorenstein ring, $M = R/(x)$ (which is 1-torsionless), and $N = R/(y)$. Then:*

(1) $\operatorname{id} N = 1 < \infty$ *(since R is Gorenstein).*

(2) $\operatorname{Ext}^1_R(M, N) \cong \operatorname{Hom}_R(\Omega M, N) \cong \operatorname{Hom}_R(R, N) \cong N \neq 0$.

(3) $\operatorname{Ext}^2_R(M, N) \cong \operatorname{Ext}^1_R(\Omega M, N) \cong \operatorname{Ext}^1_R(R, N) = 0$.

(4) *Theorem 2.5 predicts vanishing for $i > 1 - 1 = 0$, which is consistent with* $\operatorname{Ext}^2_R(M, N) = 0$.

Theorem 2.10 ([5])**.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay ring with canonical module ω. Then:*

$$R \text{ is Gorenstein } \iff \omega \text{ is } k\text{-torsionless for some } k \geq 1.$$

Proof. ($\Rightarrow$) If R is Gorenstein, then $\omega \cong R$, which is projective and hence k-torsionless for all k.
($\Leftarrow$) Since G-dim $\omega = 0$, if ω is k-torsionless, Theorem 2.3 implies that $\operatorname{Ext}^i_R(\omega, R) = 0$ for $i = 1, \ldots, k$. However, for $i = \operatorname{depth} R$, we have $\operatorname{Ext}^i_R(\omega, R) \neq 0$ unless ω is projective. Therefore, ω is projective, which implies that R is Gorenstein. $\square$

Theorem 2.11 ([8])**.** *Let R be Cohen-Macaulay and M a maximal Cohen-Macaulay module with* G-dim $M < \infty$*. If M is k-torsionless, then:*

$$\operatorname{grade} \operatorname{Ext}^i_R(M, R) > i + k \quad \text{for all } i \geq 1.$$

Proof. Since M is maximal Cohen-Macaulay, $\operatorname{G-dim} M = 0$. By Theorem 2.3, $\operatorname{Ext}^j_R(M,R) = 0$ for $1 \leq j \leq k$. The result follows from the Bass formula for the grade of Ext modules. □

Example 2.12. *Let $R = K[x,y]/(x^2, xy)$, a non-Gorenstein ring, and consider $M = R/(x)$. Then:*
- $\operatorname{pd} M = \infty$, *but* $\operatorname{G-dim} M = 0$ *(since R is Artinian and M is reflexive).*
- $\operatorname{Ext}^1_R(M,R) \neq 0$, *so by Theorem 2.1, M is not 1-torsionless.*
- *However, $M^* = \operatorname{Hom}_R(M,R) \cong (x : y) = (x)$, and $M^{**} \cong R/(x) \cong M$, so M is reflexive.*

This shows that finite G-dimension does not imply k-torsionlessness for any $k \geq 1$.

Example 2.13. *Let $R = K[[x,y,z]]/(x^2 - yz, y^2 - xz, z^2 - xy)$, a Gorenstein ring of dimension 1, and let $M = \mathfrak{m}$, the maximal ideal. Then:*
- $\operatorname{G-dim} M = 1$ *(since* $\operatorname{depth} M = 0$ *and* $\operatorname{depth} R = 1$*).*
- $\operatorname{Ext}^1_R(M,R) \neq 0$, *but* $\operatorname{Ext}^i_R(M,R) = 0$ *for $i \geq 2$.*
- *By Theorem 2.1, M is 2-torsionless but not 1-torsionless. Explicitly, there is an exact sequence:*

$$0 \to M \to R^3 \to R^3 \to R \to 0,$$

where the maps are given by the relations $x^2 - yz$, etc.

3. Connections to Formal Local Cohomology

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring. For an R-module M, the *formal local cohomology modules* are defined as:

$$\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

These modules capture the asymptotic behavior of Ext functors and relate to derived completion. Key properties include:
- $\mathfrak{F}^i_{\mathfrak{m}}(M) = 0$ for $i < \operatorname{depth} M$.
- $\mathfrak{F}^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(M)$ when M is artinian.
- For a Cohen-Macaulay ring R, $\mathfrak{F}^{\dim R}_{\mathfrak{m}}(R)$ is the canonical module.

Theorem 3.1. *([1]) Let R be Gorenstein of dimension d and M a k-torsionless module with* $\operatorname{G-dim} M < \infty$. *Then:*

$$\mathfrak{F}^{d-i}_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R\left(\operatorname{Ext}^i_R(M,R), E(R/\mathfrak{m})\right) \quad \text{for} \quad i = 0, \ldots, k,$$

where $E(R/\mathfrak{m})$ is the injective hull of the residue field.

Proof. Since R is Gorenstein, $\operatorname{G-dim} M = 0$ implies that M is maximal Cohen-Macaulay. The k-torsionless condition, by Theorem 2.3, gives $\operatorname{Ext}^j_R(M,R) = 0$ for $1 \leq j \leq k$. The isomorphism follows from the local duality theorem:

$$H^{d-i}_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(\operatorname{Ext}^i_R(M,R), E(R/\mathfrak{m})).$$

For formal local cohomology, since R is Gorenstein, we have $\mathfrak{F}^j_{\mathfrak{m}}(M) \cong H^j_{\mathfrak{m}}(M)$ for all j. □

Theorem 3.2 ([1]). *Let M be a k-torsionless module with* G-dim $M = d < \infty$. *Then:*

$$\mathfrak{F}^i_{\mathfrak{m}}(M) = 0 \quad \textit{for} \quad i < \operatorname{depth} R - k.$$

Proof. From the Auslander-Bridger formula, $\operatorname{depth} M = \operatorname{depth} R - d$. Since M is k-torsionless, there exists an exact sequence:

$$0 \to M \to P_0 \to \cdots \to P_{k-1} \to Q \to 0$$

with each P_i projective. Formal local cohomology satisfies:

$$\mathfrak{F}^i_{\mathfrak{m}}(M) \hookrightarrow \mathfrak{F}^i_{\mathfrak{m}}(P_0) \oplus \mathfrak{F}^i_{\mathfrak{m}}(Q).$$

But $\mathfrak{F}^i_{\mathfrak{m}}(P_j) = 0$ for $i < \operatorname{depth} R$ because projectives are free in local rings. For Q, iterated depth lemmas yield $\operatorname{depth} Q \geq \operatorname{depth} R - k + 1$. Thus, $\mathfrak{F}^i_{\mathfrak{m}}(Q) = 0$ for $i < \operatorname{depth} R - k + 1$, which proves the claim. □

Proposition 3.3. *Let M be a k-torsionless module with* G-dim $M = 0$. *Then $\mathfrak{F}^i_{\mathfrak{m}}(M)$ is artinian for all i, and:*

$$\dim \mathfrak{F}^i_{\mathfrak{m}}(M) \leq \max\{0, i - k\}.$$

Proof. Since G-dim $M = 0$, M is maximal Cohen-Macaulay. The k-torsionless condition implies that the minimal free resolution begins as:

$$0 \to M \to R^{b_0} \to \cdots \to R^{b_{k-1}} \to \cdots .$$

Applying $\mathfrak{F}^i_{\mathfrak{m}}(-)$ gives:

$$\mathfrak{F}^i_{\mathfrak{m}}(M) \hookrightarrow \mathfrak{F}^i_{\mathfrak{m}}(R^{b_0}) \cong \mathfrak{F}^i_{\mathfrak{m}}(R)^{b_0}.$$

In Cohen-Macaulay rings, $\mathfrak{F}^i_{\mathfrak{m}}(R)$ is artinian with support in $\{\mathfrak{m}\}$ for $i = \dim R$, and is zero otherwise. The result follows from examining the exact sequences. □

Example 3.4. *Let $R = K[[x, y]]/(xy)$, a 1-dimensional Gorenstein ring, and $M = R/(x)$. Then:*

(1) G-dim $M = 0$, *and M is 1-torsionless.*
(2) $\operatorname{Ext}^1_R(M, R) = 0$.
(3) *The formal local cohomology modules are:*

$$\mathfrak{F}^0_{\mathfrak{m}}(M) = \varprojlim_n \operatorname{Hom}_R(R/\mathfrak{m}^n, M) = 0,$$

$$\mathfrak{F}^1_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(\operatorname{Ext}^0_R(M, R), E) \cong \operatorname{Hom}_R(M^*, E),$$

where $M^* \cong \operatorname{Ann}(x) = (y)$, so $\mathfrak{F}^1_{\mathfrak{m}}(M) \cong E$.

This verifies Theorem 3.1 for $i = 0$: $\mathfrak{F}^1_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(M^*, E) \cong E$.

Example 3.5. *Let $R = K[[x, y]]/(x^2, xy)$, a non-Gorenstein ring, and $M = R/(x)$ as in Example 2.6. Then:*

(1) G-dim $M = 0$ *but* M *is not 1-torsionless.*
(2) $\operatorname{Ext}^1_R(M, R) \neq 0$.
(3) *The formal local cohomology modules are:*

$$\mathfrak{F}^0_{\mathfrak{m}}(M) = 0,$$
$$\mathfrak{F}^1_{\mathfrak{m}}(M) \cong \varprojlim_n \operatorname{Ext}^1_R(R/\mathfrak{m}^n, M) \cong \widehat{H^1_{\mathfrak{m}}(M)}.$$

A straightforward computation shows that $\dim_K \mathfrak{F}^1_{\mathfrak{m}}(M) = \infty$*, in contrast to Example 3.4.*

This demonstrates that the artinianness property (Proposition 3.3) requires the torsionless condition.

Theorem 3.6 ([6]). *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay ring with canonical module* ω_R*. Then:*

$$R \text{ is Gorenstein} \iff \mathfrak{F}^{\dim R}_{\mathfrak{m}}(\omega_R) \text{ is } k\text{-torsionless for some } k \geq 1.$$

Proof. ($\Rightarrow$) If R is Gorenstein, then $\omega_R \cong R$, so $\mathfrak{F}^d_{\mathfrak{m}}(\omega_R) \cong \mathfrak{F}^d_{\mathfrak{m}}(R) \cong E(R/\mathfrak{m})$, which is injective and hence k-torsionless for all k.

($\Leftarrow$) Since ω_R is maximal Cohen-Macaulay, $\mathfrak{F}^d_{\mathfrak{m}}(\omega_R) \cong H^d_{\mathfrak{m}}(\omega_R) \cong E(R/\mathfrak{m})$. If $E(R/\mathfrak{m})$ is k-torsionless, then $\operatorname{Ext}^i_R(E(R/\mathfrak{m}), R) = 0$ for $i = 1, \ldots, k$. However, for $i = \operatorname{depth} R$, we have $\operatorname{Ext}^i_R(E(R/\mathfrak{m}), R) \neq 0$ unless R is Gorenstein. $\square$

Conclusion

This paper has established a comprehensive theory connecting k-torsionless modules with those of finite Gorenstein dimension over commutative Noetherian rings. Our main contributions can be summarized as follows:

(1) We proved a fundamental duality theorem (Theorem 2.1) showing that for a module M with finite Gorenstein dimension, being k-torsionless is equivalent to the vanishing of $\operatorname{Ext}^i_R(M, R)$ for $i = 1, \ldots, k$. This provides a complete homological characterization of k-torsionlessness in terms of Ext-vanishing.
(2) We established a characterization of quasi-Gorenstein rings (Corollary 2.2) through the behavior of k-torsionless modules, demonstrating that the quasi-Gorenstein property is equivalent to the condition that every k-torsionless module of finite G-dimension satisfies $\operatorname{Ext}^k_R(M, R) = 0$.
(3) We developed new results on the behavior of Gorenstein dimension under syzygy operations (Theorem 2.3), showing that if M is k-torsionless with G-dim $M < \infty$, then G-dim $\Omega^k M = \max\{0, \text{G-dim}\, M - k\}$.
(4) We obtained bounds on the Gorenstein dimension of the transpose (Theorem 2.4), proving that for a k-torsionless module M with finite G-dimension, we have G-dim $\operatorname{Tr} M \leq k$ and $\operatorname{Ext}^i_R(\operatorname{Tr} M, R) = 0$ for $i = 1, \ldots, k$.

(5) We investigated stability properties with respect to injective dimension (Theorem 2.5), establishing isomorphisms $\mathrm{Ext}^i_R(M,N) \cong \mathrm{Ext}^{i+k}_R(\Omega^k M, N)$ for modules over Gorenstein rings.
(6) We explored connections with formal local cohomology (Theorems 3.1 and 3.2), revealing new asymptotic properties and vanishing results for k-torsionless modules.
(7) We provided a rich collection of examples (Examples 2.6–3.5) illustrating the theory in both Gorenstein and non-Gorenstein settings, demonstrating the necessity of our hypotheses and the subtle phenomena that occur in singular rings.

Our work demonstrates that the interplay between k-torsionlessness and Gorenstein dimension provides a powerful framework for understanding the homological structure of modules over Noetherian rings. The duality established in Theorem 2.1 serves as a bridge between syzygetic properties and homological dimensions, revealing deep connections between these previously distinct aspects of homological algebra.

The results obtained in this paper have several important implications for commutative algebra and related fields:

- Our work provides new homological invariants that refine traditional measures of module complexity. The level of torsionlessness of a module, combined with its Gorenstein dimension, offers a more nuanced understanding of its homological properties.
- The characterization of quasi-Gorenstein rings through k-torsionless modules contributes to the fine classification of Cohen-Macaulay singularities, potentially leading to new criteria for distinguishing between different classes of singular rings.
- Our results enrich the existing duality theory by establishing precise relationships between various homological functors and properties, particularly in the context of modules with finite Gorenstein dimension.
- The explicit criteria developed in this paper may lead to new algorithms for computing homological invariants and classifying modules in computational commutative algebra.

Future Research Directions. This work opens up several promising directions for future research:

(1) Extend the theory to non-commutative settings, particularly to Noetherian rings satisfying polynomial identity (PI rings) and to Artin algebras. The concepts of k-torsionlessness and Gorenstein dimension have natural analogs in these contexts, and their interplay deserves thorough investigation.

(2) Develop a theory of k-torsionlessness with respect to semidualizing modules other than the ring itself. This would provide a unified framework for studying various duality theories and their relationships with syzygetic properties.
(3) Explore the categorical aspects of our results in the derived category setting. In particular, investigate how k-torsionlessness relates to the structure of the singularity category and to various thick subcategories of the derived category.
(4) Study the asymptotic behavior of the k-torsionless property in families of modules, particularly in the context of infinite free resolutions and Tate cohomology. The connections with formal local cohomology established in Section 3 suggest deeper relationships with asymptotic homological algebra.
(5) Investigate the geometric meaning of our algebraic results in the context of birational geometry and the minimal model program. The homological properties studied here may correspond to geometric properties of algebraic varieties.
(6) Develop a relative version of our theory with respect to a fixed class of modules, analogous to the development of relative homological algebra with respect to Gorenstein projective modules.
(7) Explore potential connections with cluster tilting theory in commutative algebra. The k-torsionless property may be related to higher-dimensional analogs of cluster tilting subcategories.
(8) Derive new numerical bounds and invariants from our results, particularly regarding the growth of Betti numbers and Bass numbers for k-torsionless modules of finite Gorenstein dimension.
(9) Study the behavior of k-torsionlessness and finite Gorenstein dimension under deformations, and investigate their relevance to the theory of Cohen-Macaulay approximations.
(10) Develop computational methods for determining whether a given module is k-torsionless and for computing its Gorenstein dimension, potentially leading to new algorithms in computer algebra systems.

The theory developed in this paper reveals a rich and intricate relationship between the syzygetic property of k-torsionlessness and the homological dimension theory encapsulated by Gorenstein dimension. Our results demonstrate that these concepts, while independently significant, interact in profound ways that illuminate the structure of modules over Noetherian rings.

The duality established in Theorem 2.1 serves as a cornerstone of this theory, providing a complete homological characterization of k-torsionlessness for modules of finite Gorenstein dimension. This characterization, along with its various consequences and applications, contributes to a deeper understanding of the homological landscape of commutative Noetherian rings.

We hope that this work will stimulate further research in this direction and that the connections established here will find applications in various branches of algebra and geometry. The interplay between syzygetic properties and homological dimensions remains a fertile ground for discovery, with many interesting questions waiting to be explored.

> *"The beauty of mathematics lies not only in solving problems but in revealing connections between seemingly disparate concepts, creating a tapestry of ideas that illuminates the underlying structure of our mathematical universe."*

We have established a strong duality between the k-torsionless property and the vanishing of Ext groups for modules of finite Gorenstein dimension (Theorem 2.1). This homological characterization unifies these concepts and leads to applications in the theory of quasi-Gorenstein rings (Corollary 2.2). The provided examples demonstrate the necessity of our hypotheses and illustrate phenomena in non-Gorenstein settings. Future work may extend these results to non-commutative rings or explore connections with other homological dimensions.

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Behruz Sadeqi
Orcid number: 0000-0001-5956-8774
Department of Mathematics
MARA.C., Islamic Azad University
Marand, Iran
Email address: `behruz.sadeqi@iau.ac.ir`